Web Formalism and the IR limit of massive 2D N=(2,2) QFT

- or -

A short ride with a big machine

SCGP, Nov. 17, 2014

Gregory Moore, Rutgers University

collaboration with

Davide Gaiotto & Edward Witten

draft is ``nearly finished"…

So, why isn't it on the arXiv ?

The draft seems to have stabilized for a while at around 350 pp ….. So, why isn't it on the arXiv?

In our universe we are all familiar with the fact that

$$e^{i\pi} - 1 = -2$$

In that part of the multiverse in which we have the *refined* identity

$$e = i = \pi = -1 = -2$$

our paper has definitely been published!

Much ``written” material is available:

Several talks on my homepage.

Davide Gaiotto: Seminar at Perimeter, Fall 2013:
``Algebraic structures in massive (2,2) theories

In the Perimeter online archive of talks.

Davide Gaiotto: ``BPS webs and
Landau-Ginzburg theories,”
Talk at String-Math 2014. On the web.

Three Motivations

1. IR sector of ***massive*** 1+1 QFT with N =(2,2) SUSY

2. Knot homology.

3. Spectral networks & categorification of 2d/4d wall-crossing formula [Gaiotto-Moore-Neitzke].

(A unification of the Cecotti-Vafa and Kontsevich-Soibelman formulae.)

Summary of Results - 1

Result: When we take into account the BPS states there is an extremely rich mathematical structure.

We develop a formalism

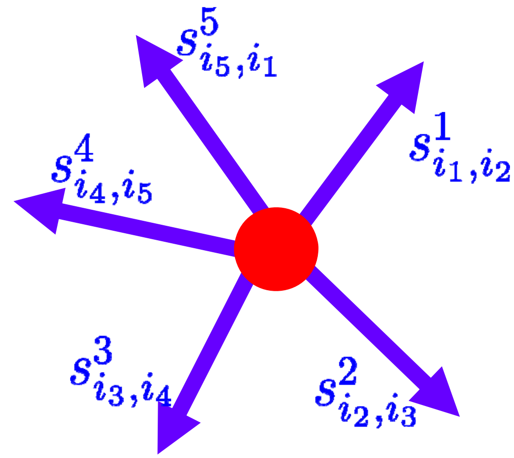
– the ``web-based formalism" –

(that's the big machine)

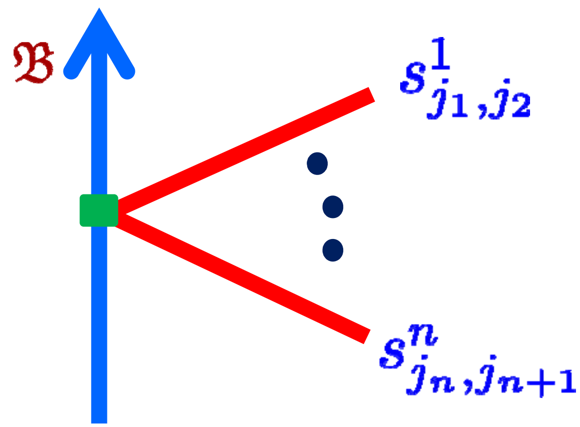
that shows:

Results - 2

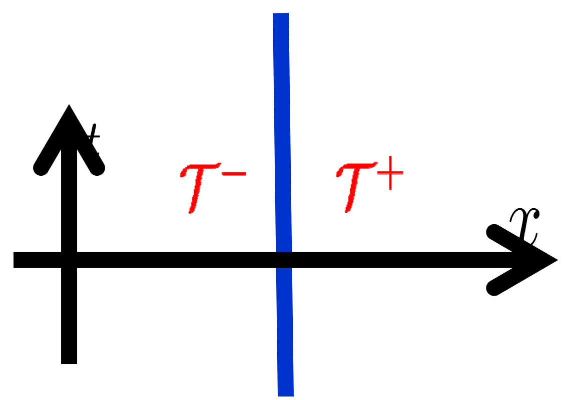
BPS states have ``interaction amplitudes'' governed by an L∞ Maurer-Cartan equation.



There is an A∞ category of branes, with amplitudes for emission of BPS particles from the boundary governed by solutions to the A∞ MC equation.



If we have a pair of theories then we can construct supersymmetric interfaces between the theories.



Results - 3

Such interfaces define A_∞ functors between Brane categories.

Theories and their interfaces form an A_∞ 2-category.

Given a continuous family of theories
(e.g. a continuous family of LG superpotentials)
we show how to construct a ``flat parallel transport''
of Brane categories.

The parallel transport of Brane categories is constructed using interfaces.

The flatness of this connection implies, and is a categorification of, the 2d wall-crossing formula.

Outline

- Introduction, Motivation, & Results
- Morse theory and LG models: The SQM approach
- Boosted solitons and ζ-webs
- Webs and their representations: L_∞
- Half-plane webs & Branes: A_∞
- Interfaces & Parallel Transport of Brane Categories
- Summary & Outlook

Basic Example: LG Models

(X,ω): Kähler manifold.

W: X $\longrightarrow \mathbb{C}$ A holomorphic Morse function

To this data we assign a 1+1 dimensional QFT

$$\phi : D \times \mathbb{R} \to X$$

$$D = \mathbb{R}, [x_\ell, \infty), (-\infty, x_r], [x_\ell, x_r], S^1$$

Morse Theory

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

M is an infinite-dimensional Kahler manifold.

Morse function:

$$h = \int_D \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

$$d\lambda = \omega \qquad \lambda = pdq$$

SQM

Morse theory is known to physicists as Supersymmetric Quantum Mechanics (Witten 1982):

Target space for SQM:

$$M = \mathrm{Map}(D, X) = \{\phi : D \to X\}$$

SQM superpotential

$$h = \int_D \left(\phi^* \lambda + \mathrm{Re}(\zeta^{-1} W) dx\right)$$

Relation to LG QFT

Plug into SQM action and recover the standard 1+1 LG model with (LG) superpotential W.

$$S = \int d\phi * d\bar{\phi} - |\nabla W|^2 + \cdots$$

Massive LG vacua are Morse critical points:

$$dW(\phi_i) = 0 \qquad W''(\phi_i) \neq 0$$

Label set of LG vacua: $\phi_i \in \mathbb{V}$

MSW Complex: Semiclassical vs. True Groundstates

MSW complex: $\mathbb{M}^{\bullet} := \oplus_{p:dh(p)=0} \mathbb{Z} \cdot \Psi(p)$

$$d(\Psi(p)) := \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$

SQM instanton equation: $\frac{d\phi}{d\tau} = \pm g^{IJ} \frac{\partial h}{\partial \phi^J}$

n(p,p') counts ``rigid instantons'' - with zero reduced moduli – $d^2=0$ thanks to broken flows at ends.

Space of SQM groundstates (BPS states) is the *cohomology*.

Apply to the LG model:

$$\frac{d\phi}{d\tau} = -\frac{\delta h}{\delta \bar{\phi}}$$

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

We call this the ζ-*instanton* equation

Time-independent: ζ-*soliton* equation:

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Physical Meaning of the ζ-instanton equation - 1

LG field theory has (2,2) supersymmetry:

$$\{Q_+, \overline{Q_+}\} = H + P$$

$$\{Q_-, \overline{Q_-}\} = H - P$$

$$\{Q_+, Q_-\} = \bar{Z}$$

$$[F, Q_+] = Q_+ \qquad [F, \bar{Q}_-] = \bar{Q}_-$$

Physical Meaning of the ζ-instanton equation - 2

We are interested in situations where two supersymmetries are unbroken:

$$U(\zeta) := Q_+ - \zeta^{-1}\overline{Q_-}$$

U(ζ)[Fermi] =0 implies the ζ-*instanton* equation:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial \tau}\right)\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

Boundary conditions for ϕ

Boundaries at infinity:

$$\phi \to \phi_i \qquad \phi \to \phi_j$$
$$x \to -\infty \qquad x \to +\infty$$

Boundaries at finite distance: Preserve ζ-susy:

$$\phi|_{x_\ell, x_r} \in \mathcal{L} \subset X$$

$$\iota_{\mathcal{L}}^*(\lambda) = dk$$

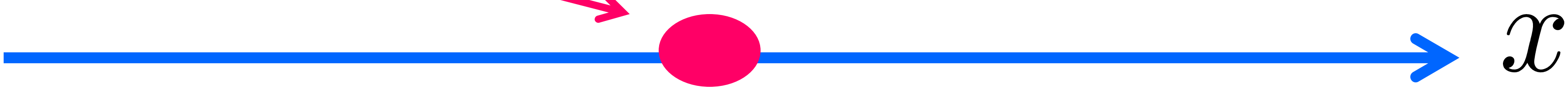
(Simplify: ω=dλ)

$$\pm \mathrm{Im}(\zeta^{-1} W) \geq \Lambda$$

Solitons For D=$\mathbb{R}$

Scale set by W

$$\frac{\partial}{\partial x}\phi^I = \zeta g^{I\bar{J}}\frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

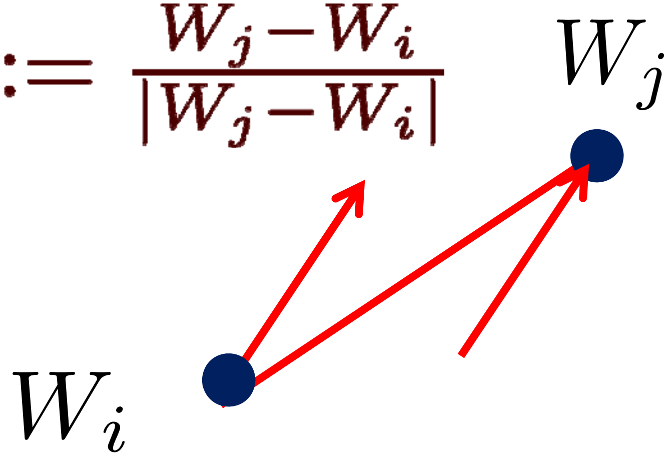


$\phi \cong \phi_i$ $\phi \cong \phi_j$

For general ζ there is no solution.

$$\zeta = \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

But for a suitable phase there is a solution



This is the classical soliton. There is one for each intersection (Cecotti & Vafa)

$$p \in L_i^{\zeta} \cap R_j^{\zeta}$$

(in the fiber of a regular value)

MSW Complex

We can discuss ij BPS states using Morse theory:

$$\frac{\delta h}{\delta \phi} = 0$$

Equivalent to the ζ-soliton equation

$$\mathbb{M}_{ij} = \oplus_{\text{solitons}} \mathbb{Z} \cdot \Psi_{ij}$$

(Taking some shortcuts here….)

$$D = \sigma^3 \mathrm{i} \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} W'' + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} \bar{W}''$$

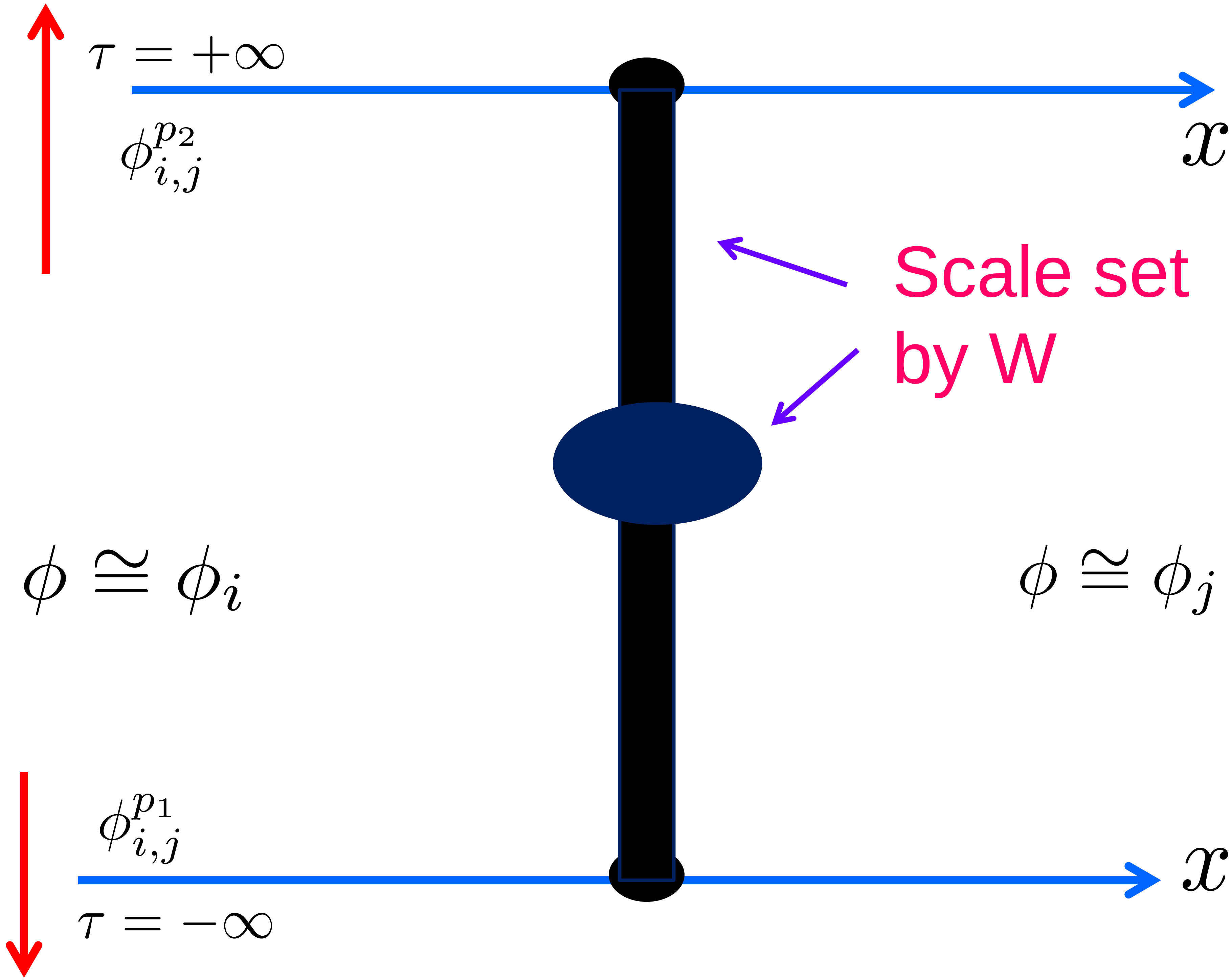
$$F = -\tfrac{1}{2} \eta (D - \epsilon)$$

A soliton of type ij preserves the supersymmetry algebra generated by:

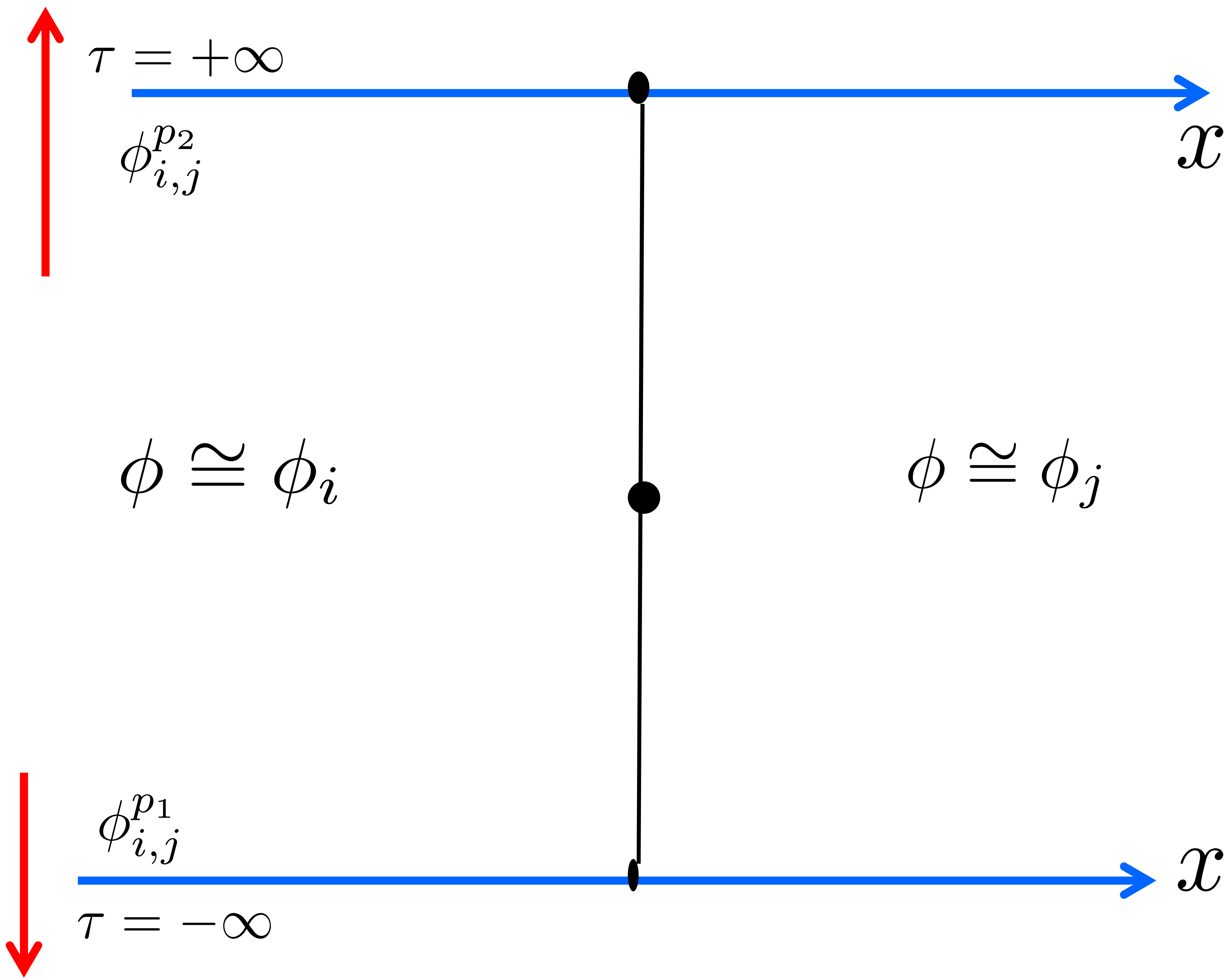
$$U(\zeta_{ji})$$

Differential obtained from counting solutions to the ζ-*instanton* equation with $\zeta = \zeta_{ji}$ and no reduced moduli:

$$d(\Psi(p)) := \sum_{p':F(p')-F(p)=1} n(p,p')\Psi(p')$$



τ = +∞
x
φ_{i,j}^{p_2}
Scale set
by W
φ ≅ φ_i
φ ≅ φ_j
φ_{i,j}^{p_1}
x
τ = −∞



$\tau = +\infty$
x
$\phi_{i,j}^{p_2}$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
$\phi_{i,j}^{p_1}$
x
$\tau = -\infty$

