The u-Plane Integral As A Tool In The Theory Of Four-Manifolds

Gregory Moore
Rutgers University

SCGP, April 26 , 2017

1. **Introduction**

2. Lightning Summary: The Physical Derivation Of Witten's Conjecture Relating Donaldson & SW

3. Brief Overview Of The World Of N=2 Theories

4. Donaldson-Witten Partition Function For General Compact Simple Lie Group

5. Theories With Matter & Superconformal Simple Type

6. Two Possible Future Directions

Introduction

Most of this talk reviews work done around 1997-1998:

Moore & Witten   Marino & Moore   Marino, Moore, & Peradze

Overlapping work: Losev, Nekrasov, & Shatashvili

Central Question: Given the successful application of N=2 SYM for SU(2) to the theory of 4-manifold invariants, are there interesting applications of OTHER N=2 field theories?

Recently re-visited with Iurii Nidaiev

1 Introduction

2 Lightning Summary: The Physical Derivation Of Witten's Conjecture Relating Donaldson & SW

3 Brief Overview Of The World Of N=2 Theories

4 Donaldson-Witten Partition Function For General Compact Simple Lie Group

5 Theories With Matter & Superconformal Simple Type

6 Two Possible Future Directions

Review: Derivation Of Witten Conjecture From SU(2) SYM

X: Smooth, compact, $\partial X = \emptyset$, oriented, $(\pi_1(X) = 0)$

Twisted N=2 SYM on X for simple Lie group G: Sum over connections $A \in \mathcal{A}(P)$ on all G_{adj} bundles $P \to X$ with fixed 't Hooft flux $\xi \in H^2\left(X; \pi_1(G_{adj})\right)$ together with various fields valued in ad P:

$\phi \in \Omega^0(adP \otimes \mathbb{C})$ $\chi \in \Pi\Omega^{2,+}(adP)$ $\eta \in \Pi\Omega^0(adP)$ $\psi \in \Pi\Omega^1(adP)$

Formally: Correlation functions of Q-invariant operators localize to integrals over the finite-dimensional moduli spaces of G-ASD conn's.

Witten's proposal: For $G = SU(2)$ correlation functions of Q-invariant operators are the Donaldson polynomials.

Local Observables

$$U \in Inv(\mathfrak{g}) \Rightarrow \ U(\phi)$$

$$\begin{tikzcd}
H_*(X, \mathbb{Z}) \arrow[rr, "\text{Descent formalism}"] \arrow[dr, "\mu"'] & & H^*(Fieldspace; Q) \arrow[dl, "\text{Localization identity}"] \\
& H^*(\mathcal{M}) &
\end{tikzcd}$$

$\mathfrak{g} = \mathfrak{su}(2)$ $\qquad U = Tr_2\left(\frac{\phi^2}{8\pi^2}\right)$ $\qquad U(S) \sim \int_S Tr(\phi F + \psi^2)$

Donaldson-Witten Partition Function

$$Z_{DW}^{\xi}(p,s) = \langle e^{2p\, U + s^a U(S_a)} \rangle_\Lambda$$

$$= \Lambda^{-\frac{3}{4}(\chi+\sigma)} \sum \frac{(\Lambda^2 p)^\ell (\Lambda s)^r}{\ell!\, r!} \wp_D(pt^\ell S^r)$$

Mathai-Quillen & Atiyah-Jeffrey:
Path integral formally localizes:

$$\mathcal{M} \hookrightarrow \mathcal{A}/\mathcal{G}$$

Strategy: Evaluate in LEET:
Integrate over vacua on $\mathbb{R}^4$

Spontaneous Symmetry Breaking

$SU(2) \rightarrow U(1)$ by vev of adjoint Higgs field ϕ:

Order parameter:
$u = \langle\, U(\phi)\, \rangle$

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C}/W \cong \mathbb{C}$

$adP \rightarrow L^2 \oplus \mathcal{O} \oplus L^{-2}$

Photon: Connection A on L

$U(1)\ \ VM$: $(a^q, A, \chi, \psi, \eta)$

a^q: complex scalar field on $\mathbb{R}^4$:

Do path integral of quantum fluctuations around
$a^q(x) = a + \delta a(x)$

What is the relation of $a = \langle a^q(x) \rangle$ to u ?

What are the couplings in the LEET for the U(1) VM ?

LEET: Constraints of N=2 SUSY

General result on N=2 abelian gauge theory with Lie algebra $\mathfrak{t} \cong \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)$: Action determined by a family of Abelian varieties and an ``N=2 central charge function'':

$$\mathfrak{A} \to \mathfrak{t} \otimes \mathbb{C} - \mathcal{D} \qquad \Gamma := H_1(\mathfrak{A}; \mathbb{Z})$$

$$Z: \Gamma \to \mathbb{C} \qquad \langle dZ, dZ \rangle = 0$$

Duality Frame: $\Gamma \cong \Gamma^{electric} \oplus \Gamma^{magnetic}$

$$a^I = Z(\alpha^I) \quad a_{D,I} := Z(\beta_I) = \left(\frac{\partial \mathcal{F}}{\partial a^I}\right) \qquad \tau_{IJ} := \frac{\partial a_{D,I}}{\partial a^J}$$

$$Action \sim \int_X \bar{\tau}(F^+)^2 + \tau(F^-)^2 + da^q * (Im\,\tau) d\,\overline{a^q} + \cdots$$

Seiberg-Witten Theory:

