Midterm Exam, Quantum Mechanics 501, Rutgers

October 21, 2015

1. **Short questions:**

   (a) What is a pure state? How does its density matrix look like?

   Ans: A Pure state is a state described by a single wave function $|\psi\rangle$. Density matrix is $\rho = |\psi\rangle \langle\psi|$.

   (b) When you make measurement on a pure state, are you assured of getting a precise value for an observable?

   Ans.: No. Pure state is not necessary an eigenstate of an observable.

   (c) What is the form of the density matrix for mixed (non-pure) state?

   Ans.: $\rho = \sum_i P_i |\psi_i\rangle \langle\psi_i|$

   (d) If $|i\rangle$ and $|j\rangle$ are eigenkets of Hermitian operator A. Under what conditions is $|i\rangle + |j\rangle$ an eigenket of A?

   Ans.: If $|i\rangle$ and $|j\rangle$ have degenerate eigenvalues of operator A.

   (e) Without explicit calculation, sketch the wavefunction of the lowest two eigenstates of a particle in a potential

   $$V(x) = \begin{cases} kx & x \geq 0 \\ \infty & x < 0 \end{cases} \tag{1}$$

   taking care to show how the shape and amplitude vary with position.

2. Two quantum operators have the matrix representation

   $$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

   (a) A system is in quantum state $|\psi\rangle$ that is in an eigenfunction of operator A, corresponding to eigenvalue -1. Then for this state, what are $\langle A\rangle$ and ΔA?

   Ans.: $\langle A\rangle = -1$, and $\Delta A = 0$, because observable is sharply defined in an eigenstate.

   (b) First B is measured and the result is $b = -1$. What is the state of the system after the measurement?

   Ans.: It is in the eigenstate of B with eigenvalue -1, which is $|b = -1\rangle = \frac{1}{\sqrt{2}}(0, 1, -1)$.

(c) Immediately afterwards, A is measured. What is the probability to find $a = 1$? There are two states with eigenvalue $a = 1$. These are

$$|a_1\rangle = \frac{1}{\sqrt{2}}(1,1,0) \tag{3}$$

$$|a_2\rangle = (0,0,1) \tag{4}$$

The probability is therefore $P(1) = |\langle a_1|b\rangle|^2 + |\langle a_2|b\rangle|^2 = \frac{3}{4}$

(d) Assuming that $a = 1$ was indeed found in (c), what is the state of the system after the measurement of A?

Ans.: We will find the system in the projected state

$$|\psi_{final}\rangle = (|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2|)\,|b\rangle \propto \frac{1}{\sqrt{6}}(1,1,-2)$$

3. The wavefunction of a particle of mass m is in a 1D potential $V(x)$ is

$$\psi(x) = \begin{cases} Axe^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{5}$$

(a) Assuming the particle is in an eigenstate of the Hamiltonian, find the potential $V(x)$ and the total energy E for this state.

Ans.: It needs to satisfy the Schroedinger Equation. The second derivative is

$$\psi''(x) = \psi(x)(a^2 - 2\frac{a}{x}) \tag{6}$$

which gives for the Schroedinger equation

$$-\frac{\hbar^2}{2m}(a^2 - 2\frac{a}{x})\psi(x) + V(x)\psi(x) = E\psi(x) \tag{7}$$

For the equation to be satisfied, we need $V(x) = -\frac{\hbar^2 a}{m}\frac{1}{x} + C$. Without loss of generality, we can set $C = 0$, which gives $E = -\frac{\hbar^2 a^2}{2m}$. Note that this holds only for $x > 0$, as the wave function vanishes at $x = 0$ (with finite derivative) and potential is therefore infinite at $x < 0$.

(b) Find the potential energy expectation value $\langle V\rangle$ for this state

Ans.:

$$\langle V\rangle = \int_0^\infty V(x)\psi(x)^2 = A^2\int_0^\infty dx x^2 e^{-2ax}\left(-\frac{\hbar^2 a}{mx}\right) = -\frac{\hbar^2}{m}\frac{A^2}{4a} \tag{8}$$

and

$$1 = \int_0^\infty dx\psi(x)^2 = A^2\int_0^\infty dx x^2 e^{-2ax} = \frac{A^2}{4a^3} \tag{9}$$

hence $\langle V\rangle = -\frac{\hbar^2 a^2}{m}$.

(c) Find the expectation value of the kinetic energy for this state.

Ans.: $\langle K \rangle = E - \langle V \rangle = \frac{\hbar^2 a^2}{2m}$

4. The eigenstates, which are accesible to a single electron, have energies ε_0, ε_1 and ε_2 and their states are $|0\rangle$, $|1\rangle$ and $|2\rangle$. When two electrons are introduced in such system, what are possible wave-functions of the system of two electrons, if we neglect interaction between the two electrons?

(a) How many possible states can you write down, which have correct statistics? Write them down.

Ans.:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \tag{10}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |2\rangle - |2\rangle \otimes |0\rangle) \tag{11}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle) \tag{12}$$

(b) What are the energies of these states?

Ans.: $\varepsilon_0 + \varepsilon_1$, $\varepsilon_0 + \varepsilon_2$, $\varepsilon_1 + \varepsilon_2$.

(c) Is the state $|0\rangle \otimes |1\rangle$ a valid wave function of such system? Why (not)?

Ans.: No. It does not satisfy fermionic statistics for identical particles.