Example of a categorified WCF:

BPS Index

The BPS index is the Witten index:

$$\mu_{ij} := \mathrm{Tr}_{\mathcal{H}_{ij}^{BPS}} F(-1)^F$$

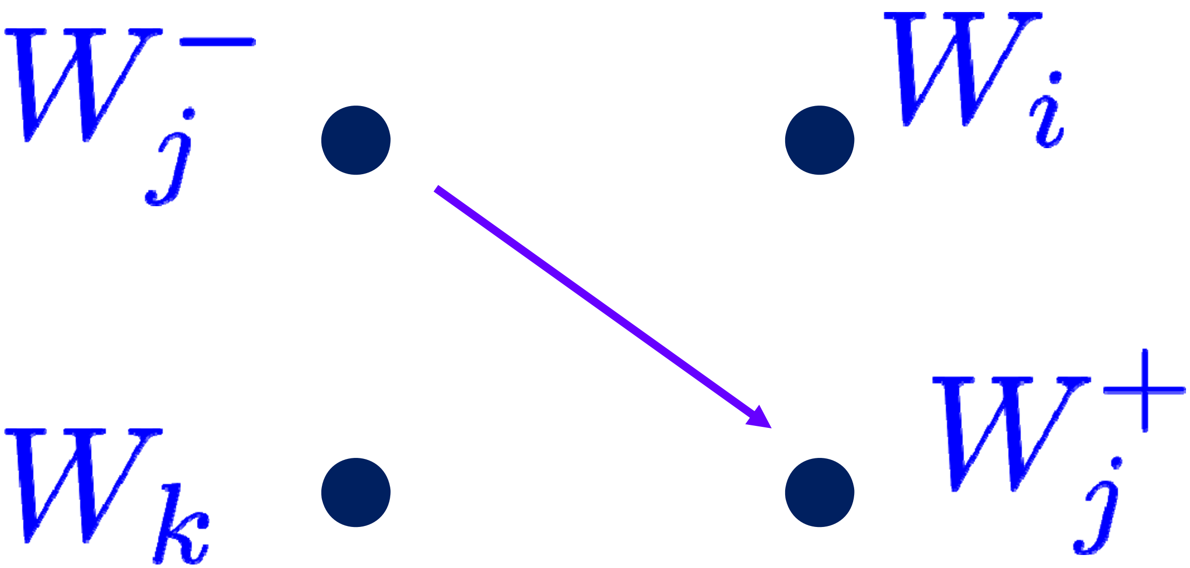
``New supersymmetric index'' of Fendley & Intriligator;
Cecotti, Fendley, Intriligator, Vafa; Cecotti & Vafa c. 1991

Remark: It can be computed as a signed sum over classical solitons:

$$\mu_{ij} = \sum_{p \in L_i^\zeta \cap R_j^\zeta} (-1)^{\iota(p)}$$

These BPS indices were studied by [Cecotti, Fendley, Intriligator, Vafa and by Cecotti & Vafa]. They found the wall-crossing phenomena:

Given a one-parameter family of W's:



$$\mu_{ik}^- \to \mu_{ik}^+ = \mu_{ik}^- + \mu_{ij}\mu_{jk}$$

One of our goals is to ``categorify'' this wall-crossing formula.

That means understanding what actually happens to the ``off-shell complexes'' whose cohomology gives the BPS states.

We just defined the relevant complexes:

$$(\mathbb{M}_{ij}, d)$$

$$\mathcal{H}_{i,j}^{\mathrm{BPS}} = H^*(\mathbb{M}_{ij}, d)$$

Replace wall-crossing for indices:

$$\mu_{ik}^{+} = \mu_{ik}^{-} + \mu_{ij}\mu_{jk}$$

$$\left(\mathbb{M}_{ik}^{0} - \mathbb{M}_{ik}^{1}\right)^{+} =?$$

$$= \left(\mathbb{M}_{ik}^{0} - \mathbb{M}_{ik}^{1}\right)^{-}$$

$$+ \left(\mathbb{M}_{ij}^{0} - \mathbb{M}_{ij}^{1}\right) \otimes \left(\mathbb{M}_{jk}^{0} - \mathbb{M}_{jk}^{1}\right)$$

Sometimes categorification is not always so straightforward:

An example is provided by studying BPS states on the interval $[x_l, x_r]$.

BPS Solitons on half-line D:

Boundary condition preserves U(ζ)

U(ζ) -preserving BPS states must be solutions of

$$\frac{\partial \phi^I}{\partial x} = \zeta g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}$$

$$\phi|_{x_\ell} \in \mathcal{L}$$

$$\phi \to \phi_j$$
$$x \to \infty$$

Classical solitons on the positive half-line are labeled by:

$$p \in \mathcal{L} \cap R_j^\zeta$$

BPS States on half-line D:

MSW complex: $\mathbb{M}_{\mathcal{L},j} = \oplus_p \mathbb{Z} \cdot \Psi_{\mathcal{L},j}(p)$

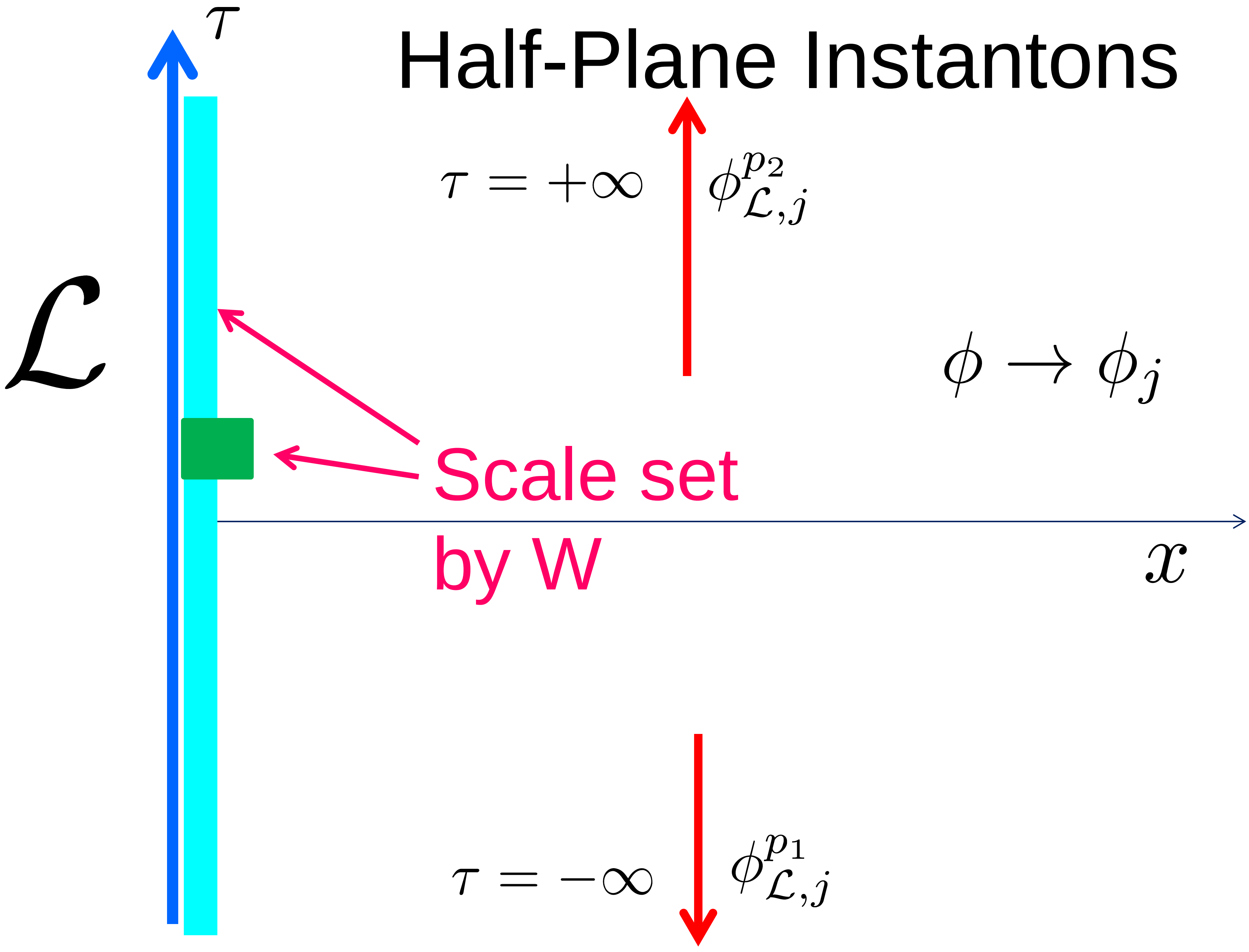
Grading on complex?

Assume X is CY and that we can find a logarithm:

$$w = \mathrm{Im} \log \frac{\iota^*(\Omega^{d,0})}{\mathrm{vol}(\mathcal{L})}$$

$$F = -\frac{1}{2}(\eta(D) - w)$$

$$\mu_{\mathcal{L},i} = \mathrm{Tr}_{\mathcal{H}^{\mathrm{BPS}}_{\mathcal{L},j}} (-1)^F e^{-\beta H}$$

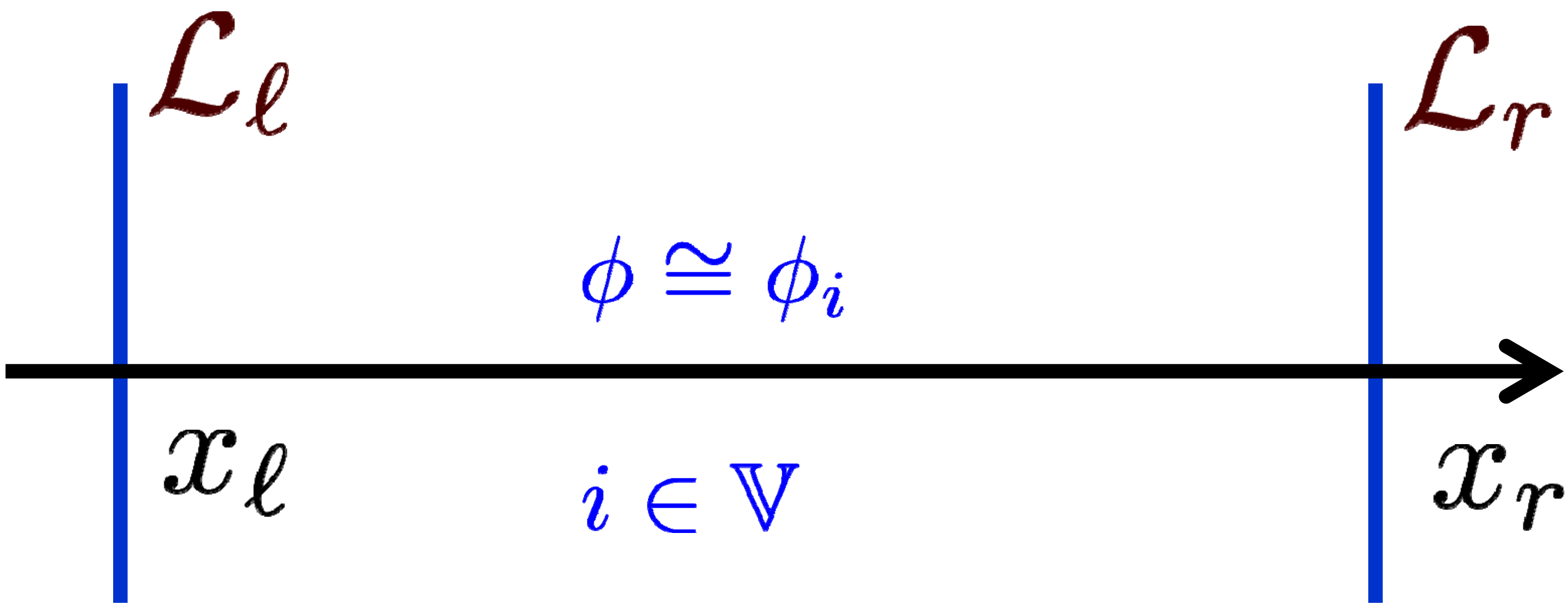


τ
Half-Plane Instantons
$\tau = +\infty$
$\phi^{p_2}_{\mathcal{L},j}$
$\mathcal{L}$
$\phi \to \phi_j$
Scale set
by W
x
$\tau = -\infty$
$\phi^{p_1}_{\mathcal{L},j}$

What is the space of BPS states on an interval ?

The theory is massive:

For a susy state, the field in the middle of a large interval is close to a vacuum:



Witten index on the interval

$$\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_{i\in\mathbb{V}} \mu_{\mathcal{L}_\ell,i} \cdot \mu_{i,\mathcal{L}_r}$$

Naïve categorification?

$$\mathcal{H}^{\mathrm{BPS}}_{\mathcal{L}_\ell,\mathcal{L}_r} \overset{?}{\neq} \sum_{i\in\mathbb{V}} \mathcal{H}^{\mathrm{BPS}}_{\mathcal{L}_\ell,i} \otimes \mathcal{H}^{\mathrm{BPS}}_{i,\mathcal{L}_r}$$

No!