For G=SU(2) SYM $\mathfrak{A}$ is a family of elliptic curves:

$$E_u: \quad y^2 = x^2(x-u) + \frac{\Lambda^4}{4}x \qquad\qquad u \in \mathbb{C}$$

$$Z(\gamma) = \oint_\gamma \lambda \qquad\qquad \lambda = \frac{dx}{y}(x-u)$$

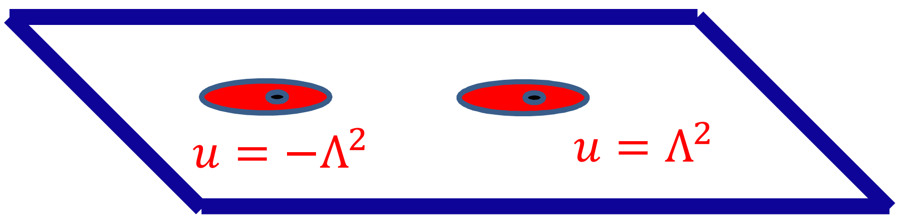
$u \to \infty$: *Invariant cycle A*: $\qquad a(u) = \oint_A \lambda$

Choose B-cycle: $\Rightarrow \tau(a) \Rightarrow$ Action for LEET

LEET breaks down at $u = \pm\Lambda^2$ where $Im(\tau) \to 0$

Seiberg-Witten Theory - II

LEET breaks down because there are new massless fields associated to BPS states



$$U(1)_D \ VM: \quad (a_D, A_D, \chi_D, \psi_D, \eta_D)$$

Near $\mathcal{U}_{\Lambda^2}$:

$$+$$

Charge 1 HM: $(M = q \oplus \tilde{q}^*, \cdots)$

$$Z_{DW}^{\xi}(p,s) = Z_u + Z_{\Lambda^2} + Z_{-\Lambda^2}$$

u-Plane Integral Z_u

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$

$$Z_u = \int da\, d\bar{a} \;\left(\frac{du}{da}\right)^{\frac{\chi}{2}} \Delta^{\frac{\sigma}{8}}\; e^{2pu+S^2T(u)}\;\Theta$$

$$\Delta = (u - \Lambda^2)(u + \Lambda^2)$$

Contact term: $T(u) = \left(\frac{du}{da}\right)^2 E_2(\tau) - 8\,u$

Θ: Sum over line bundles for the U(1) photon.

Photon Theta Function

$$\Theta = e^{y^{-1}\left(\frac{du}{da}\right)^2 S_+^2} \sum_{\lambda=\lambda_0+H^2(X,\mathbb{Z})} y^{-\frac{1}{2}} e^{-i\,\pi\bar{\tau}\lambda_+^2 - i\,\pi\tau\lambda_-^2}$$

$$\underbrace{(-1)^{\mathrm{w}_2(\mathrm{X})\cdot(\lambda-\lambda_0)}}_{}\,\underbrace{\mathrm{e}^{-\mathrm{i}\left(\frac{\mathrm{du}}{\mathrm{da}}\right)S\cdot\lambda_-}}_{}\left(\frac{\mathrm{d}\bar{\tau}}{d\bar{a}}\right)\left(\lambda_+ + \frac{1}{4\pi y}S_+\left(\frac{du}{da}\right)\right)$$

$$\tau = x + i\,y$$

$2\lambda_0$ is an integral lift of $\xi = w_2(P)$

Metric dependent! $\lambda = \lambda_+ + \lambda_-$

Contributions From $\mathcal{U}_{\Lambda^2}$

Path integral for $U(1)_D$ VM + HM:

General considerations imply:

$$\sum_{\lambda \in \frac{1}{2} w_2(X) + H^2(X,\mathbb{Z})} SW(\lambda) e^{2\pi i \, \lambda \cdot \lambda_0} \, R_\lambda(p,S)$$

$$R_\lambda(p,S) = Res\left[\left(\frac{d a_D}{a_D^{1+\frac{d(\lambda)}{2}}} \right) e^{2pu + S^2 T(u) + i \left(\frac{du}{da_D}\right) S\cdot\lambda} C(u)^{\lambda^2} P(u)^{\sigma} E(u)^{\chi} \right]$$

$$d(\lambda) = \frac{(2\lambda)^2 - c_1^2}{4}$$

$$u = \Lambda^2 + Series \; a_D$$

$$c_1^2 = 2\chi + 3\,\sigma$$

C,P,E : Universal functions. In principle computable.

Deriving C,P,E From Wall-Crossing

$$\frac{d}{dg_{\mu\nu}} Z_u = \int Tot\ deriv = \oint_{\infty} du(\ldots.) + \oint_{\Lambda^2} du(\ldots.) + \oint_{-\Lambda^2} du(\ldots.)$$

Z_u piecewise constant: Discontinuous jumps across walls:

$$\Delta_{\infty} Z_u: \;\; W(\lambda): \quad \lambda_+ = 0 \quad \lambda = \lambda_0 + H^2(X, \mathbb{Z})$$

Precisely matches formula of Göttsche!

$$\Delta_{\pm\Lambda^2} Z_u: \;\; W(\lambda): \quad \lambda_+ = 0 \quad \lambda = \frac{1}{2} w_2(X) + H^2(X, \mathbb{Z})$$

