Hmw2_Kn_recursion

December 31, 2020

1 Recursion for K_n functions

$$K_n(z,\alpha,a,b) = \int_a^b dx \frac{x^n}{z+\alpha\, x} \tag{1}$$

This integral comes from expanding correlation function in complete basis set of Legendre Polinomials $< P_n|G|P_m >$, where $G(x) = 1/(z-x)$

To find recursion, we start with K_{n+1} and rewrite it in terms of K_n:

$$K_{n+1} = \int_a^b dx \frac{(x+z/\alpha)x^n - z/\alpha\; x^n}{z+\alpha\, x} = \frac{1}{\alpha}\int_a^b x^n dx - \frac{z}{\alpha}K_n = \frac{b^{n+1}-a^{n+1}}{\alpha(n+1)} - \frac{z}{\alpha}K_n \tag{2}$$

If $|\frac{\alpha}{z}| > 1$ upward recursion stable: $K_{n+1} = \frac{1}{\alpha(n+1)} - \frac{z}{\alpha}K_n$

If $|\frac{\alpha}{z}| < 1$ downward recursion stable: $K_n = -\frac{\alpha}{z}K_{n+1} + \frac{1}{z(n+1)}$

Because recursion is not homogeneous (it is however linear), we can not multiply all terms with an arbitrary constant to normalize the entire series. Instead, we need to start with a very precise value of K_n for some n.

This can be achieved by Taylor expansion:

$$K_n = \frac{1}{z}\int_a^b dx x^n (1+\frac{\alpha}{z}x)^{-1} = \frac{1}{z}\int_a^b dx x^n \sum_{m=0}(-\frac{\alpha}{z}x)^m = \sum_{m=0}\frac{(-\alpha)^m}{z^{m+1}}\frac{b^{n+m+1}-a^{n+m+1}}{n+m+1} \tag{3}$$

Lets take $b = 1$ and $a = 0$, which gives

$$K_n = \sum_{m=0}^{M}\frac{(-\alpha)^m}{z^{m+1}}\frac{1}{n+m+1} \tag{4}$$

For $|\alpha/z| < 1$ the terms beyond some large M are arbitrary small, hence we can stop the sum at some large M, approximately $(\alpha/z)^M < \epsilon$ and $M \approx \frac{\log(\epsilon)}{\log(\alpha/z)}$. Hence we can compute K_n for the largest n with Taylor expansion, and then use downward recursion to obtain any other n.

[28]:
```
from scipy import *
from numpy import *

def GetPolyInt(z, alpha, n):
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