Solitons On The Interval

When the interval is much longer than the scale set by W the MSW complex is

$$\mathbb{M}_{\mathcal{L}_\ell,\mathcal{L}_r} = \oplus_{i\in\mathbb{V}}\mathbb{M}_{\mathcal{L}_\ell,i}\otimes\mathbb{M}_{i,\mathcal{L}_r}$$

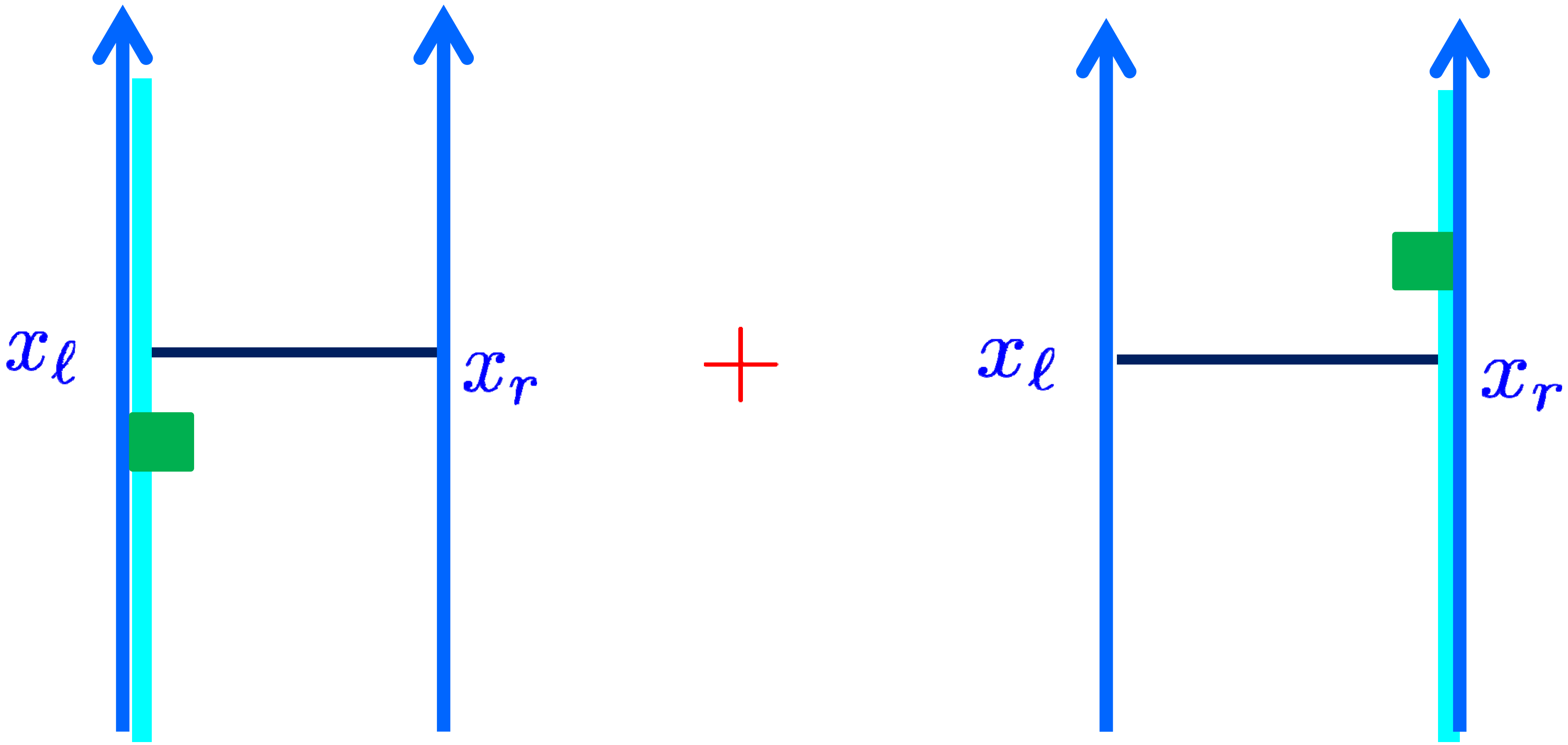
So Witten index factorizes nicely:

$$\mu_{\mathcal{L}_\ell,\mathcal{L}_r} = \sum_i \mu_{\mathcal{L}_\ell,i}\mu_{i,\mathcal{L}_r}$$

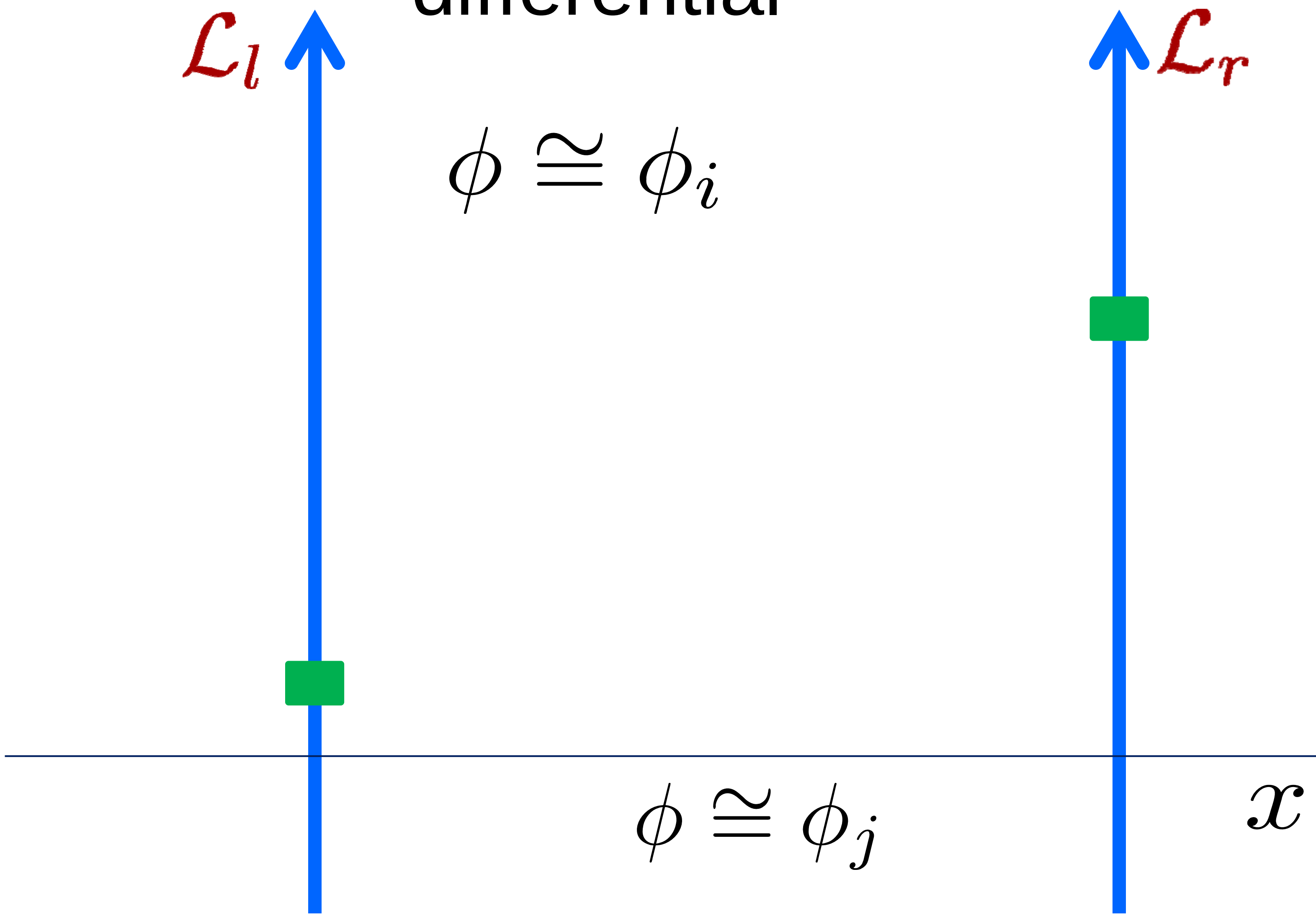
But the differential $d_{\mathcal{L}_\ell,i}\otimes 1 + 1\otimes d_{i,\mathcal{L}_r}$

is too naïve !

$$\sum_i \left(d_{\mathcal{L}_\ell, i} \otimes 1 + 1 \otimes d_{i,\mathcal{L}_r}\right)$$



Instanton corrections to the naïve differential



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The Boosted Soliton - 1

We are interested in the ζ-instanton equation for a fixed generic ζ

We can still use the soliton to produce a solution for phase ζ

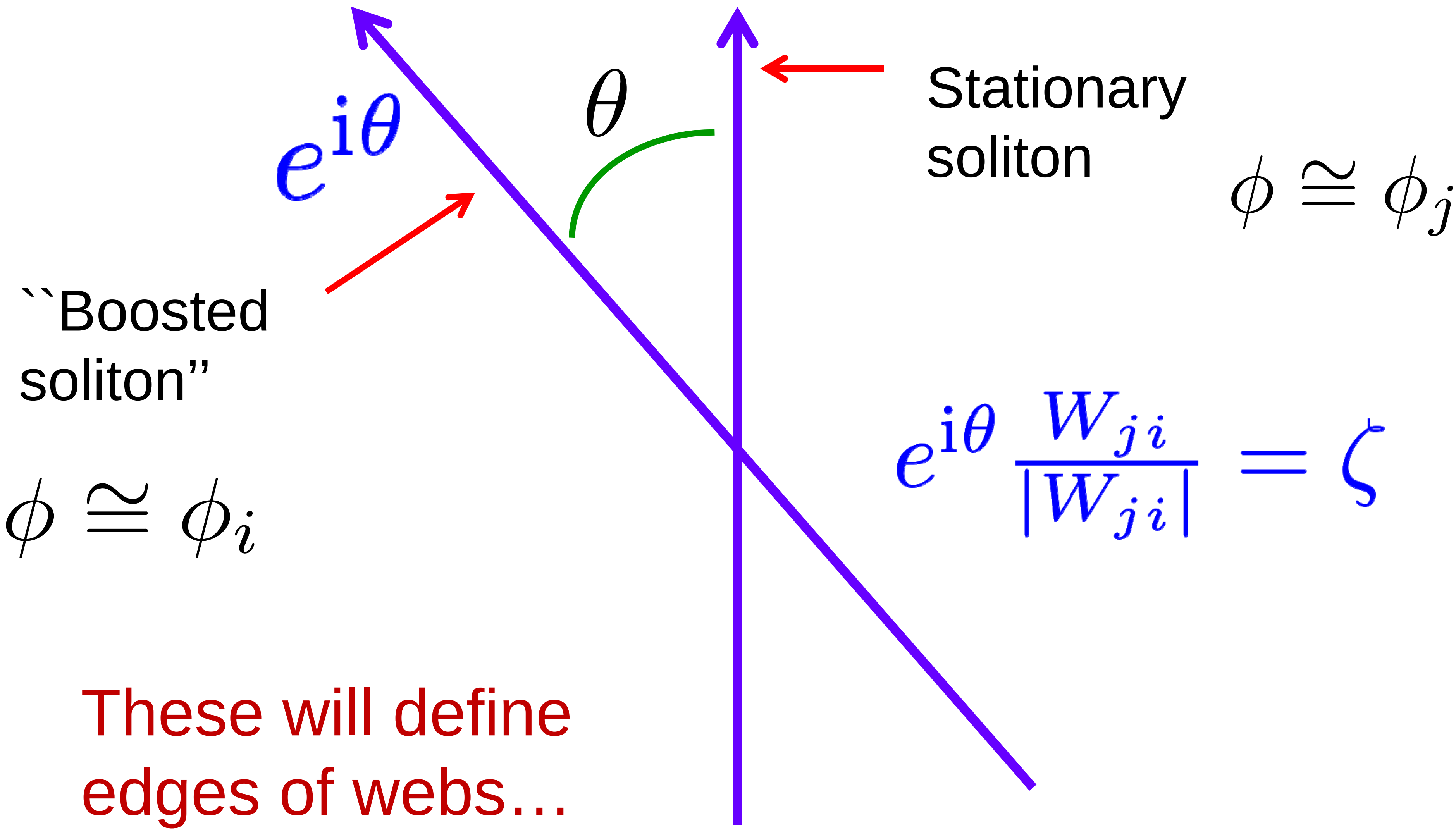
$$\phi_{ij}^{\rm inst}(x,\tau) := \phi_{ij}^{\rm sol}(\cos\theta x + \sin\theta\tau)$$

$$\left(\frac{\partial}{\partial x} + {\rm i}\frac{\partial}{\partial\tau}\right)\phi_{ij}^{\rm inst} = e^{{\rm i}\theta}(\phi_{ij}^{\rm sol})' = e^{{\rm i}\theta}\zeta_{ji}\frac{\partial\bar{W}}{\partial\bar{\phi}}$$

Therefore we produce a solution of the instanton equation with phase ζ if

$$\zeta = e^{{\rm i}\theta}\zeta_{ji} \qquad\qquad \zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|}$$

The Boosted Soliton -2



These will define edges of webs…

The Boosted Soliton - 3

Put differently, the stationary soliton in ***Minkowski*** space preserves the supersymmetry:

$$Q_+ - \zeta_{ji}^{-1} \overline{Q_-}$$

So, a boosted soliton preserves supersymmetry :

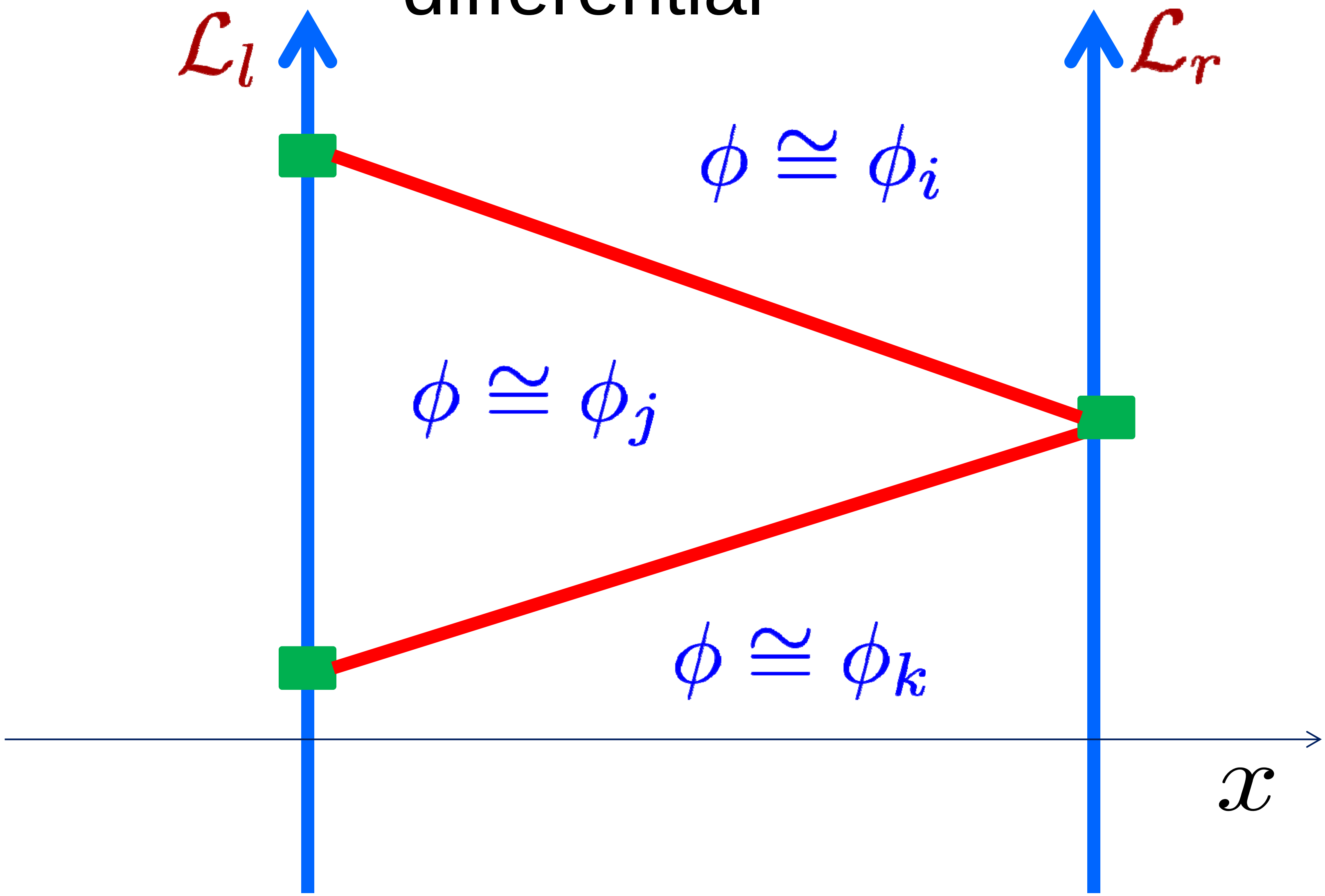
$$e^{\beta/2} Q_+ - \zeta_{ji}^{-1} e^{-\beta/2} \overline{Q_-}$$