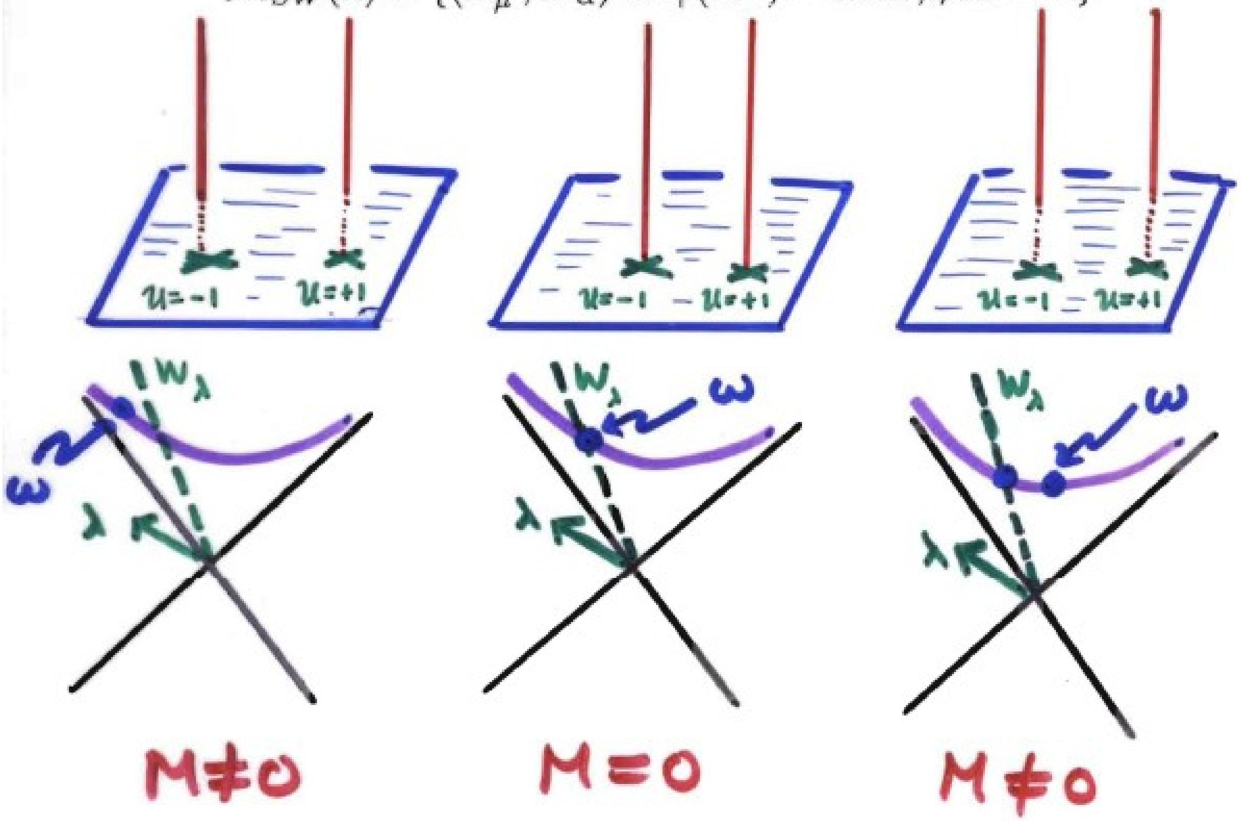
$$\Delta_{\Lambda^2} Z_u + \Delta Z_{\Lambda^2} = 0 \Rightarrow \;\; C(u), P(u), E(u)$$

$$\boxed{Z_{DW} = \langle e^{p\mathcal{O}+I(S)} \rangle_{\rm micro} = Z_{\rm Coulomb} + Z_{\rm Higgs} = Z_u + Z_{SW}}$$

Donaldson polynomials do *not* jump at SW walls $\Rightarrow$

$$\underline{0 = \delta Z_{DW} = \delta Z_{\rm Coulomb} + \delta Z_{\rm Higgs}}$$

$$\mathcal{M}_{SW}(\lambda) = \{(A^D_\mu, M_\alpha) : F_+(A^D) = \bar{M}M, \not{D}M = 0\}$$



Witten Conjecture

Now, with C,P,E known one takes $b_2^+(X) > 1$ and SWST to recover the Witten conjecture:

$$Z_{DW}^{\xi}(p,s) = 2^{c^2-\chi_h}\left(e^{\frac{1}{2}S^2+2p}\sum_{\lambda} SW(\lambda) e^{2\pi i\,\lambda\cdot\lambda_0} e^{2S\cdot\lambda} + e^{-\frac{1}{2}S^2-2p}\sum_{\lambda} SW(\lambda) e^{2\pi i\,\lambda\cdot\lambda_0} e^{-2iS\cdot\lambda}\right)$$

