Final Exam, Quantum Mechanics 501, Rutgers

December 15, 2015

1. (a) Construct the spin singlet ($S = 0$) state and the spin triplet ($S = 1$) states of a two electron system.

   Answ.: singlet:

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{1}$$

   triplets:

$$|1,1\rangle = |\uparrow\uparrow\rangle \tag{2}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{3}$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle \tag{4}$$

   (b) In the experiment we have two electrons, which are in the spin-singlet state. They move in the opposite direction along the y-axis, and two observers A and B measure the spin state of each electron. A measures the spin component along the z axis, and B measures the spin component along an axis making an angle θ with the z axis in the xz-plane. Suppose that A's measurement yields a spin down state and subsequently B makes a measurement. What is the probability that B's measurement yields an up spin (measured along an axis making an angle θ with the z-axis)?

   The explicit formula for the representation of the rotation operator $\exp(-i\mathbf{S}\cdot\hat{\mathbf{n}}\theta/\hbar)$ in the spin space is given by the spin 1/2 Wigner matrix

$$D^{(1/2)}(\hat{\mathbf{n}},\theta) = \begin{pmatrix} \cos(\theta/2) - in_z\sin(\theta/2) & (-in_x - n_y)\sin(\theta/2) \\ (-in_x + n_y)\sin(\theta/2) & \cos(\theta/2) + in_z\sin(\theta/2) \end{pmatrix} \tag{5}$$

   and $\hat{\mathbf{n}} = n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z$ ($|\hat{\mathbf{n}}| = 1$) is the axis of rotation.

Answ.: Since the state of two electrons is singlet, and we know that the first electron points down, the second has to point up in the same coordinate system. But observer B is rotated by θ around y axis, hence we need to find how spin-up looks in the rotated coordinate system. We thus apply $D^{(1/2)}(\vec{e}_y,\theta)$ on $(1,0)$ to get

$$|\psi_B\rangle = (cos(\theta/2), sin(\theta/2)) \tag{6}$$

The probability for up-spin is thus $P(|\uparrow\rangle) = \cos^2(\theta/2)$ and for down-spin $P(|\downarrow\rangle) = \sin^2(\theta/2)$.

2. The Wigner-Eckart theorem s given by

$$\langle n'j'm'|T_q^{(l)}|njm\rangle = \langle j'm'|lq, jm\rangle \frac{\langle\langle n'j'|T^{(l)}|nj\rangle\rangle}{\sqrt{2j+1}} \tag{7}$$

(a) Explain the meaning of the two terms on the right hand side.

Answ.: The first term is the Clebsch-Gordan coefficient, which encodes the geometric properties of the matrix element under rotation. The second is the reduced matrix element, which is a common coefficient for all m,m' quantum numbers.

(b) The interaction of the electromagnetic field with a charged particle is given by

$$\Delta H = \frac{e}{2m}\mathbf{A}\cdot\mathbf{p}$$

If the electromagnetic fields are in the form of a plane wave, then $\mathbf{A} = A_0\ \hat{\varepsilon}e^{i\mathbf{kr}}$, where $\hat{\varepsilon}$ is the polarization of the plane wave. Assuming that the wavelength $\lambda = 2\pi/k$ is much larger than the atomic size, we may write

$$\mathbf{A} = A_0\hat{\varepsilon}(1 + i\mathbf{k}\cdot\mathbf{r} + \cdots)$$

such that

$$\Delta H \approx \frac{e}{2m}A_0\ \hat{\varepsilon}\cdot\mathbf{p}(1 + i\mathbf{k}\cdot\mathbf{r})$$

Her we kept both the dipole, and the quadrupole terms.

If the field is polarized along the x-axis ($\hat{\varepsilon} = \vec{e}_x$), and the wave propagation is along the z-axis ($\mathbf{k} = k\vec{e}_z$) express the Hamiltonian in terms of spherical harmonics. Note that $\mathbf{p}$ is a vector operator, and transforms under rotation as $\mathbf{r}$. For symmetry consideration you may therefore replace $\mathbf{p}$ by $C\mathbf{r}$

Answ.: The Hamiltonian for the above configuration is

$$\Delta H = \frac{e}{2m}A_0C(x + ik\ xz) \tag{8}$$

Using the expressions for $Y_{lm}'s$ we can get

$$x = \sqrt{\frac{2\pi}{3}}r(Y_{1,-1} - Y_{1,1}) \tag{9}$$

$$xz = \sqrt{\frac{2\pi}{15}}r^2(Y_{2,-1} - Y_{2,1}) \tag{10}$$

hence

$$\Delta H = \frac{e}{2m}A_0C\sqrt{\frac{2\pi}{3}}r(Y_{1,-1} - Y_{1,1} + i\frac{kr}{\sqrt{5}}(Y_{2,-1} - Y_{2,1})) \tag{11}$$

(c) For the above configuration, derive the selection rules for the dipole and the quadrupole transitions, by considering the transition probability matrix elements $|\langle\psi_f|\Delta H|\psi_i\rangle|^2 = |\langle l_f m_f|\Delta H|l_i m_i\rangle|^2$.