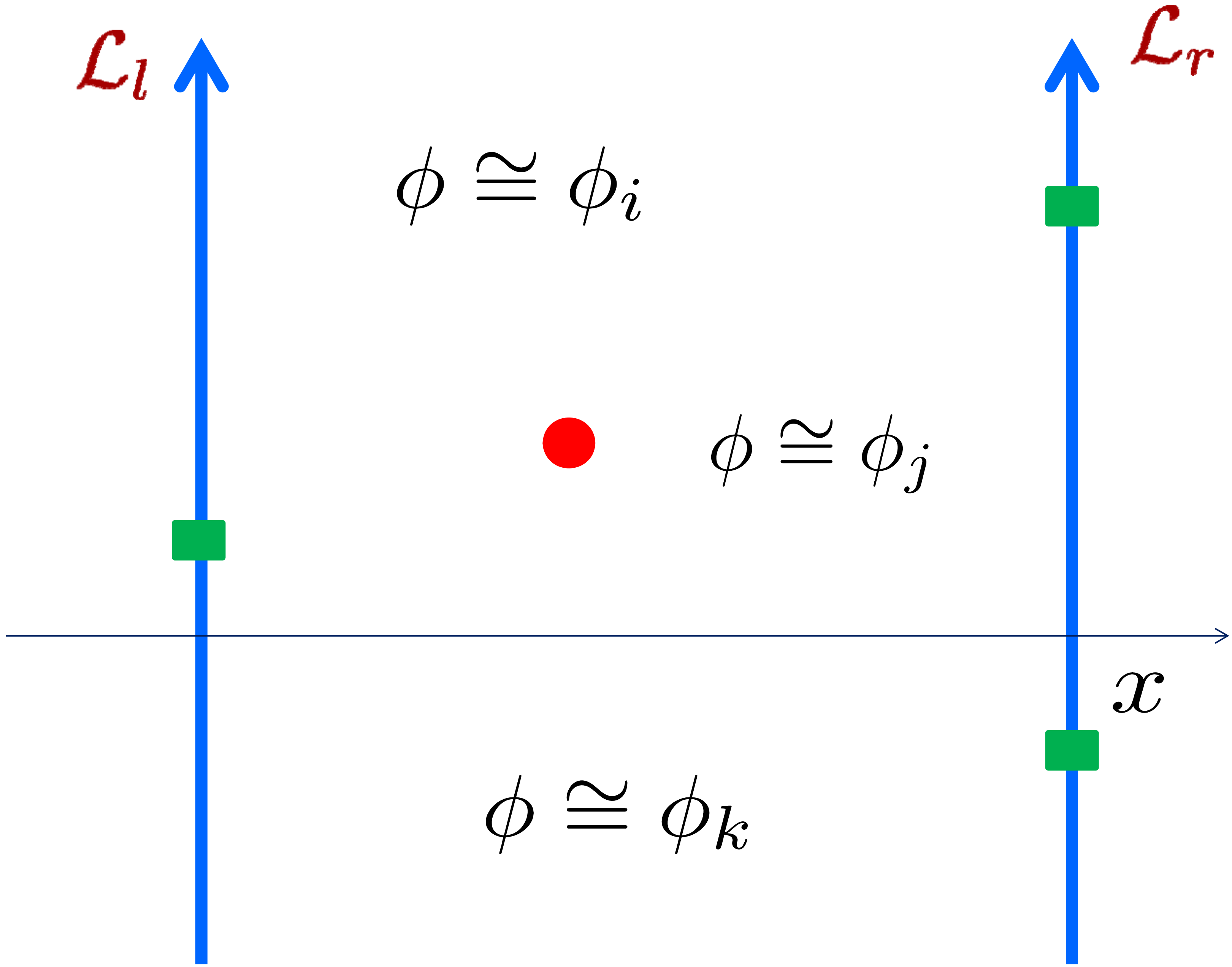
β is a real boost: In ***Euclidean*** space this becomes a rotation:

$$e^{i\theta/2} Q_+ - \zeta_{ji}^{-1} e^{-i\theta/2} \overline{Q_-}$$

And for suitable θ this will preserve U(ζ)-susy

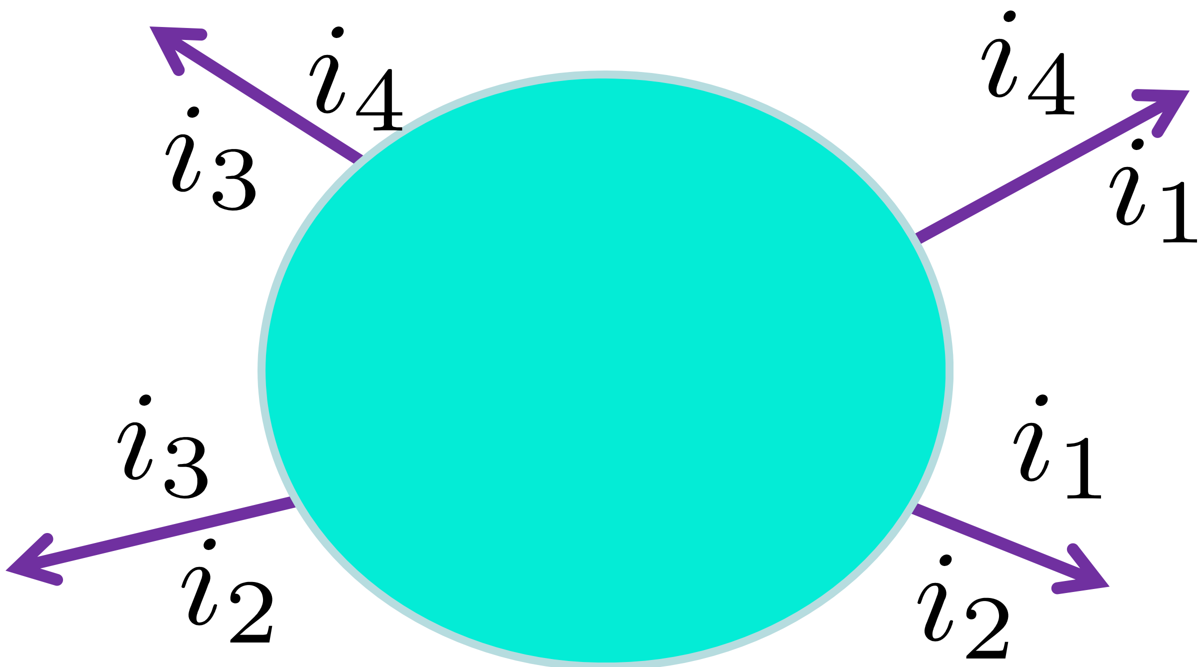
More corrections to the naïve differential





$\mathcal{L}_l$
$\mathcal{L}_r$
$\phi \cong \phi_i$
$\phi \cong \phi_j$
x
$\phi \cong \phi_k$

Path integral on a large disk



Choose boundary conditions preserving ζ-supersymmetry:

Consider a cyclic ``fan of solitons''

$$\mathcal{F} = \{\phi^{\rm inst}_{i_1 i_2}, \cdots, \phi^{\rm inst}_{i_n i_1}\}$$

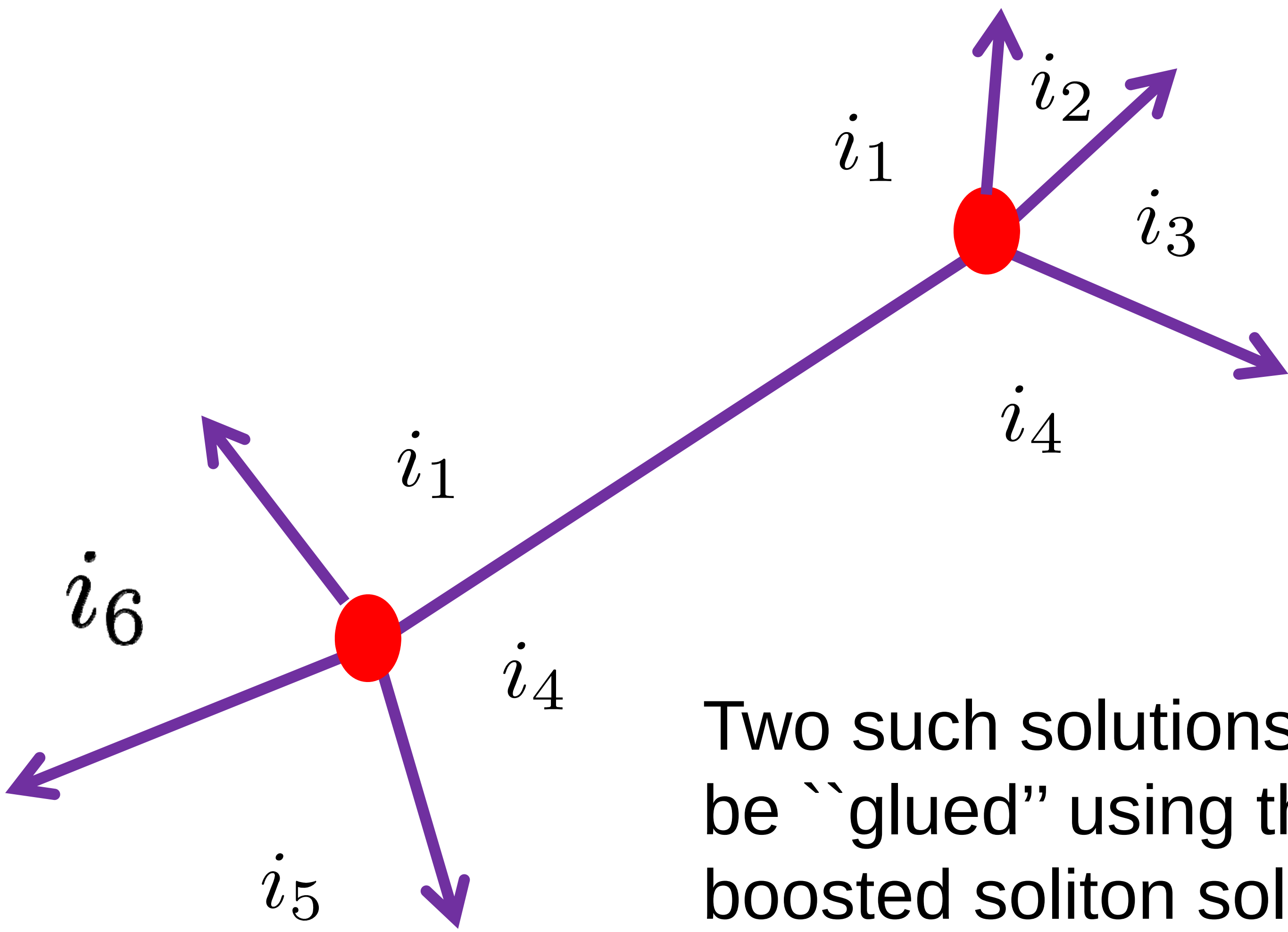
Localization

The path integral of the LG model with these boundary conditions localizes on moduli space of ζ-instantons:

$$\mathcal{M}(\mathcal{F})$$

We assume the mathematically nontrivial statement that, when the ``fermion number'' of the boundary condition at infinity is positive then the moduli space is nonempty.

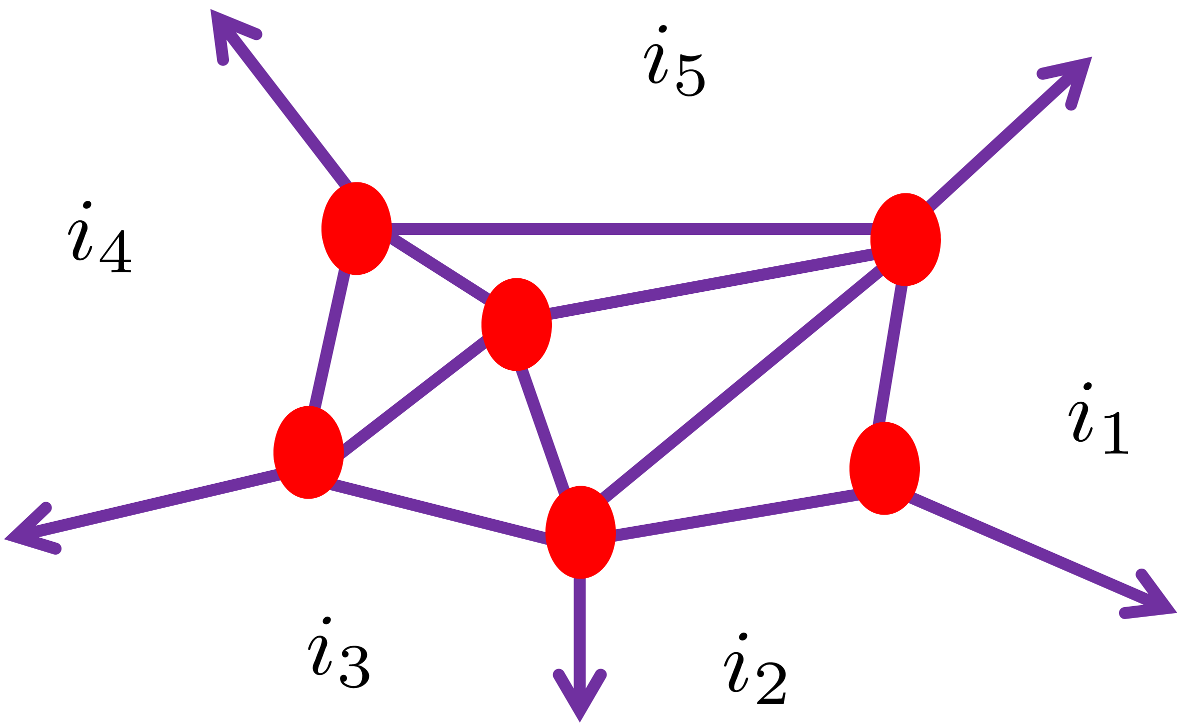
Gluing



Two such solutions can be ``glued'' using the boosted soliton solution -

Ends of moduli space

This moduli space has several "ends" where solutions of the ζ-instanton equation look like



We call this picture a ζ - *web*: $\mathfrak{w}$

ζ-Vertices & Interior Amplitudes

The red vertices represent solutions from the ***compact*** and ***connected*** components of

$$\mathcal{M}(\mathcal{F})$$

The contribution to the path integral from such components are called ``*interior amplitudes*.'' For the zero-dimensional moduli spaces they count (with signs) the solutions to the ζ-instanton equation.

Path Integral With Fan Boundary Conditions

Just as in the Morse theory proof of $d^2=0$ using ends of moduli space corresponding to broken flows, here the broken flows correspond to webs $\mathfrak{w}$

The state created by the path integral with fan boundary conditions should be $U(\zeta)$-invariant.

⟹

L_∞ identities on the interior amplitudes

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Definition of a Plane Web

We now give a purely mathematical construction.

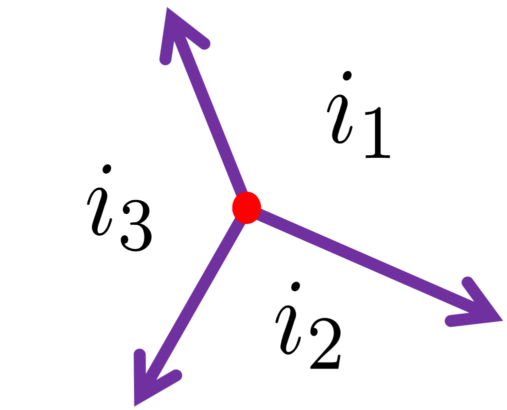
It is motivated from LG field theory.