$$\chi_h = \frac{\chi+\sigma}{4} \qquad c^2 = 2\chi + 3\,\sigma$$

1 Introduction

2 Lightning Summary: The Physical Derivation Of Witten's Conjecture Relating Donaldson & SW

3 Brief Overview Of The World Of N=2 Theories

4 Donaldson-Witten Partition Function For General Compact Simple Lie Group

5 Theories With Matter & Superconformal Simple Type

6 Two Possible Future Directions

N=2 Theories

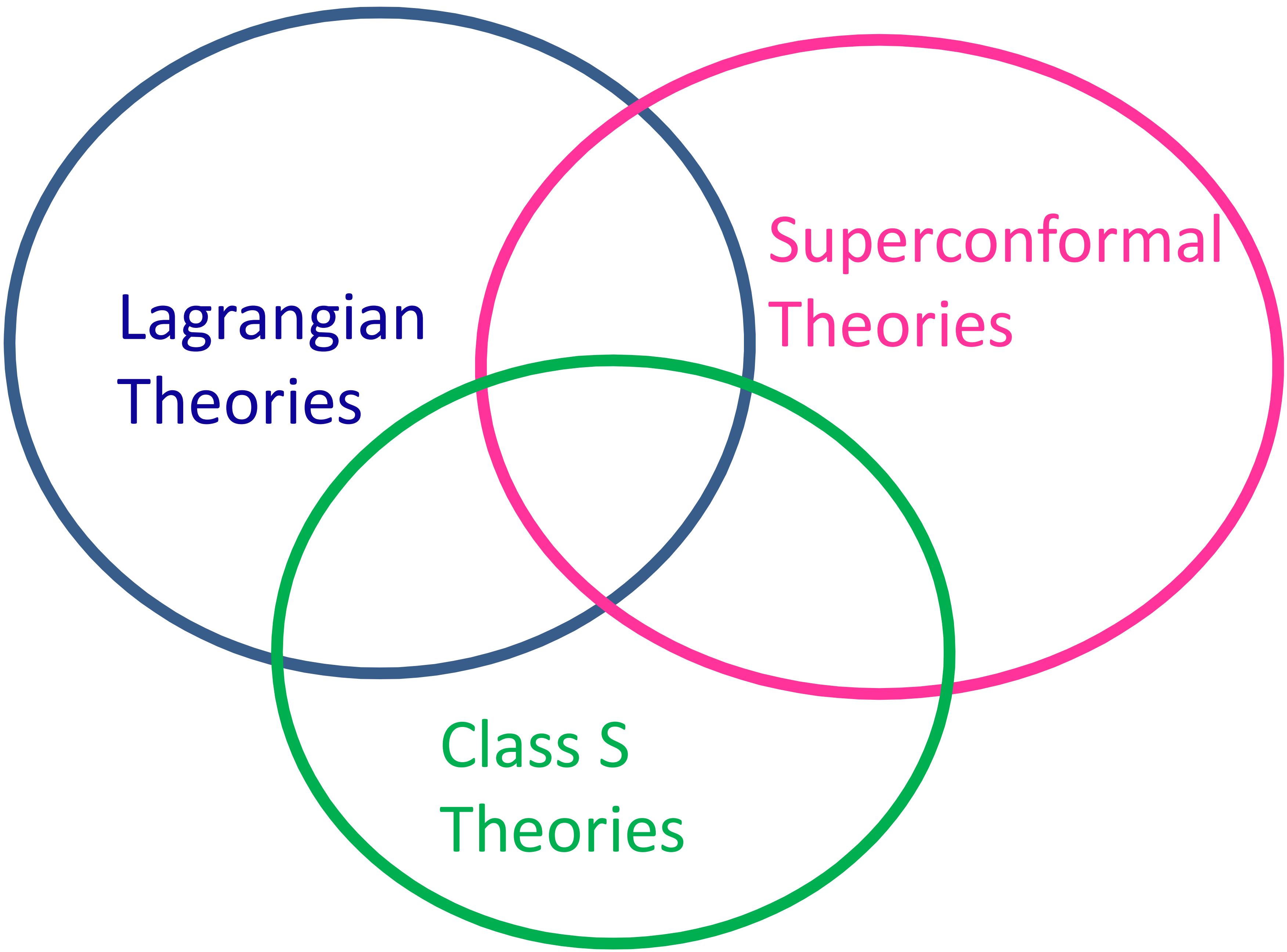
Lagrangian theories: Compact Lie group G, quaternionic representation $\mathcal{R}$ with G-invariant metric,

$$\tau_0 \in \prod_{simple\ factors} \mathcal{H} \qquad m \in Lie(G_f) \qquad G_f = Z(G) \subset O(\mathcal{R})$$

Class S: Theories associated to Hitchin systems on Riemann surfaces.

Superconformal theories

Couple to N=2 supergravity


Lagrangian
Theories
Superconformal
Theories
Class S
Theories

1 Introduction

2 Lightning Summary: The Physical Derivation Of Witten's Conjecture Relating Donaldson & SW

3 Brief Overview Of The World Of N=2 Theories

4 Donaldson-Witten Partition Function For General Compact Simple Lie Group

5 Theories With Matter & Superconformal Simple Type

6 Two Possible Future Directions

G-Donaldson Invariants

Pure VM theory for G a compact simple Lie group of rank r

0,2 observables derived from independent invariant polynomials $U, V \in Inv(\mathfrak{g})$

$$V(S) \sim \int_S V_{ab}\,(\phi^a F^b + \psi^a \psi^b)$$

$$Z_{DW}^{G,\xi}\big(U, V(S)\big) = \langle e^{U+V(S)} \rangle_\Lambda$$

Formally the path integral localizes to G-ASD moduli space

Generating function of generalization of Donaldson polynomials for any G.

