Solution of homework I

$$\Phi_{xx}(\omega) = \int \frac{d^2k}{(2\pi)^2} \left(\frac{d\varepsilon_{\mathbf{k}}}{dk_x}\right)^2 \delta(\omega - \varepsilon_{\mathbf{k}}) \tag{1}$$

$$\frac{d\varepsilon_{\mathbf{k}}}{dk_x} = 2t \sin k_x \tag{2}$$

- $0 < x < 2$

$$\Phi_{xx}(x) = \frac{16t^2}{2t(2\pi)^2} \int_0^{\arccos(x-1)} \frac{\sin^2 k_x dk_x}{\sqrt{1-(x-\cos k_x)^2}}$$

- $-2 < x < 0$

$$\Phi_{xx}(x) = \frac{16t^2}{2t(2\pi)^2} \int_{\arccos(x+1)}^{\pi} \frac{\sin^2 k_x dk_x}{\sqrt{1-(x-\cos k_x)^2}}$$

Standart change of variables $\cos k_x = u$ gives

- $0 < x < 2$

$$\Phi_{xx}(x) = \frac{2t}{\pi^2}\int_{x-1}^{1}\frac{\sqrt{1-u^2}du}{\sqrt{1-(x-u)^2}}$$

- $-2 < x < 0$

$$\Phi_{xx}(x) = \frac{2t}{\pi^2}\int_{-1}^{x+1}\frac{\sqrt{1-u^2}du}{\sqrt{1-(x-u)^2}}$$

By inspection we see $\Phi_{xx}(-x) = \Phi_{xx}(x)$. Need to consider $x > 0$ and use $|x|$.

We have only one pole at the beginning of interval at $u = x - 1$.

Change of variable

If $0 < x < 1$, blindly changing the variable $t = \sqrt{1-(x-u)^2}$ does not work because t is not unique function of u (two u's give the same t!) $\longrightarrow$ need to split the interval of integration

a) $0 < x < 1$

$$\Phi_{xx}(x) = \frac{2t}{\pi^2}\int_{x-1}^{0}\frac{\sqrt{1-u^2}du}{\sqrt{1-(x-u)^2}} + \frac{2t}{\pi^2}\int_{0}^{1}\frac{\sqrt{1-u^2}du}{\sqrt{1-(x-u)^2}} \tag{3}$$

The second part is free of divergencies. Finished!

In the first part, change the variable $t = \sqrt{1-(x-u)^2}$.

$$\Phi_{xx}(x) = \frac{2t}{\pi^2}\int_{0}^{\sqrt{1-x^2}}\frac{\sqrt{1-(x-\sqrt{1-t^2})^2}dt}{\sqrt{1-t^2}} + \frac{2t}{\pi^2}\int_{0}^{1}\frac{\sqrt{1-u^2}du}{\sqrt{1-(x-u)^2}} \tag{4}$$

b) $1 < x < 2$

$t = \sqrt{1-(x-u)^2}$ works since $t(u)$ is unique. We get

$$\Phi_{xx}(x) = \frac{2t}{\pi^2}\int_{0}^{\sqrt{1-(x-1)^2}}\frac{\sqrt{1-(x-\sqrt{1-t^2})^2}dt}{\sqrt{1-t^2}} \tag{5}$$

Subtraction of divergent part

Let us Taylor expand around the pole $u = x - 1 + \epsilon$

$$\sqrt{\frac{1-(x-1+\epsilon)^2}{1-(1-\epsilon)^2}} = \sqrt{\frac{2x-x^2}{2\epsilon}} + \sqrt{\frac{2\epsilon}{2x-x^2}}\frac{4-2x-x^2/4}{8} \tag{6}$$

This expansion is very precise close to the pole but bad far from the pole, therefore we will use it only in one half of the interval

$$\Phi_{xx}(x) = \frac{2t}{\pi^2}\int_{x/2}^{1}\frac{\sqrt{1-u^2}du}{\sqrt{1-(x-u)^2}}$$
$$+\frac{2t}{\pi^2}\int_{x-1}^{x/2}du\left[\frac{\sqrt{1-u^2}}{\sqrt{1-(x-u)^2}} - \sqrt{\frac{2x-x^2}{2(1+u-x)}} - \sqrt{\frac{2(1+u-x)}{2x-x^2}}\frac{4-2x-x^2}{8}\right]$$
$$+\frac{2t}{\pi^2}\frac{(2-x)(4+22x-x^2)}{24\sqrt{x}} \tag{7}$$

FInally, one can test the normalization of the transport function. From the definition it follows