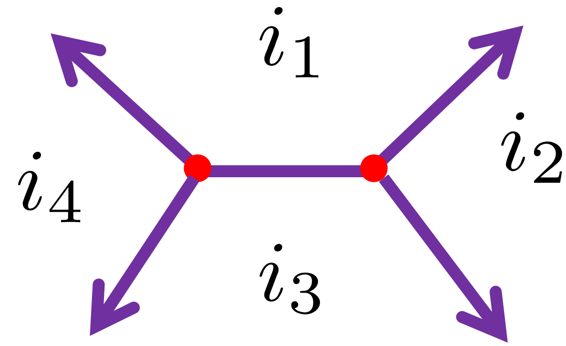
Vacuum data:

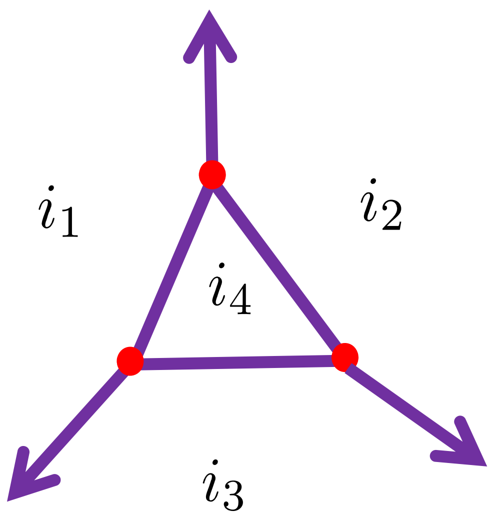
1. A finite set of ``vacua'': $i, j, k, \cdots \in \mathbb{V}$

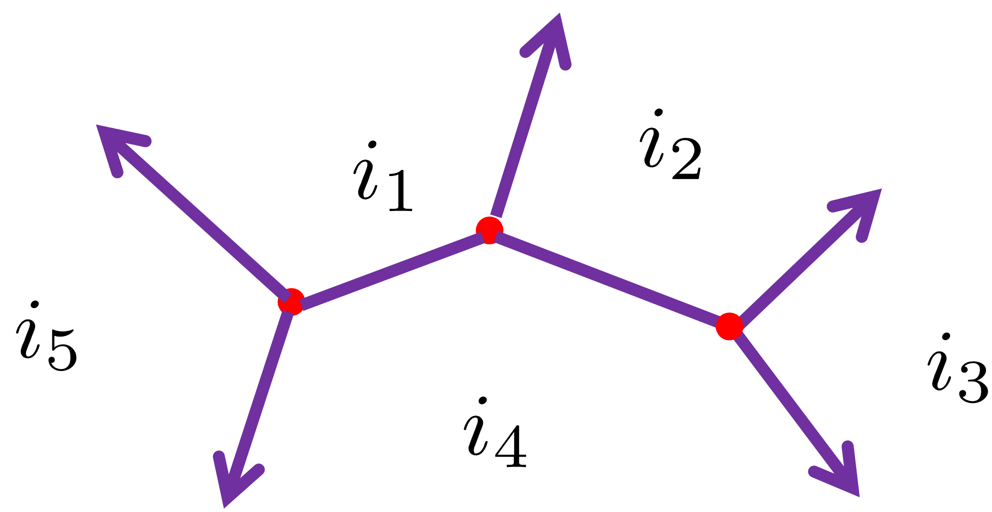
2. A set of weights $z : \mathbb{V} \to \mathbb{C}$

Definition: A *plane web* is a graph in $\mathbb{R}^2$, together with a coloring of faces by vacua (so that across edges labels differ) and if an edge is oriented so that *i* is on the left and *j* on the right then the edge is parallel to $z_{ij} = z_i - z_j$. (Option: Require vertices at least 3-valent.)


i_1
i_3
i_2

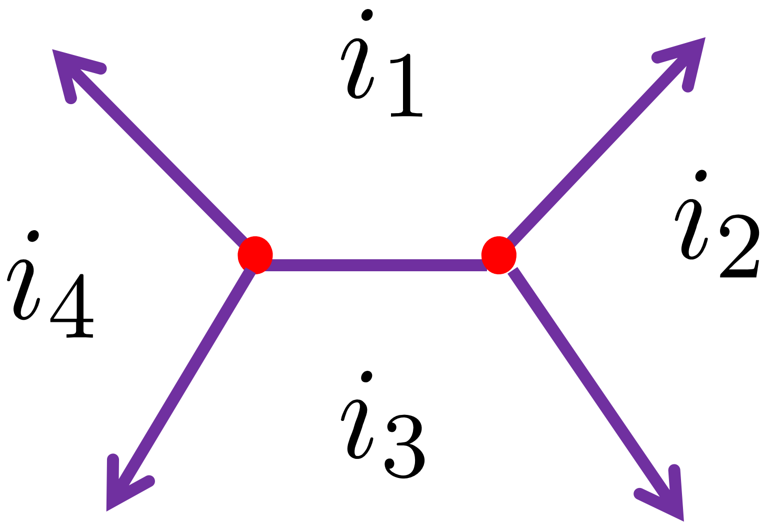

i_1
i_2
i_4
i_3

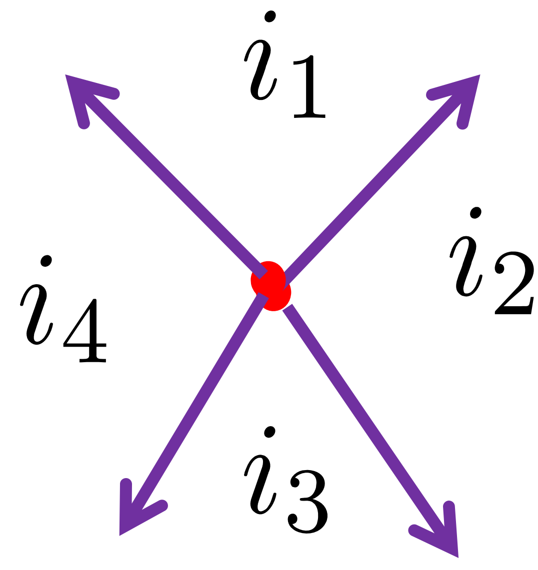

i_1
i_2
i_4
i_3


i_2
i_1
i_5
i_3
i_4

Deformation Type

Equivalence under translation and stretching (but not rotating) of edges subject to slope constraints defines ***deformation type***.





Moduli of webs with fixed deformation type

$$\mathcal{D}(\mathfrak{w}) \subset (\mathbb{R}^2)^{V(\mathfrak{w})}$$

$$\dim \mathcal{D}(\mathfrak{w}) = 2V(\mathfrak{w}) - E(\mathfrak{w})$$

(z_i in generic position)

$$\mathcal{D}^{\mathrm{red}}(\mathfrak{w}) = \mathcal{D}(\mathfrak{w})/\mathbb{R}^2_{\mathrm{transl}}$$

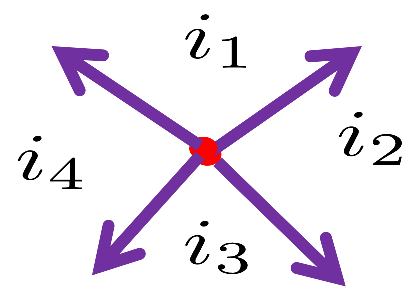
Cyclic Fans of Vacua

Definition: A cyclic fan of vacua is a cyclically-ordered set

$$I = \{i_1, \dots, i_n\}$$

so that the rays $z_{i_k, i_{k+1}} \mathbb{R}_+$ are ordered clockwise

$$I = \{i_1, i_2, i_3, i_4\}$$

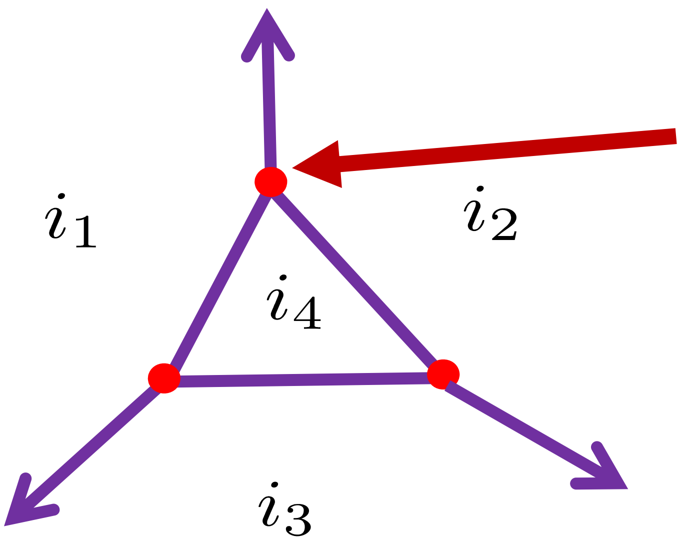


Fans at vertices and at ∞

For a web $\mathfrak{w}$ there are two kinds of cyclic fans we should consider:

Local fan of vacua at a vertex *v*: $I_v(\mathfrak{w})$

Fan of vacua ∞ : $I_\infty(\mathfrak{w})$



$$I_v(\mathfrak{w}) = \{i_1, i_2, i_4\}$$

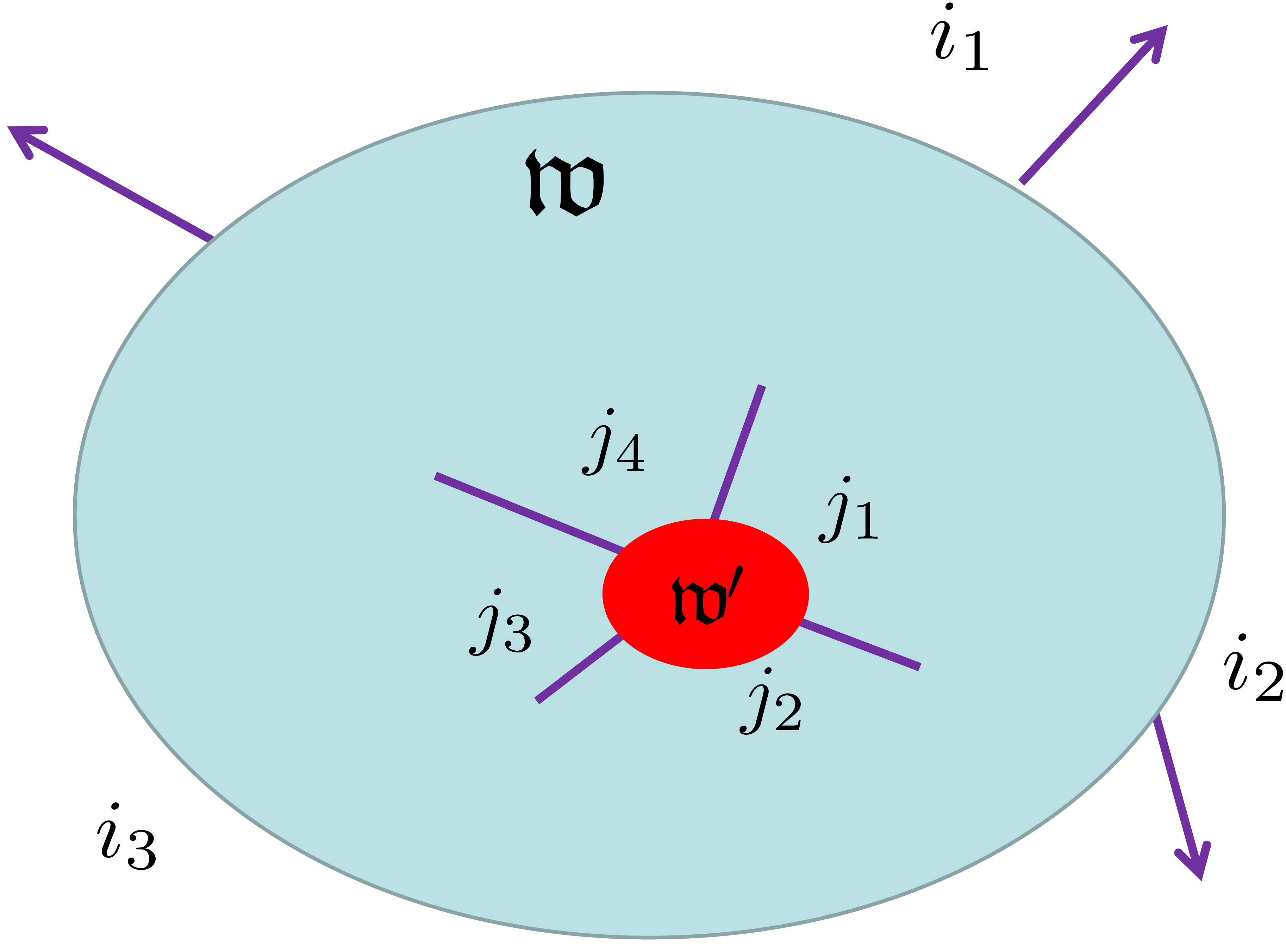
$$I_\infty(\mathfrak{w}) = \{i_1, i_2, i_3\}$$

Convolution of Webs

Definition: Suppose $\mathfrak{w}$ and $\mathfrak{w}'$ are two plane webs and $v \in \mathcal{V}(\mathfrak{w})$ such that

$$I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}')$$

The *convolution of $\mathfrak{w}$ and $\mathfrak{w}'$*, denoted $\mathfrak{w} *_v \mathfrak{w}'$ is the deformation type where we glue in a copy of $\mathfrak{w}'$ into a small disk cut out around *v*.



i_1
$\mathfrak{w}$
j_4
j_1
$\mathfrak{w}'$
j_3
i_2
j_2
i_3

The Web Ring

$\mathcal{W}$ Free abelian group generated by oriented deformation types of plane webs.

``oriented'': Choose an orientation o($\mathfrak{w}$) of $\mathcal{D}^{\text{red}}(\mathfrak{w})$

$$* : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}$$

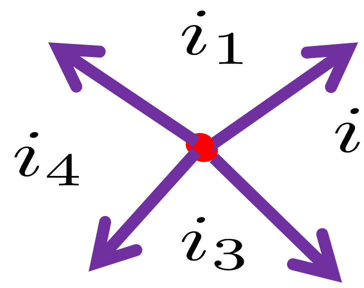
$$I_v(\mathfrak{w}_1) \neq I_\infty(\mathfrak{w}_2) \Rightarrow \mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$$

$$\mathfrak{w}_1 * \mathfrak{w}_2 := \sum_{v \in \mathcal{V}(\mathfrak{w}_1)} \mathfrak{w}_1 *_v \mathfrak{w}_2$$

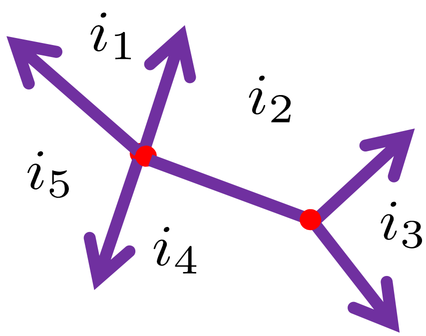
$$o(\mathfrak{w} *_v \mathfrak{w}') = o(\mathfrak{w}) \wedge o(\mathfrak{w}')$$

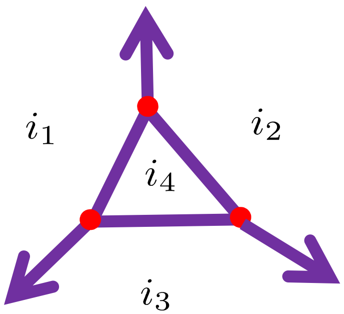
Rigid, Taut, and Sliding

A ***rigid web*** has d($\mathfrak{w}$) = 0.
It has one vertex:

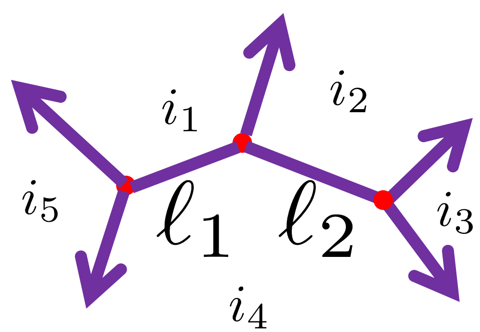


A ***taut web*** has
d($\mathfrak{w}$) = 1:





A ***sliding web*** has
d($\mathfrak{w}$) = 2



The taut element

Definition: The taut element $\mathfrak{t}$ is the sum of all taut webs with standard orientation

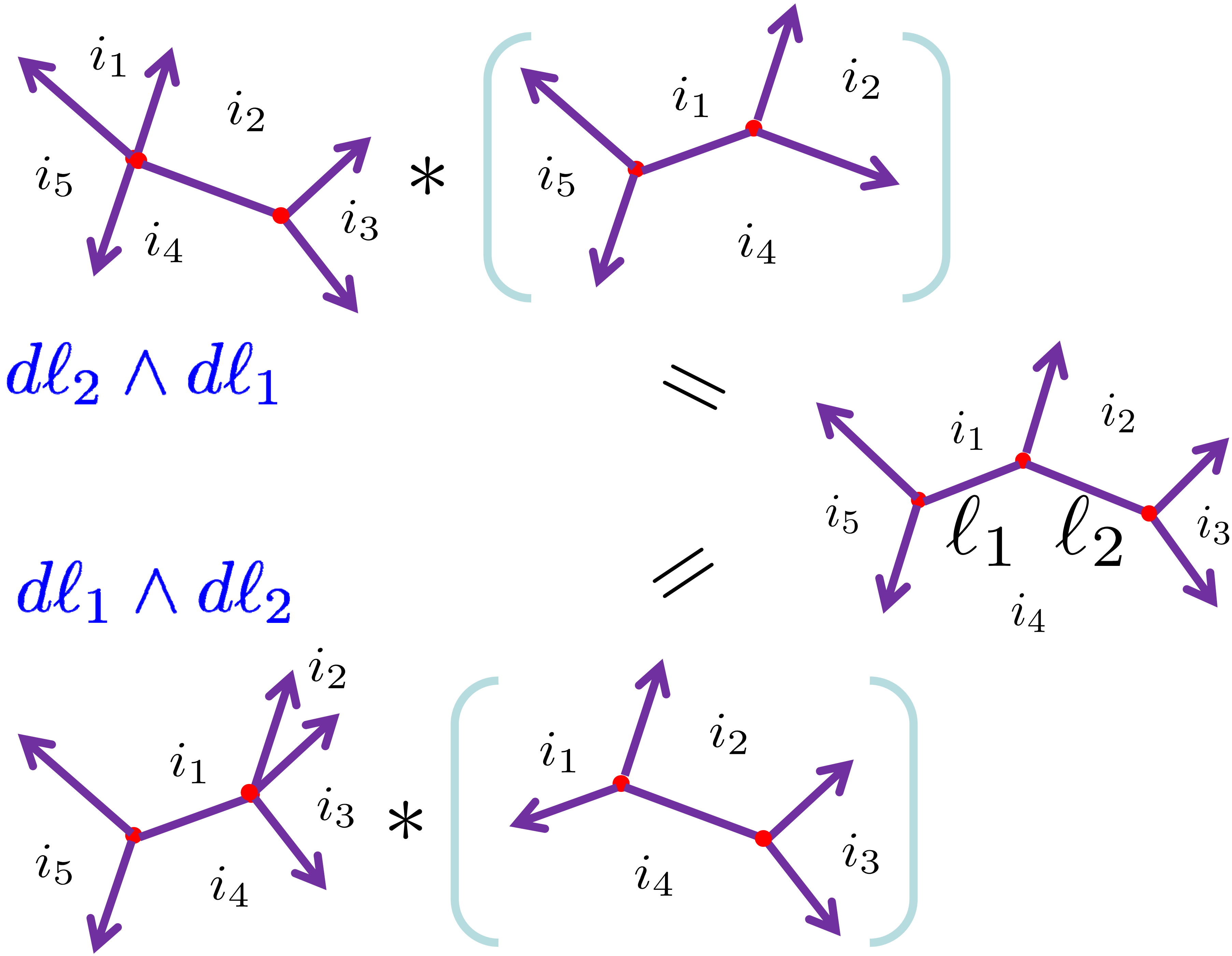
$$\mathfrak{t} := \sum_{d(\mathfrak{w})=1} \mathfrak{w}$$

Theorem: $\mathfrak{t} * \mathfrak{t} = 0$

Proof: The terms can be arranged so that there is a cancellation of pairs:

$$\mathfrak{w}_1 * \mathfrak{w}_2 \qquad \mathfrak{w}_3 * \mathfrak{w}_4$$

Representing two ends of a moduli space of sliding webs



i1 i2 i3 i4 i5
*
i1 i2 i4 i5
dℓ2 ∧ dℓ1
=
i1 i2 i3 i4 i5
ℓ1 ℓ2
dℓ1 ∧ dℓ2
=
i1 i2 i3 i4 i5
*
i1 i2 i3 i4

Web Representations

Definition: A *representation of webs* is

a.) A choice of $\mathbb{Z}$-graded $\mathbb{Z}$-module R_{ij} for every ordered pair ij of distinct vacua.