Rigorous setup: Kronheimer & Mrowka

LEET On Coulomb Branch

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C} \ /W$

SSB: $G \to T \ \Rightarrow$ Abelian VM's valued in T

Example of SW Geometry : G=SU(N)

Σ_u: $\quad y^2 = P(x)^2 - \Lambda^{2N} \qquad P(x) = x^N + u_2 x^{N-2} + \cdots + u_N$

$\mathfrak{A}_u = Jac(\Sigma_u) \qquad \Gamma_u = H_1(\Sigma_u; \mathbb{Z})$

$$Z(\gamma) = \oint_\gamma \lambda \qquad \lambda = x\, d \log \frac{y + P}{y - P}$$

u-Plane Integral

Can compute u-plane integral explicitly from QFT:

$\exists$ almost canonical duality frame in weak-coupling region at ∞

$\Rightarrow$ $\mathfrak{t}$ −valued VM: (a, A, χ, η, ψ)

$$Z_u = \int_{\mathfrak{t}_c} [da]\, A^{\chi}\, B^{\sigma}\, e^{U+S^2 T_V}\, \Theta$$

$$A = \alpha\left(Det\left(\frac{\partial u^I}{\partial a^J}\right)\right)^{\frac{1}{2}} \qquad B = \beta\Delta^{\frac{1}{8}}$$

Δ: Holomorphic function vanishing along ``discriminant locus'' $\mathcal{D}$

T_V: Contact term. General theory Losev-Nekrasov-Shatashvili; Edelstein, Gomez-Reino, Marino

For quadratic Casimir: $T_{u_2} \sim 2\, u_2 \,-a^I \dfrac{\partial u_2}{\partial a^I}$

Theta Function

Θ: Theta function for abelian gauge fields remaining after SSB

Reduction of structure group $G_{adj} \rightarrow T$

Classes for fluxes in a torsor for $H^2\left(X; \Lambda_{wt}(\mathfrak{g})\right) \cong \Lambda_{wt}(\mathfrak{g}) \otimes H^2(X; \mathbb{Z})$

$$[F] = 4\pi\lambda \qquad \lambda \in \Lambda_{wt}(\mathfrak{g}) \otimes H^2(X; \mathbb{Z}) + \lambda_0 \;\coloneqq \Lambda$$

$$\Theta \sim \sum_{\lambda \in \Lambda} e^{-i\pi\lambda_+ \bar{\tau} \lambda_+ - i\pi\lambda_- \tau \lambda_-} \, e^{i\pi(\lambda - \lambda_0)\cdot\rho \otimes w_2(X)} \, e^{i \frac{\partial V}{\partial a^I} \lambda_-^I \cdot S}$$

Metric dependent $\Rightarrow$ Possible wall-crossing

Discriminant Locus

Just as in rank 1, the integrand is singular along a ``discriminant locus'' $\mathcal{D}$ where BPS states become massless and some $U(1) \subset T$ becomes strongly coupled.

$$\mathcal{D} = \cup_i \ \mathcal{D}_i$$

$\mathcal{D}_i$ generalizes $u = \pm\Lambda^2$

Higher rank: complicated intersections where multiple BPS states become massless, i.e. multiple periods of the curve vanish.

Integral Z_u must be regularized by cutting out tubular regions around $\mathcal{D}_i$

General Form Of Z_{DW}

$$Z_{DW}^{G,\xi}(p,s) = Z_u + \sum_i Z_{\mathcal{D}_i} + \sum_{i \neq j} Z_{\mathcal{D}_{ij}} + \cdots + Z_{mx}$$

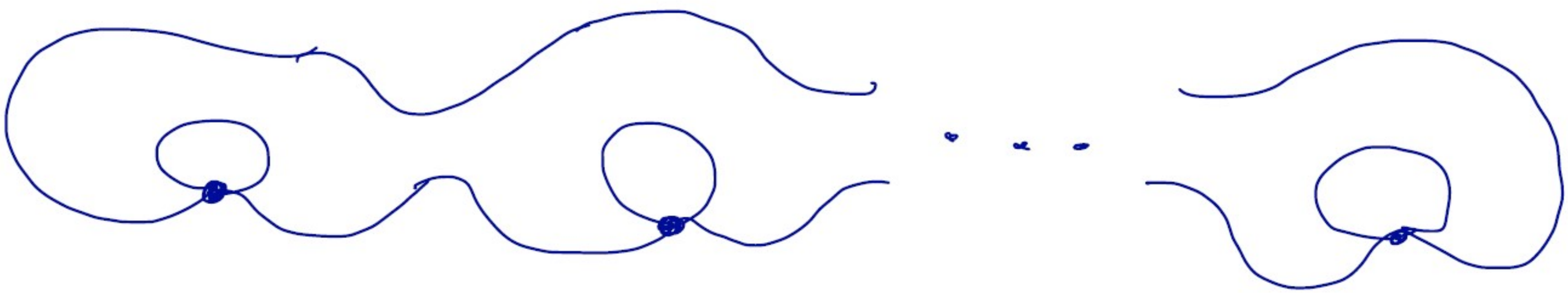
Cancelling wall-crossing inductively determines $Z_{\mathcal{D}_i}$ from Z_u and $Z_{\mathcal{D}_{ij}}$ from $Z_{\mathcal{D}_i}$, etc.

All $Z_{\mathcal{D}_{i_1 \dots i_k}}$ vanish for $b_2^+ > 1$ EXCEPT Z_{mx}

So for $b_2^+ > 1$ the answer is given entirely by Z_{mx}

The ``N=1 Vacua''

$\mathcal{D}_{mx}$ contains $h = h^{\vee}(G)$ isolated points $v_{\mathfrak{a}}$ permutated by spontan. broken $\mathbb{Z}/h\mathbb{Z}$ R-symmetry



$b_2^+ > 1$ $$Z_{DW}^{\mathrm{G},\xi}(\mathrm{U},\mathrm{V}) = \sum_{v_{\mathfrak{a}}} Z_{DW}^{G,\xi}(U,V;v_{\mathfrak{a}})$$

In principle, other maximal degenerations – corresponding to superconformal points- might have contributed.