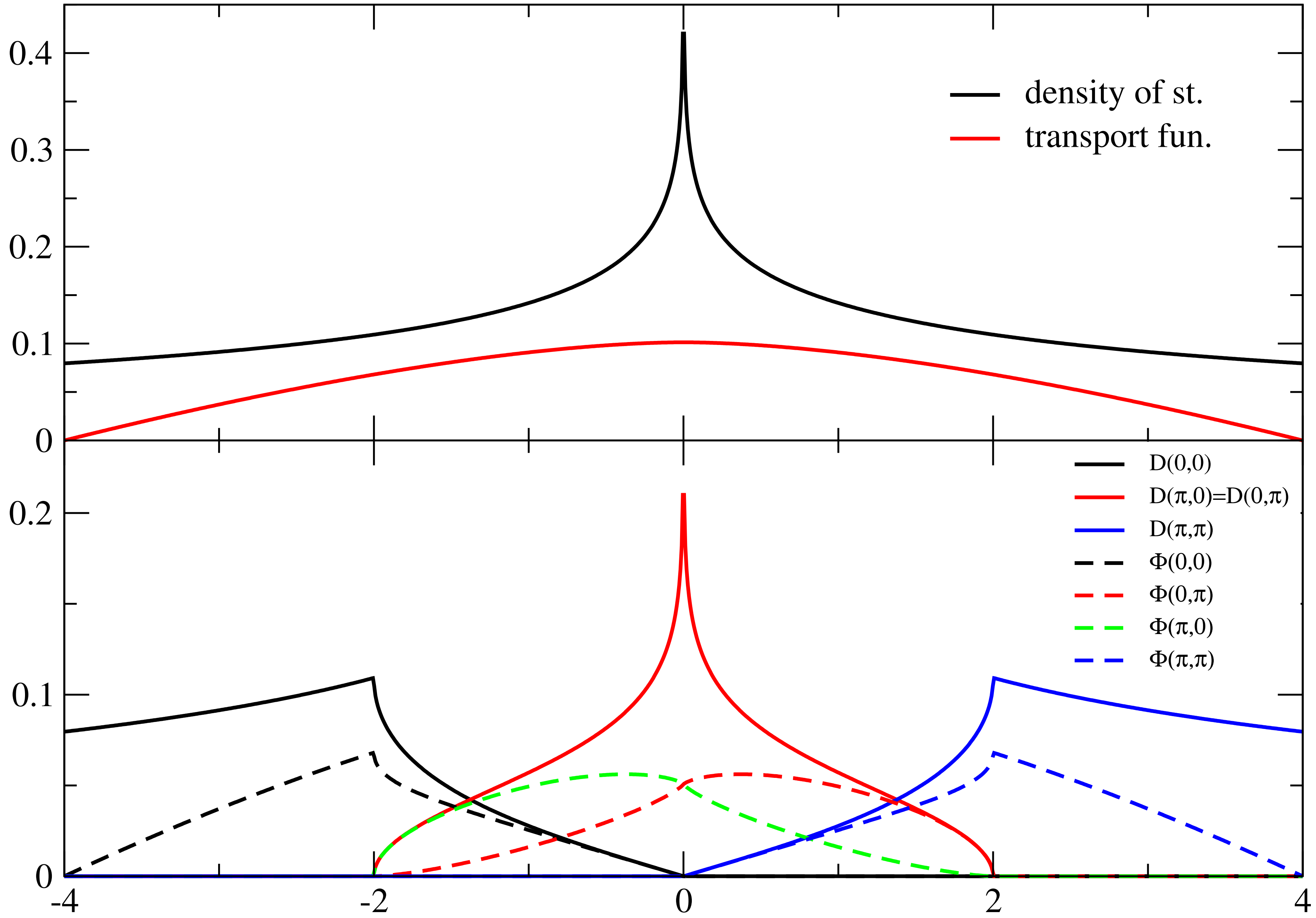
$$\int d\omega \Phi_{xx}(\omega) = \int \frac{d^2k}{(2\pi)^2}(2t\sin k_x)^2 = (2t)^2\frac{1}{2} = 2t^2 \tag{8}$$

In terms of Elliptic integrals:

$$D(y) = \frac{1}{2t\pi^2}\frac{1}{|y|}K(1-1/y^2) \quad (9)$$

$$\Phi_{xx}(y) = \frac{2t}{\pi^2}\left[2|y|E(1-1/y^2) + 2K(1-1/y^2) - 2\Pi(1-1/|y|, 1-1/y^2)\right] \quad (10)$$

Here, $K(x)$, $E(x)$ and $\Pi(x)$ are complete elliptic integrals of first, second and third kind and $y = \omega/(4t)$.



0.4
0.3
0.2
0.1
0
density of st.
transport fun.
D(0,0)
D(π,0)=D(0,π)
D(π,π)
Φ(0,0)
Φ(0,π)
Φ(π,0)
Φ(π,π)
0.2
0.1
0
-4
-2
0
2
4