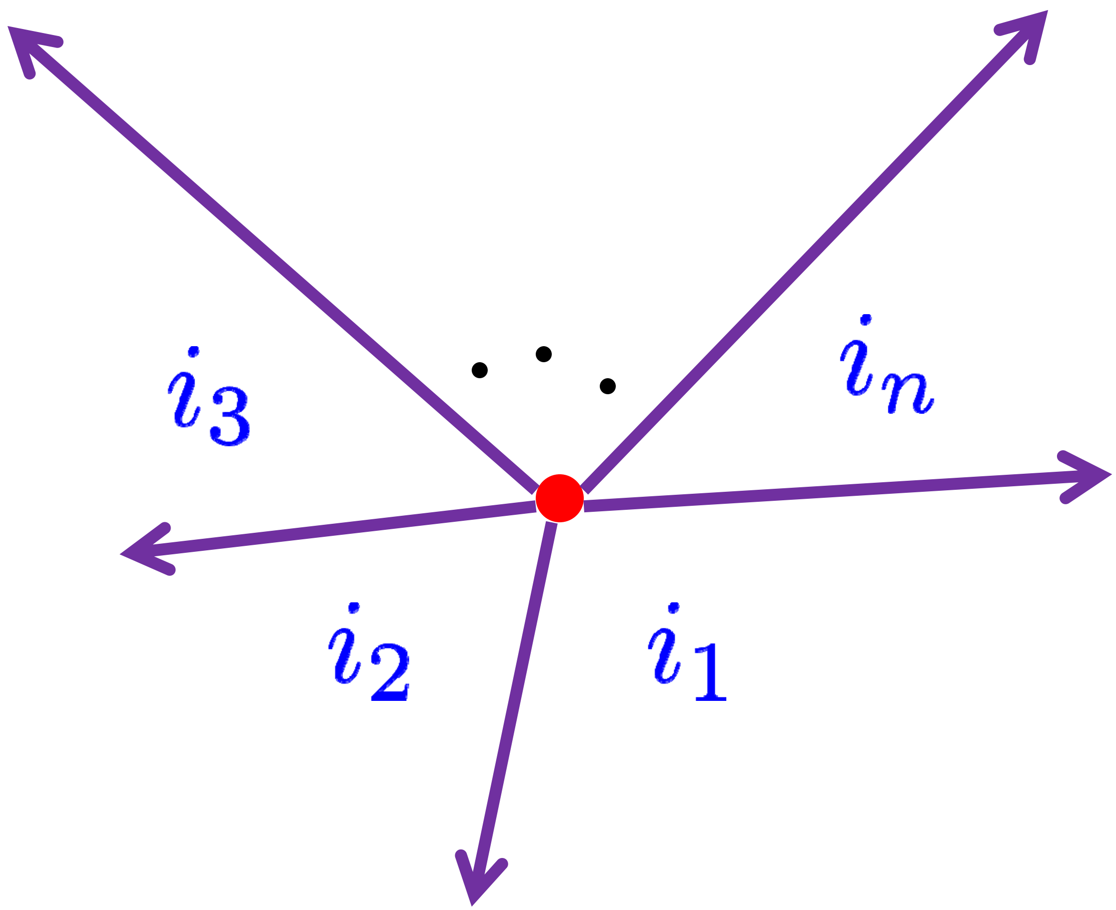
b.) A symmetric degree = -1 perfect pairing

$$K : R_{ij} \otimes R_{ji} \to \mathbb{Z}$$

For every cyclic fan of vacua introduce a *fan representation*:

$$I = \{i_1, \dots, i_n\} \Rightarrow$$

$$R_I := R_{i_1,i_2} \otimes \cdots \otimes R_{i_n,i_1}$$



$$R_I := R_{i_1,i_2} \otimes \cdots \otimes R_{i_n,i_1}$$

Web Rep & Contraction

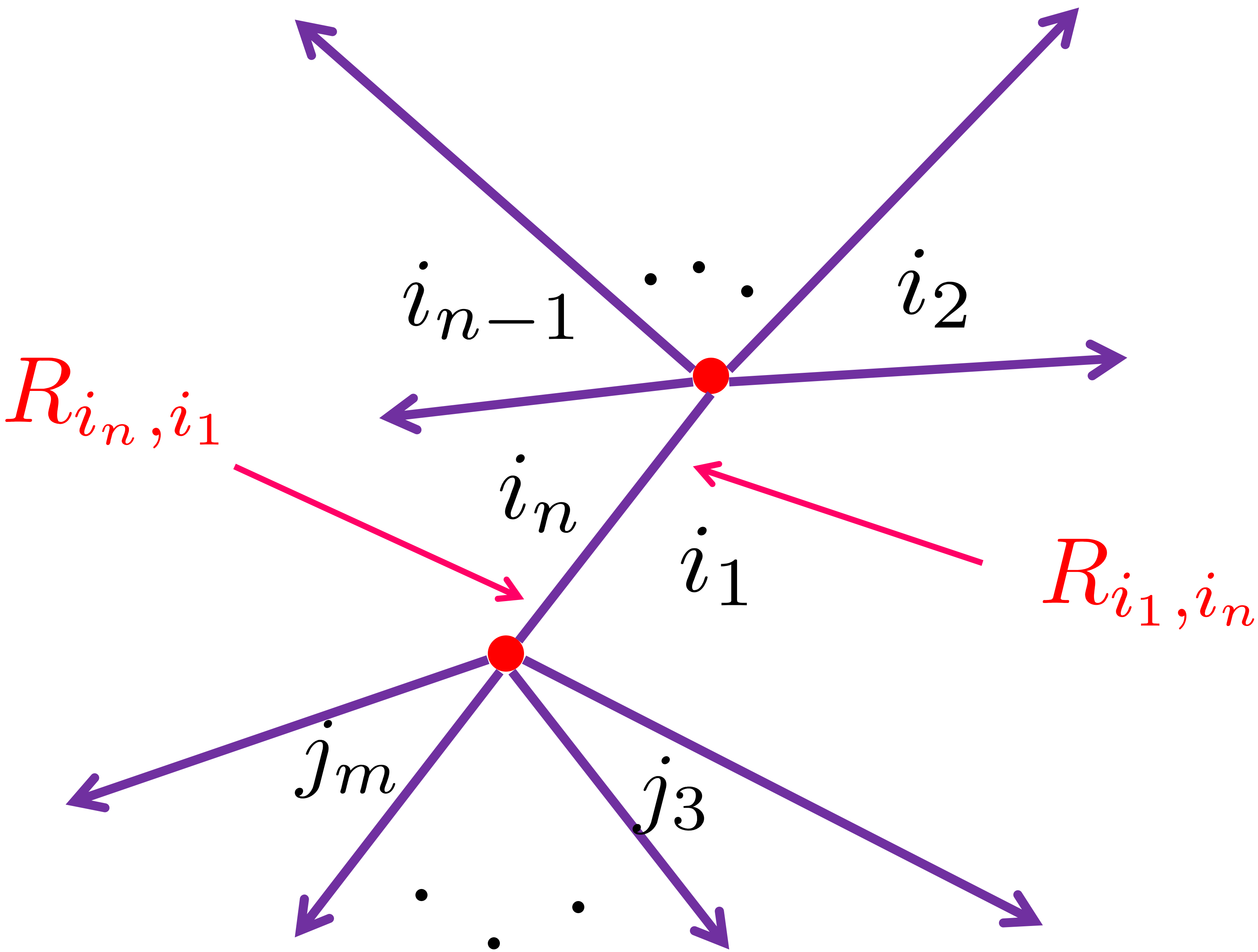
Given a rep of webs and a deformation type $\mathfrak{w}$ we define the *representation of $\mathfrak{w}$*:

$$R(\mathfrak{w}) := \otimes_{v \in \mathcal{V}(\mathfrak{w})} R_{I_v(\mathfrak{w})}$$

There is a natural contraction operator:

$$\rho(\mathfrak{w}) : R(\mathfrak{w}) \to R_{I_\infty(\mathfrak{w})}$$

by applying the contraction K to the pairs R_{ij} and R_{ji} on each internal edge:


i_{n-1}
i_2
R_{i_n,i_1}
i_n
i_1
R_{i_1,i_n}
j_m
j_3

Extension to Tensor Algebra

$$R^{\text{int}} := \oplus_I R_I$$ Rep of all vertices.

$$\rho(\mathfrak{w}) : TR^{\text{int}} \to R^{\text{int}}$$

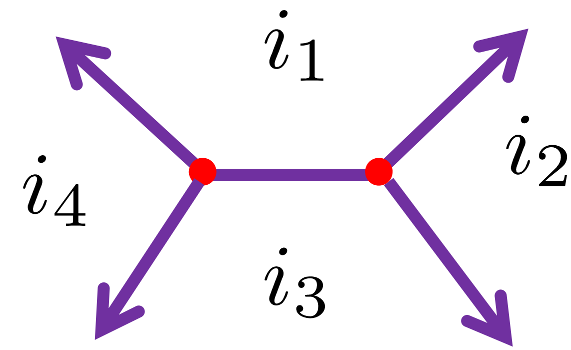
$$r^{(1)} \otimes \cdots \otimes r^{(n)} \in R_{I_1} \otimes \cdots \otimes R_{I_n}$$

$$\rho(\mathfrak{w})[r^{(1)}, \ldots, r^{(n)}]$$

vanishes, unless

$$\{R_{I_1}, \ldots, R_{I_n}\} \Longleftrightarrow \{R_{I_v(\mathfrak{w})}\}$$

Example



$$R(\mathfrak{w}) = R_{i_1 i_3 i_4} \otimes R_{i_1 i_2 i_3}$$

$$\rho(\mathfrak{w})[\underline{r_{i_1 i_3}} r_{i_3 i_4} r_{i_4 i_1} \otimes r_{i_1 i_2} r_{i_2 i_3} \underline{r_{i_3 i_1}}]$$

$$\pm K(r_{i_1 i_3}, r_{i_3 i_1}) r_{i_1 i_2} r_{i_2 i_3} r_{i_3 i_4} r_{i_4 i_1}$$

$$\in R_{i_1 i_2 i_3 i_4}$$

L_∞ -algebras

$$\rho(\mathfrak{t}) : TR^{\text{int}} \to R^{\text{int}}$$

$$\mathfrak{t} * \mathfrak{t} = 0 \quad \Longrightarrow$$

$$\sum_{\text{Sh}_2(S)} \epsilon \, \rho(\mathfrak{t})[\rho(\mathfrak{t})[S_1], S_2] = 0.$$

$$S = \{r_1, \ldots, r_n\} \quad r_i \in R^{\text{int}}$$

$$S = S_1 \amalg S_2 \qquad \epsilon \in \{\pm 1\}$$

L∞ and A∞ Algebras

If A is a vector space (or Z-module) then an ∞-algebra structure is a series of multiplications:

$$m_n(a_1, \ldots, a_n) \in A$$

Which satisfy quadratic relations:

$$S = \{a_1, \ldots, a_n\}$$

$$L_\infty : \quad \sum_{\mathrm{Sh}_2(S)} \epsilon m_{s_1+1}(m_{s_2}(S_2), S_1) = 0$$

$$A_\infty : \sum_{\mathrm{Pa}_3(S)} \epsilon m_{s_1+1+s_3}(S_1, m_{s_2}(S_2), S_3)) = 0$$

The Interior Amplitude

Sum over cyclic fans: $R^{\mathrm{int}} := \oplus_I R_I$

$$\rho(\mathfrak{t}) : TR^{\mathrm{int}} \to R^{\mathrm{int}}$$

Interior amplitude: $\beta \in R^{\mathrm{int}}$

Satisfies the L_∞ ``Maurer-Cartan equation''

$$\rho(\mathfrak{t})(e^{\beta}) = 0$$

$$e^{\beta} = 1 + \beta + \frac{1}{2!}\beta \otimes \beta + \cdots$$

``Interaction amplitudes for solitons''

Definition of a Theory

By a ***Theory*** we mean a collection of data

$$(\mathbb{V}, z, R_{ij}, K, \beta)$$

``Physics Theorem''

The LG model with massive superpotential defines a Theory in the above sense.

In particular, the interior amplitudes β_I defined by counting the number of solutions of the ζ-instanton equation with no reduced moduli define solutions to the L_∞ Maurer-Cartan equation.

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- Half-plane webs & Branes: A_∞
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Half-Plane Webs

Same as plane webs, but they sit in a half-plane $\mathcal{H}$.

Some vertices (but no edges) are allowed on the boundary.

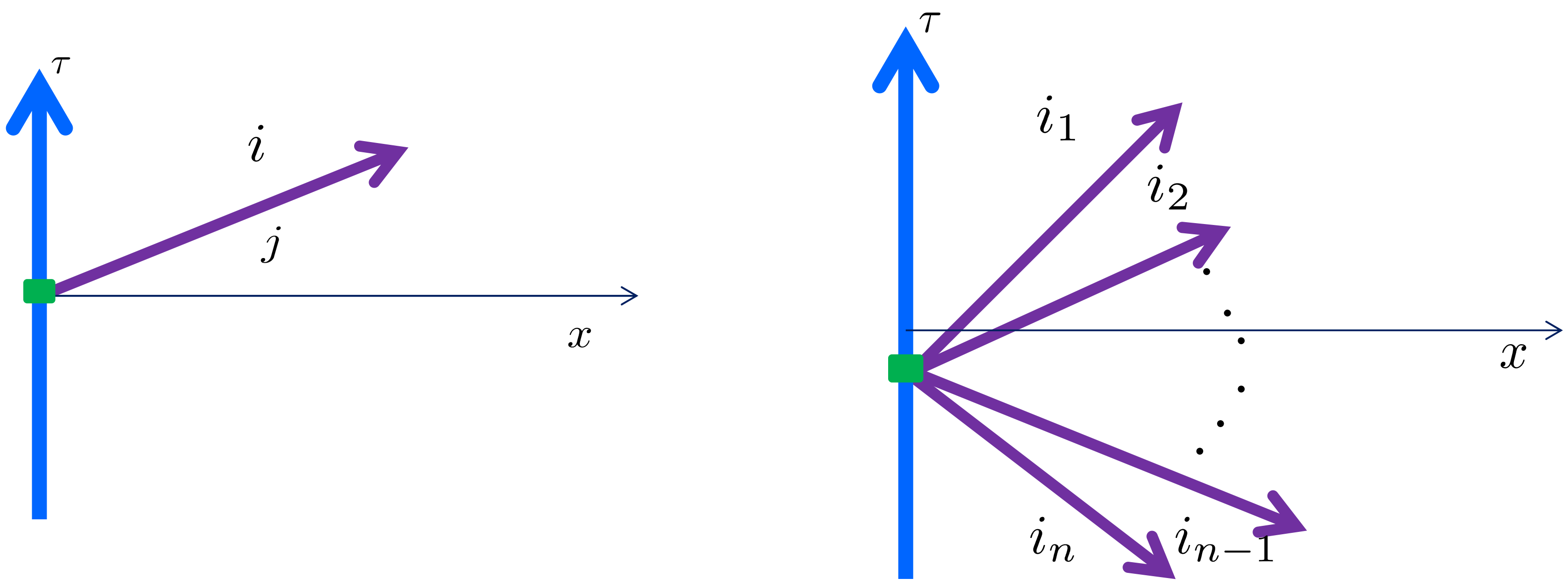
$\mathcal{V}_i(\mathfrak{u})$ Interior vertices

$\mathcal{V}_\partial(\mathfrak{u}) = \{v_1, \ldots, v_n\}$ ***time-ordered*** boundary vertices.

deformation type, reduced moduli space, etc. ….