But detailed analysis shows they do not for G=SU(3) and it is natural to conjecture that this is the case for all G.

Analog Of Witten Conjecture

In duality frame where max degeneration is $a_D^I = 0$

the Θ function is a sum over

$$\lambda \in \frac{1}{2}\rho \otimes w_2(X) + \Lambda_{\mathrm{wt}}(\mathfrak{g}) \otimes H^2(X;\mathbb{Z})$$

r independent spin-c structures : $f_I \in \frac{1}{2} w_2(X) + H^2(X;\mathbb{Z})$

$$SW(\lambda) := \prod_I SW(f_I)$$

$$Z_{DW}^{G,\xi}(U,V;v_{\mathfrak{a}}) = e^{i\,\theta_{\mathfrak{a}}} \sum_{\lambda} e^{(2\pi i \lambda\cdot\lambda_0)} SW(\lambda)\, R_\lambda(U;V)$$

$$R_\lambda(U,V) = Res[\ (da_D^1 \wedge \cdots \wedge da_D^r)/\prod_I (a_D{}^I)^{1+\frac{d(f_I)}{2}})\ \mathcal{E}(a_D^I) e^{Us+S^2 T_V + i\frac{\partial V}{\partial a_D^I} S\cdot f_I}$$

All computable from the degenerate curve and its first order variation.

Example Of SU(N)

Thus we can derive the $SU(N)$ Donaldson invariants.

Corollary: X of simple type, $b_1 = 0, b_2^+ > 1$:

Only the $\mathcal{N} = 1$ points contribute. Local analysis near $\mathcal{N} = 1$ points $\Rightarrow$

$$\boxed{\begin{aligned}\langle \mathrm{e}^{U+I_2(S)}\rangle_{SU(N)} =& \widetilde{\alpha}_N^{\chi}\widetilde{\beta}_N^{\sigma} \sum_{k=0}^{N-1} \sum_{\lambda^I} \omega^{k(N^2-1)\delta} \Big(\prod_{I=1}^{N-1} SW(\lambda^I)\Big) \\ \cdot \exp\Big[& \sum_{s=1}^{[\frac{N-1}{2}]} p_{2s}\omega^{2ks} u_{2s} + 2\omega^{2k} S^2 + 4\omega^k \sum_{I=1}^{N-1} (S, \lambda^I) \sin \frac{\pi I}{N}\Big]\end{aligned}}$$

$\omega = \exp[i\pi/N]$ $\qquad$ $\delta = (\chi + \sigma)/4$ $\qquad$ $u_{2s} = 4^s \binom{2s}{s} N$

The sum $\sum_{\lambda^I}$ is over the finite set of SW classes with: $4\lambda^2 = 3\chi + 2\sigma$

1. Introduction

2. Lightning Summary: The Physical Derivation Of Witten's Conjecture Relating Donaldson & SW

3. Brief Overview Of The World Of N=2 Theories

4. Donaldson-Witten Partition Function For General Compact Simple Lie Group

5. Theories With Matter & Superconformal Simple Type

6. Two Possible Future Directions

Including Matter

Now consider the general Lagrangian theory:

Data: $G, \mathcal{R}, m, \Lambda \; or \;\; q = e^{2\pi i \, \tau}$

Twisted N=2 theory is again of MQ form:

Localize on moduli space of generalized monopole equations.

Q: Differential for G_f- equivariant cohomology
with parameters $m \in Lie(G_f)$

Labastida & Marino; Losev-Nekrasov-Shatashvili

$$Z(U, V(S)) = \sum_{\ell, r} \frac{1}{\ell! \, r!} \int_{\mathcal{M}} \omega_U^{\ell} \, \omega_{V(S)}^{r} \, Eul(Cok(\mathbb{F}))$$

SU(2) With Fundamental Hypers

Moore & Witten

$$\mathcal{R} = N_{fl}(2 \oplus 2^*) \qquad Spin(2N_{fl}) \subset G_f$$

Mass parameters $m_f \in \mathbb{C},\ f = 1, \dots, N_{fl}$

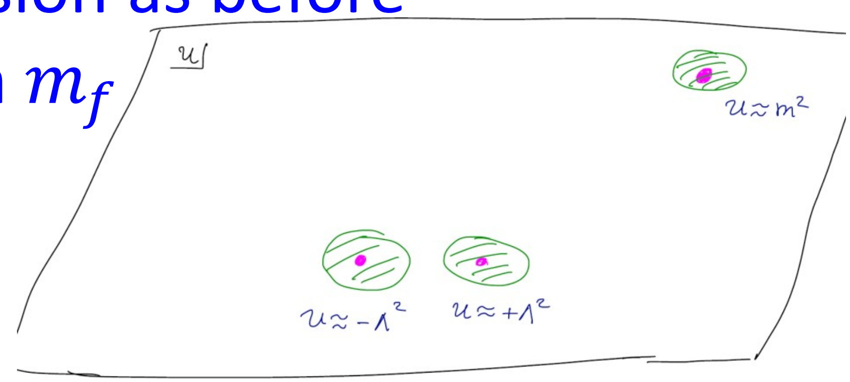
Must take $\xi = w_2(X)$

Seiberg-Witten: $E_u: \quad y^2 = x^3 + a_2 x^2 + a_4 x + a_6$

a_k: Polynomials in Λ, m_f, u

Z_u has exactly the same expression as before but now, e.g. da/du depends on m_f

New ingredient: $\mathcal{D}$ has $2 + N_{fl}$ points u_i: $\Delta = \prod_i (u - u_i)$