Answ.: The dipole matrix elements are proportional to

$$\langle l_f m_f|\Delta H_1|l_i m_i\rangle \propto \langle l_f m_f|Y_{1,-1} - Y_{1,1}|l_i m_i\rangle \propto \langle l_f m_f|11, l_i m_i\rangle - \langle l_f m_f|1-1, l_i m_i\rangle \quad (12)$$

hence $|m_f - m_i| = 1$, and $|l_f - l_i| \leq 1$.
The quadrupole matrix elements are

$$\langle l_f m_f|\Delta H_2|l_i m_i\rangle \propto \langle l_f m_f|Y_{2,-1} - Y_{2,1}|l_i m_i\rangle \propto \langle l_f m_f|21, l_i m_i\rangle - \langle l_f m_f|2-1, l_i m_i\rangle \quad (13)$$

hence $|m_f - m_i| = 1$, and $|l_f - l_i| \leq 2$.

The explicit expressions for the spherical harmonics for $l = 1, 2$ are given by

$$Y_{1,1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r} \qquad Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \quad (14)$$

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{r^2} \qquad Y_{2,1} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)z}{r^2} \qquad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\frac{2z^2 - x^2 - y^2}{r^2} \quad (15)$$

and $Y_{l,-m} = (-1)^m Y^*_{l,m}$.

3. A particle of reduced mass $\mu = 200\, MeV/c^2$ is moving in a spherical potential well of range a and depth $V_0 = -150\, MeV$. $[V(\mathbf{r}) = V_0$ for $|\mathbf{r}| < a$ and $V(\mathbf{r}) = 0$ for $|\mathbf{r}| > a]$.

   The particle is bound in the $1s$ ground state with binding energy $E = -5\, MeV$. (This is supposed to be a very simple model of the deuteron). Note: $\hbar c = 197.327\, MeV\, fm$.

   (a) Solve the Schroedinger equation for both $r < a$ and for $r > a$.

   (b) Using the boundary conditions at $r = a$, extract the size of the "potential range" a.

   (c) Calculate the probability that a measurement of r will find $r > a$, i.e. the particle is outside the range of the potential (which is of course forbidden classically).

   Answ.: The radial wave function for $l = 0$ solution is

$$\psi(r < a) = A\frac{\sin(kr)}{r} \quad (16)$$

$$\psi(r > a) = C\frac{e^{-\kappa r}}{r} \quad (17)$$

where

$$k = \sqrt{\frac{2\mu(E - V_0)}{\hbar^2}}$$

and

$$\kappa = \sqrt{\frac{2\mu|E|}{\hbar^2}}$$

Given the numbers in the text, we can get

$$k = 1.22/fm$$

$$\kappa = 0.227/fm$$

The continuity of the wave function and its derivative at $r = a$ gives the following set of equations

$$A \sin(ka) = Ce^{-\kappa a} \tag{18}$$
$$A\,k \cos(ka) = -C\kappa e^{-\kappa a} \tag{19}$$

which is satisfied if

$$\tan(ka) = -\frac{k}{\kappa}. \tag{20}$$

This equation can be solved for range parameter a, and the first solution ($1s$) gives:

$$a = \frac{1}{k}(\pi - \arctan(k/\kappa)) \approx 1.44fm$$

The probability for the particle to be outside the well is

$$P(r > a) = \frac{\int_a^\infty |\psi(r)|^2 r^2 dr}{\int_0^\infty |\psi(r)|^2 r^2 dr} \tag{21}$$

The integration inside the well gives $A^2 \int_0^a \sin^2(kr)dr = A^2 \frac{a}{2}(1 - \frac{\sin(2ka)}{2ka})$ and integration outside the box gives $C^2 \int_a^\infty e^{-2\kappa r} dr = C^2 \frac{e^{-2\kappa a}}{2\kappa}$. We also have $C/A = e^{\kappa a} \sin(ka)$
The ratio that describes the probability $P(r > a)$ is

$$\frac{\sin^2(ka)}{\sin^2(ka) + \kappa a(1 - \frac{\sin(2ka)}{2ka})} \approx 0.73 \tag{22}$$