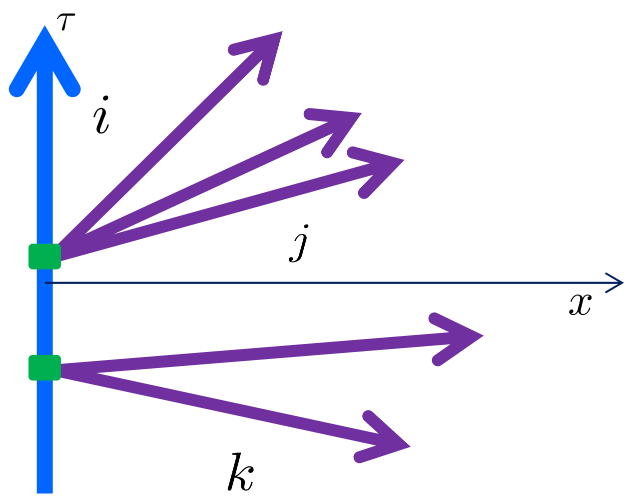
$$d(\mathfrak{u}) := 2V_i(\mathfrak{u}) + V_\partial(\mathfrak{u}) - E(\mathfrak{u}) - 1$$

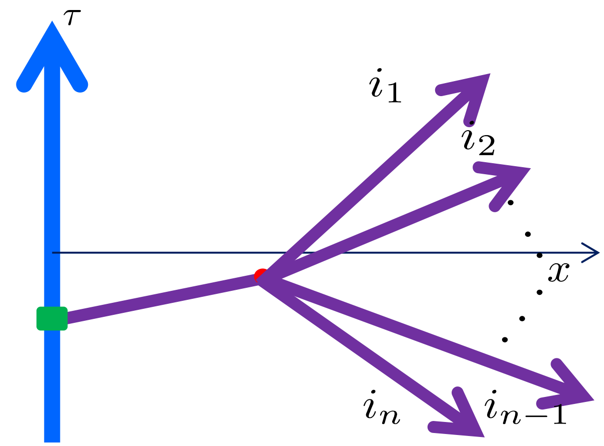
Rigid Half-Plane Webs



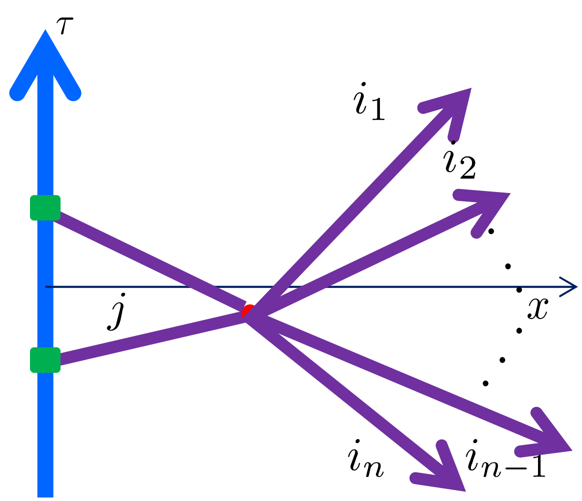
$$d(\mathfrak{u}) = 0$$

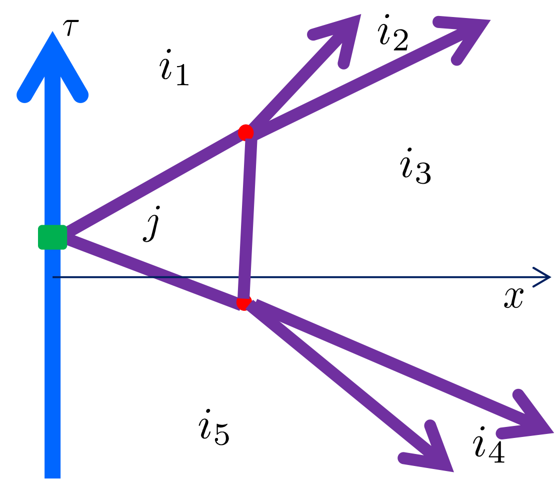
Taut Half-Plane Webs



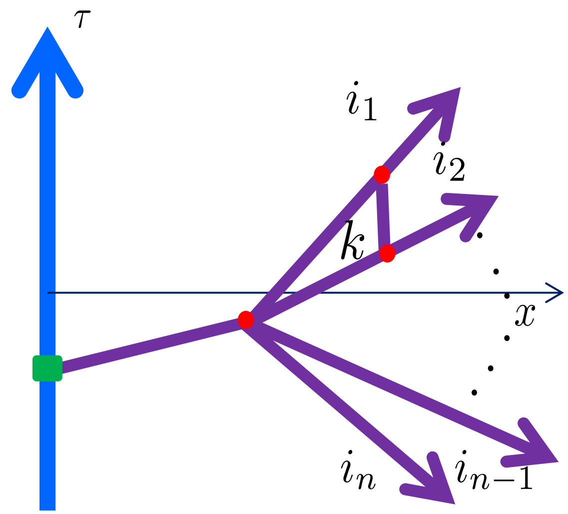


$$d(\mathfrak{u}) = 1$$

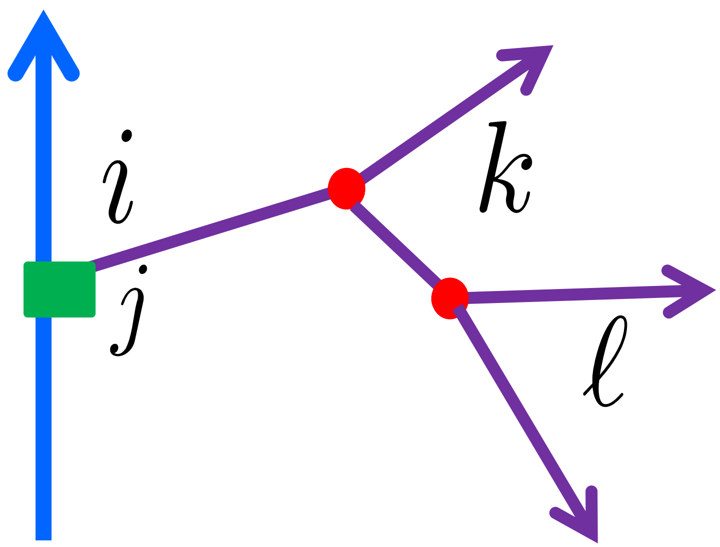


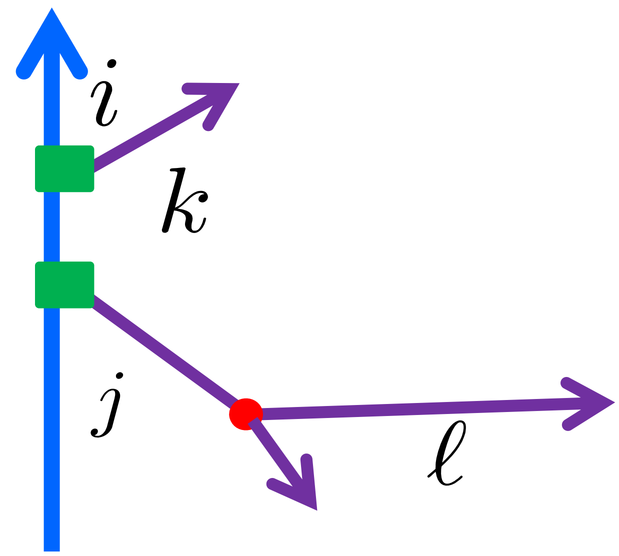


Sliding Half-Plane webs



$$d(\mathbf{u}) = 2$$





Half-Plane fans

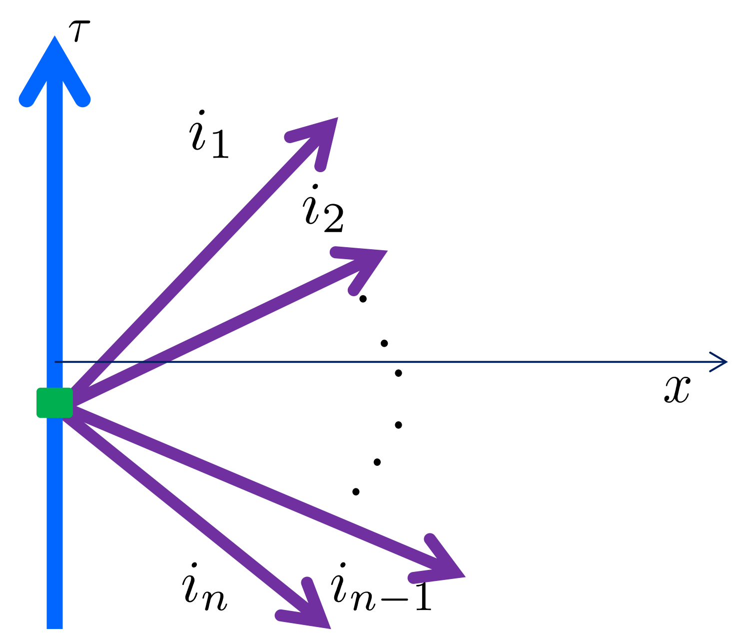
A half-plane fan is an ordered set of vacua,

$$J = \{i_1, \dots, i_n\}$$

such that successive vacuum weights:

$$z_{i_s, i_{s+1}}$$

are ordered clockwise and in the half-plane:



Convolutions for Half-Plane Webs

We can now introduce a convolution at boundary vertices:

Local half-plane fan at a boundary vertex *v*: $J_v(\mathfrak{u})$

Half-plane fan at infinity: $J_\infty(\mathfrak{u})$

$\mathcal{W}_{\mathcal{H}}$ Free abelian group generated by oriented def. types of half-plane webs

There are now two convolutions:

$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W}_{\mathcal{H}} \to \mathcal{W}_{\mathcal{H}}$$

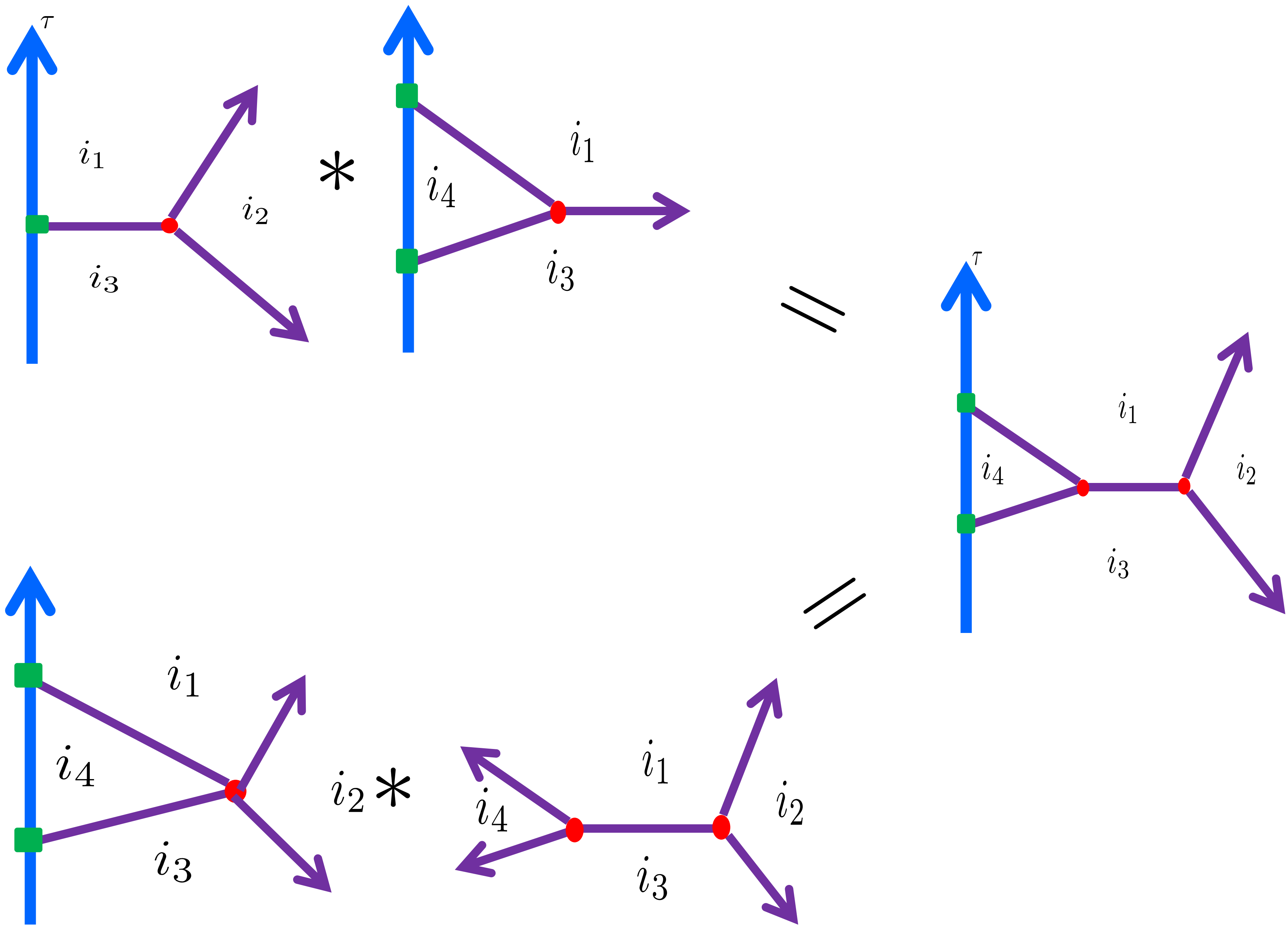
$$\mathcal{W}_{\mathcal{H}} \times \mathcal{W} \to \mathcal{W}_{\mathcal{H}}$$

Convolution Theorem

Define the half-plane taut element: $$t_{\mathcal{H}} := \sum_{d(u)=1} u$$

Theorem: $$t_{\mathcal{H}} * t_{\mathcal{H}} + t_{\mathcal{H}} * t_{p} = 0$$

Proof: A sliding half-plane web can degenerate (in real codimension one) in two ways: Interior edges can collapse onto an interior vertex, or boundary edges can collapse onto a boundary vertex.



τ
i_1
i_2
i_3
*
i_4
i_1
i_3
=
τ
i_1
i_4
i_2
i_3
=
i_1
i_4
i_3
i_2
*
i_1
i_4
i_2
i_3

Half-Plane Contractions

A rep of a half-plane fan: $J = \{j_1, \ldots, j_n\}$

$$R_J := R_{j_1,j_2} \otimes \cdots \otimes R_{j_{n-1},j_n}$$

$\rho(\mathfrak{u})$ now contracts R($\mathfrak{u}$):

$$\bigotimes_{v \in \mathcal{V}_\partial(\mathfrak{u})} R_{J_v(\mathfrak{u})} \bigotimes_{v \in \mathcal{V}_i(\mathfrak{u})} R_{I_v(\mathfrak{u})}$$

$$\rightarrow R_{J_\infty(\mathfrak{u})}$$

The Vacuum A_∞ Category

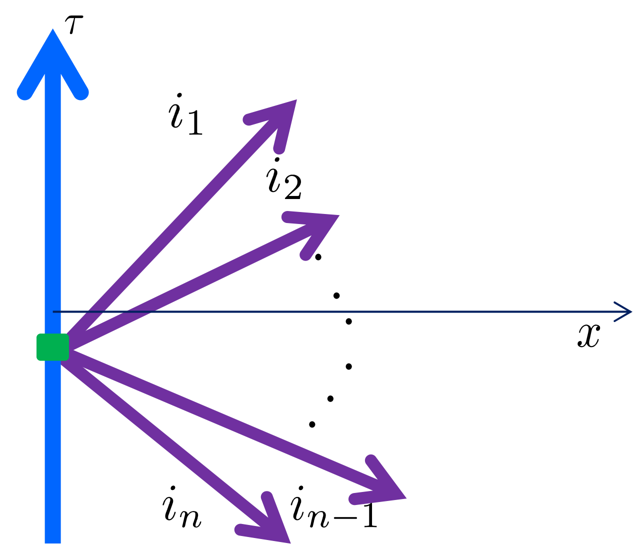
(For $\mathcal{H}$ = the positive half-plane)

Objects: i $\in \mathbb{V}$.