Analog Of Witten Conjecture

$$b_2^+ > 1 \qquad Z(p;s;m_f) = \sum_{j=1}^{2+N_{fl}} Z(p,s;m_f;u_j)$$

$$Z(p,s;m_f;u_j) = \tilde{\alpha}^{\chi} \tilde{\beta}^{\sigma} \sum_{\lambda} SW(\lambda) e^{2\pi i\, \lambda\cdot\lambda_0} R_j(p,s)$$

X is SWST $\Rightarrow$

$$R_j(p,s) = \kappa_j^{\chi_h} \left(\frac{du}{da}\right)^{\chi_h+\sigma} \exp\left(2p\, u_j + S^2 T(u_j) - i\, \left(\frac{du}{da}\right)_j S\cdot\lambda \right)$$

$$u = u_j + \kappa_j q_j + \mathcal{O}(q_j^2)$$

Everything computable explicitly as functions of the masses from first order degeneration of the SW curve.

Superconformal Points

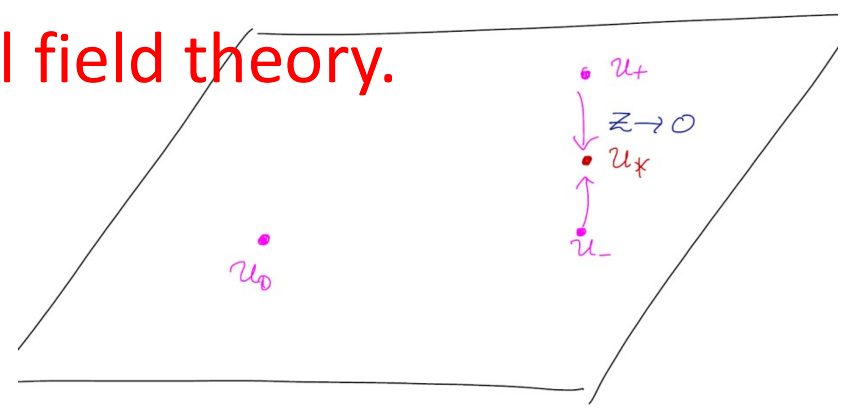
Consider $N_{fl} = 1$. At a critical point $m = m_*$ two singularities $u_\pm$ collide at $u = u_*$ and the SW curve becomes a cusp: $y^2 = x^3$ [Argyres,Plesser,Seiberg,Witten]

Two mutually nonlocal BPS states have vanishing mass:

$$\oint_{\gamma_1} \lambda \to 0 \quad \oint_{\gamma_2} \lambda \to 0 \qquad \gamma_1 \cdot \gamma_2 \neq 0$$

Physically: No local Lagrangian for the LEET :
Signals a nontrivial superconformal field theory.

$$m = m_* + z$$



Superconformal Simple Type – 1/2

Analyze contributions at the two colliding points $u_{\pm}$

$$R_j(p,s) = \kappa_j^{\chi_h} \left(\frac{du}{da}\right)^{\chi_h+\sigma} \exp\left(2p\, u_j + S^2 T(u_j) - i \left(\frac{du}{da}\right)_j S\cdot\lambda \right)$$

$$= const.\, e^{2pu_* + S^2 T(u_*)} e^{i\,\theta_{\mp}}\; z^{\frac{c^2-\chi_h}{2}} (1 + Series\ in\ z^{\frac{1}{2}})$$

$$\exp\left(e^{i\,\theta_{\pm}}\, z^{\frac{1}{4}}\, \left(1 + Series\ in\ z^{\frac{1}{2}}\right) S\cdot\lambda\right)$$

$$\frac{c^2-\chi_h}{2} = \frac{7\,\chi + 11\,\sigma}{8} < 0$$

Perfectly reasonable!

Physics: $\lim_{z\to 0} Z_{DW}\ (p,s;m_* + z) < \infty$

Superconformal Simple Type – 2/2

Physics: $\lim_{z\to 0} Z_{DW}\ (p, s; m_* + z)\ < \infty$

No IR divergences on X No noncompact moduli spaces of vacua

Form of explicit answer implies the only way this can hold for all polynomials in pnt and S is for a series expansion in z with coefficients made from $SW(\lambda)$ to be regular

Theorem [MMP]: There is no divergence in Z_{DW} if :

a.) $\chi_h - c^2 - 3 \leq 0$

b.) $\sum_\lambda SW(\lambda) e^{2\pi i \lambda\cdot\lambda_0} \lambda^k = 0 \quad 0 \leq k \leq \chi_h - c^2 - 4$

Conditions a,b define SST.

MMP checked that all known (c. 1998) 4-folds with $b_2^+ > 1$ are SST.

1 Introduction

2 Lightning Summary: The Physical Derivation Of Witten's Conjecture Relating Donaldson & SW

3 Brief Overview Of The World Of N=2 Theories

4 Donaldson-Witten Partition Function For General Compact Simple Lie Group

5 Theories With Matter & Superconformal Simple Type

6 Two Possible Future Directions

Two Possible Future Directions

Invariants for families of 4-manifolds

New invariants
(or new facts about old invariants)
from superconformal theories??

Families Of Four-Manifolds – 1/5

Donaldson invariants can be generalized to families of four-manifolds: Donaldson, Durham lectures 1989

Naïve attempt at a physical approach:

Couple N=2 field theory to N=2 supergravity:

$$g_{\mu\nu}, \psi^A_{\mu\alpha}, \bar{\psi}^A_{\mu\dot{\alpha}}, \dots$$

Topological twist: $\Rightarrow g_{\mu\nu}, \Psi_{\mu\nu}, \phi^\mu, \dots$

$$Q g_{\mu\nu} = \Psi_{\mu\nu} \;, Q\Psi_{\mu\nu} = D_\mu \phi_\nu + D_\nu \phi_\mu \;, Q\phi^\mu = 0 \,, \dots$$

Superfields describe (Cartan model) for

$\mathrm{d}iff(X)$ – equivariant cohomology of $\Omega^*\left(Met(X)\right)$

Families Of Four-Manifolds – 2/5

$$S = \{ Q, V \} + const \int tr\, F \wedge F$$

$$T_{\mu\nu} = \{ Q, \Lambda_{\mu\nu} \} \qquad D^{\mu}\Lambda_{\mu\nu} = \{ Q, Z_{\nu} \}$$

$$Q\left(S + \int_X vol(g)\, \Psi^{\mu\nu}\Lambda_{\mu\nu} + vol(g)\phi^{\mu} Z_{\mu} \right) = 0$$

(For a fixed volume form $vol(g)$.)

Families Of Four-Manifolds – 3/5

$$Z[g_{\mu\nu}, \Psi_{\mu\nu}, \phi^{\mu}] = \int d[A, \phi, \chi, \psi, \eta]\, exp(S + \int_X \Psi^{\mu\nu} \Lambda_{\mu\nu} + \phi^{\mu} Z_{\mu})$$

$Q -$ closed diff(X)-equivariant differential form on $Met(X)$

Diffeomorphism invariant

Descends to cohomology class $\in H^*(\frac{Met(X)}{Diff(X)})$

Conjecture: These are the family Donaldson invariants

n – parameter families of metrics have wall-crossing in the degree n component for $b_2^+(X) \leq n + 1$

Four-Manifold Families – 4/5

$b = b_2^+(X)$ Singularities of (b-1)-form component for b-dimensional families are associated with classes $\lambda \in H^2(X;\mathbb{Z})$

Suppose $\lambda \in H^2(X;\mathbb{Z})$ is ASD for a metric $g^{(0)}$

Perturb : $g(t) = g^{(0)} + \sum_{\alpha=1}^{b} t^{\alpha} p_{\alpha}$

$$Z^{sing} \sim c\left(\frac{\lambda^2}{2}\right) \omega_{b-1} + d\,(*)$$

ω_{b-1} angular form in t^{α} around the point t=0 .

For G=SU(2) c(n) are the coefficients of the same modular form that appears in the standard Donaldson WCF.

Four-Manifold Families – 5/5

One can also couple $g_{\mu\nu}, \Psi_{\mu\nu}, \phi^{\mu}$ to the LEET around $\mathcal{U}_{\Lambda^2}$

It is natural to expect that this will give the family SW invariants formulated by T.-J. Li & A.-K. Liu.

… and moreover that there is an analog of the Witten conjecture for the family Donaldson invariants.

Superconformal Theories – 1/4

Basic question:

There are lots of interesting superconformal theories.

(Some of them don't even have Lagrangian descriptions.)

Nevertheless, they can be topologically twisted and have Q-invariant operators.

Is this a source of new four-manifold invariants?

Superconformal Theories – 2/4

Important lesson from $SU(2)\ N_{fl} = 4$

$\tau(u; m_a)$ approaches a FINITE limit as $u \to \infty$

Completely changes the wall-crossing story.

$$\frac{d}{dg_{\mu\nu}} Z_u \sim \sum_{\ell,r} \wp^{\ell} S^{r} \lim_{R\to\infty} \oint_{|u|=R} du\ u^{\frac{\sigma+1+2\ell+r}{2}} \Theta_{\ell,r}(\tau_0)(1 + Series\ \frac{1}{u}, \frac{1}{\bar{u}})$$

$\frac{1}{2}(\sigma + 1 + 2\ell + r) < -1$ No wall-crossing at $b_2^+ = 1$

$\frac{1}{2}(\sigma + 1 + 2\ell + r) \geq -1$ ***Continuous*** metric dependence!

TFT fails utterly !!

Superconformal Theories – 3/4

Now consider SU(2) $N_{fl} = 1$ at $m = m_*$

$$\lim_{m \to m_*} Z_u \neq \int du\, d\bar{u} \lim_{m \to m_*} Measure(u, \bar{u}; m) := Z_u^*$$

Z_u^* : continuous or no metric dependence from singularity!

Continuous metric dependence for $\sigma + r + 6\ell < -7$

No metric dependence for $\sigma + r + 6\,\ell \geq -7$

Conjecture: $Z_u^* + Z_{u_0} + \langle\, e^{p\,\mathcal{O}_1 + \mathcal{O}_2(S)}\, \rangle_{AD3\ Theory}$

is topologically invariant except for wall-crossing at

$u = \infty$ for $b_2^+(X) = 1$

Superconformal Theories – 4/4

The truth of this conjecture would suggest that the superconformal theories might provide new four-manifold invariants, at least in some range of r & ℓ

The truth of this conjecture would then strongly motivate an investigation of the u-plane integral for general class S.

Much of the structure of Z_u is known – follows pattern of higher rank.

Some important details remain to be understood more clearly.

Can, in principle, be derived from a 2d (2,0) QFT derived from reduction of abelian 6d (2,0) theory along a four-manifold.