Morphisms: $$\mathrm{Hom}(j,i) = \begin{cases} \widehat{R}_{ij} & \mathrm{Re}(z_{ij}) > 0 \\ \mathbb{Z} & i = j \\ 0 & \mathrm{Re}(z_{ij}) < 0 \end{cases}$$

$$\widehat{R}_{i_1,i_n} := \oplus_J R_J$$

$$J = \{i_1, \dots, i_n\}$$



Categorified Spectrum Generator/Stokes Matrix

The morphism spaces can be defined by a Cecotti-Vafa/Kontsevich-Soibelman-like product:

Suppose $\mathbb{V} = \{ 1, \dots, K \}$.
Introduce the elementary K x K matrices e_{ij}

$$\underbrace{\otimes_{\mathrm{Re}(z_{ij})>0}}_{\text{phase ordered!}} \left(\mathbb{Z}\mathbf{1} \oplus R_{ij} e_{ij} \right) = \mathbb{Z}\mathbf{1} \oplus_{i,j} \widehat{R}_{ij} e_{ij}$$

Taking the index produces the matrix S of Cecotti-Vafa.

A_∞ Multiplication

Interior amplitude:

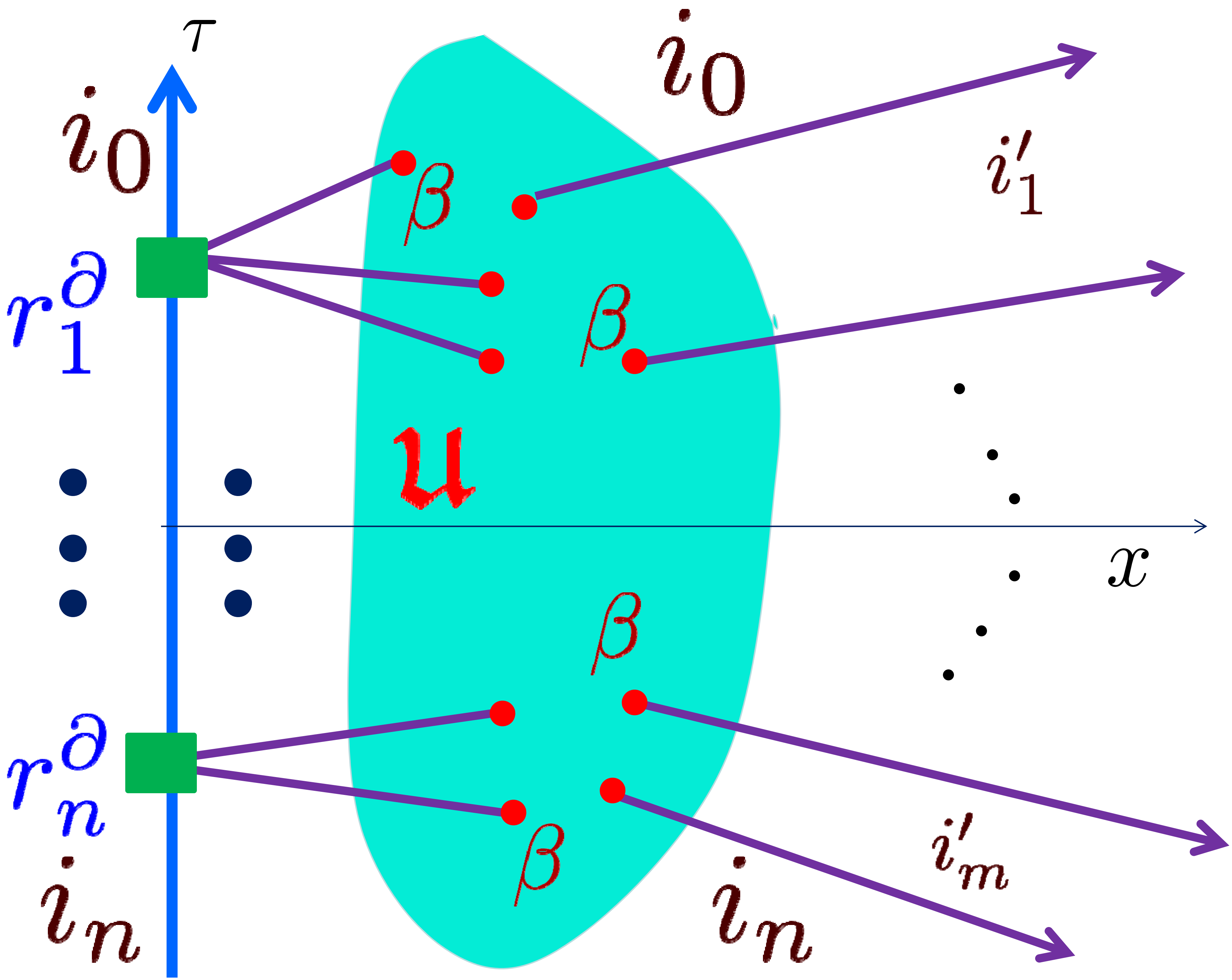
$$\beta \in R^{\mathrm{int}}$$

Satisfies the L_∞ ``Maurer-Cartan equation''

$$\rho(\mathfrak{t}_p)(e^{\beta}) = 0$$

$$m_n^{\beta}[r_1^{\partial}, \dots, r_n^{\partial}] := \rho(\mathfrak{t}_{\mathcal{H}})[r_1^{\partial}, \dots, r_n^{\partial}; e^{\beta}]$$

$$r_s^{\partial} \in \mathrm{Hom}(i_{s-1}, i_s)$$



τ
i_0
r_1^{∂}
β
i_0
i'_1
β
$\mathfrak{U}$
x
β
r_n^{∂}
β
i'_m
i_n
i_n

Enhancing with CP-Factors

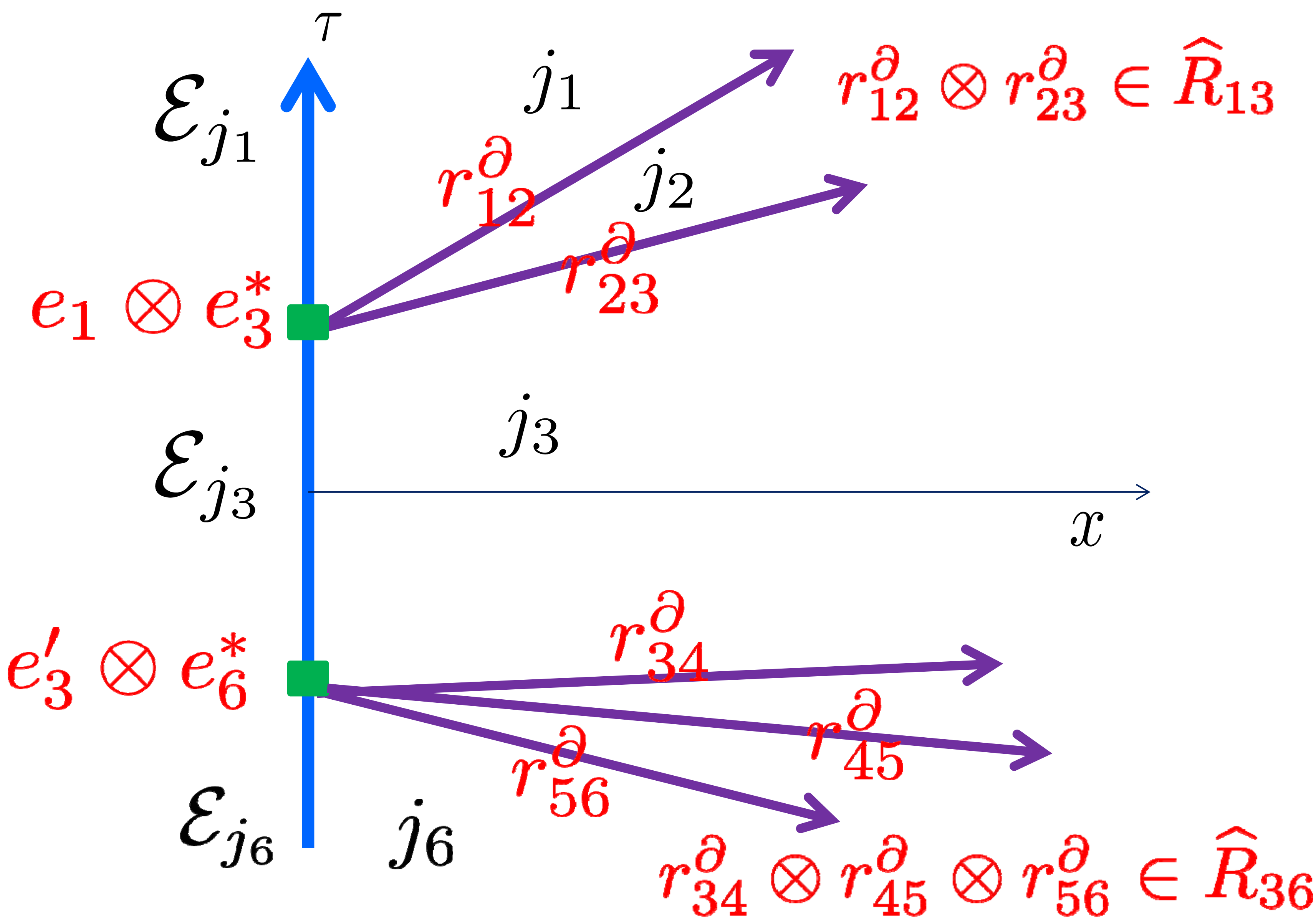
CP-Factors: $i \in \mathbb{V}$ $\longrightarrow$ $\mathcal{E}_i$ $\mathbb{Z}$-graded module

$$\mathrm{Hop}(i,j) \Longrightarrow \mathcal{E}_i \otimes \mathrm{Hop}(i,j) \otimes \mathcal{E}_j^*$$

$$m_n^\beta \Longrightarrow m_n^\beta \otimes m_2^{\mathrm{CP}}$$

Enhanced A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta; \mathcal{E})$

Example: Composition of two morphisms



Proof of A_∞ Relations

$$\mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_{\mathcal{H}} + \mathfrak{t}_{\mathcal{H}} * \mathfrak{t}_p = 0 \quad \Rightarrow$$

$$\sum \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon\, \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{r_1, \dots, r_m\} \qquad S = S_1 \amalg S_2$$

$$P = \{r_1^\partial, \dots, r_n^\partial\} \qquad P = P_1 \amalg P_2 \amalg P_3$$

$$r_a \in R^{\mathrm{int}} \qquad r_s^\partial \in \widehat{R}_{i_{s-1}, i_s}$$

$$\sum \epsilon \; \rho(\mathfrak{t}_{\mathcal{H}})[P_1, \rho(\mathfrak{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2]$$

$$+ \sum \epsilon \; \rho(\mathfrak{t}_{\mathcal{H}})[P; \rho(\mathfrak{t}_p)[S_1], S_2] = 0.$$

$$S = \{\beta, \ldots, \beta\}$$

and the second line vanishes.

Hence we obtain the A∞ relations for :

$$m^{\beta}[P] := \rho(\mathfrak{t}_{\mathcal{H}})[P; e^{\beta}]$$

Defining an A∞ category : $\mathfrak{Vac}(\mathbb{V}, z, R, K, \beta, \mathcal{E})$

Boundary Amplitudes

A ***Boundary Amplitude*** $\mathcal{B}$ (defining a Brane) is a solution of the A_∞ MC:

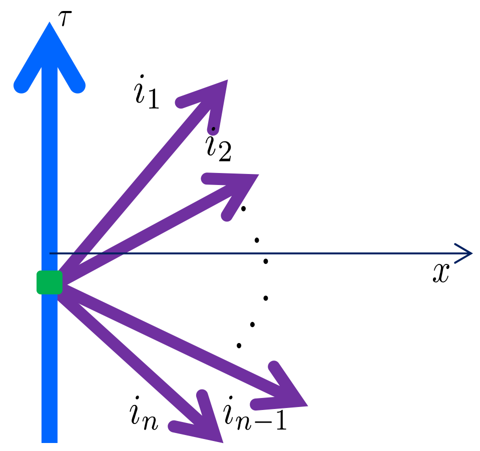
$$\mathcal{B} \in \oplus_{i,j} \mathrm{Hop}^{\mathcal{E}}(i,j)$$

$$\mathcal{B} \in \oplus_{\mathrm{Re}(z_{ij})>0} \mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^*$$

$$\sum_{n=1}^{\infty} m_n^{\beta}[\mathcal{B}^{\otimes n}] = 0$$

$$\rho(\mathfrak{t}_{\mathcal{H}})[\frac{1}{1-\mathcal{B}}; e^{\beta}] = 0$$

``Emission amplitude" from the boundary:



Category of Branes

The Branes themselves are objects in an A_∞ category

$$\mathfrak{Br}(\mathbb{V}, z, R, K, \beta)$$

$$\mathrm{Hop}(\mathcal{B}_1, \mathcal{B}_2)$$

$$= \oplus_{i,j\in\mathbb{V}} \mathcal{E}_i^1 \otimes \mathrm{Hop}(i,j) \otimes (\mathcal{E}_j^2)^*$$

$$M_n(\delta_1, \dots, \delta_n) = \dots.$$

("Twisted complexes": Analog of the derived category.)

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Families of Data

Now suppose the data of a Theory varies ***continuously*** with space:

$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$

We have an interface or Janus between the theories at x_{in} and x_{out}.

?? How does the Brane category change??

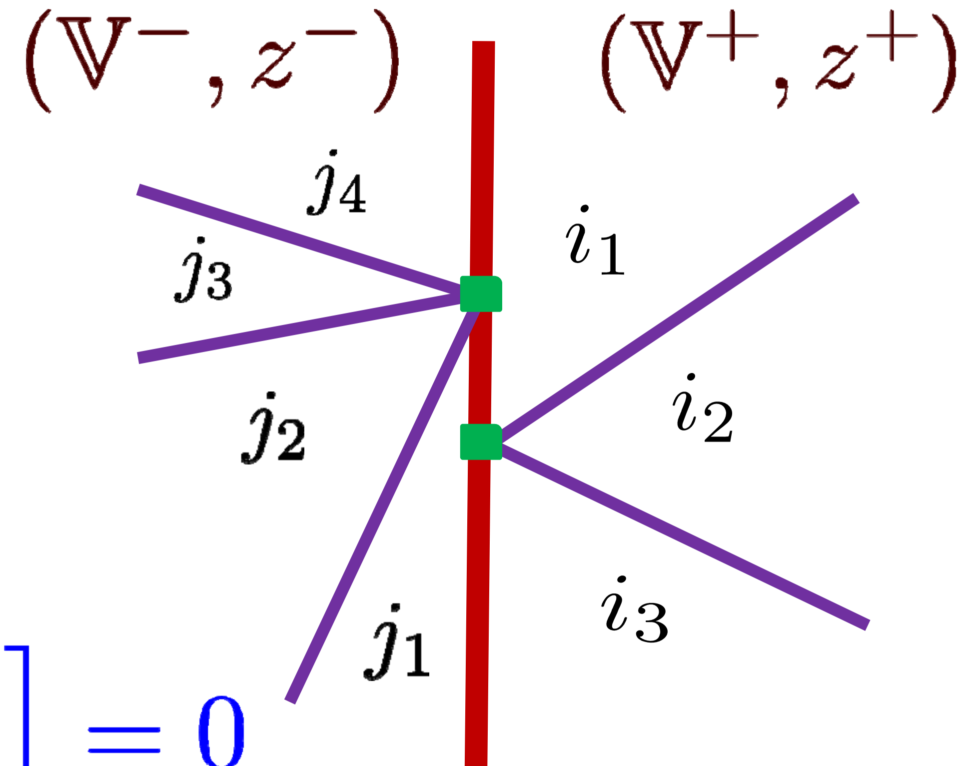
We wish to define a ``flat parallel transport'' of Brane categories. The key will be to develop a theory of supersymmetric interfaces.

Interface webs & amplitudes

Given data $\mathcal{T}^{\pm} = (\mathbb{V}, z, R, K, \beta)^{\pm}$

Introduce a notion of ``interface webs''

These behave like half-plane webs and we can define an *Interface Amplitude* to be a solution of the MC equation:



$$\rho(\mathbf{t}^{-,+})\left[\frac{1}{1-\mathcal{B}^{-,+}}; e^{\beta}\right] = 0$$

Category of Interfaces

Interfaces are very much like Branes,

Chan-Paton: $\mathcal{E}(\mathfrak{I})_{i^-,j^+}$

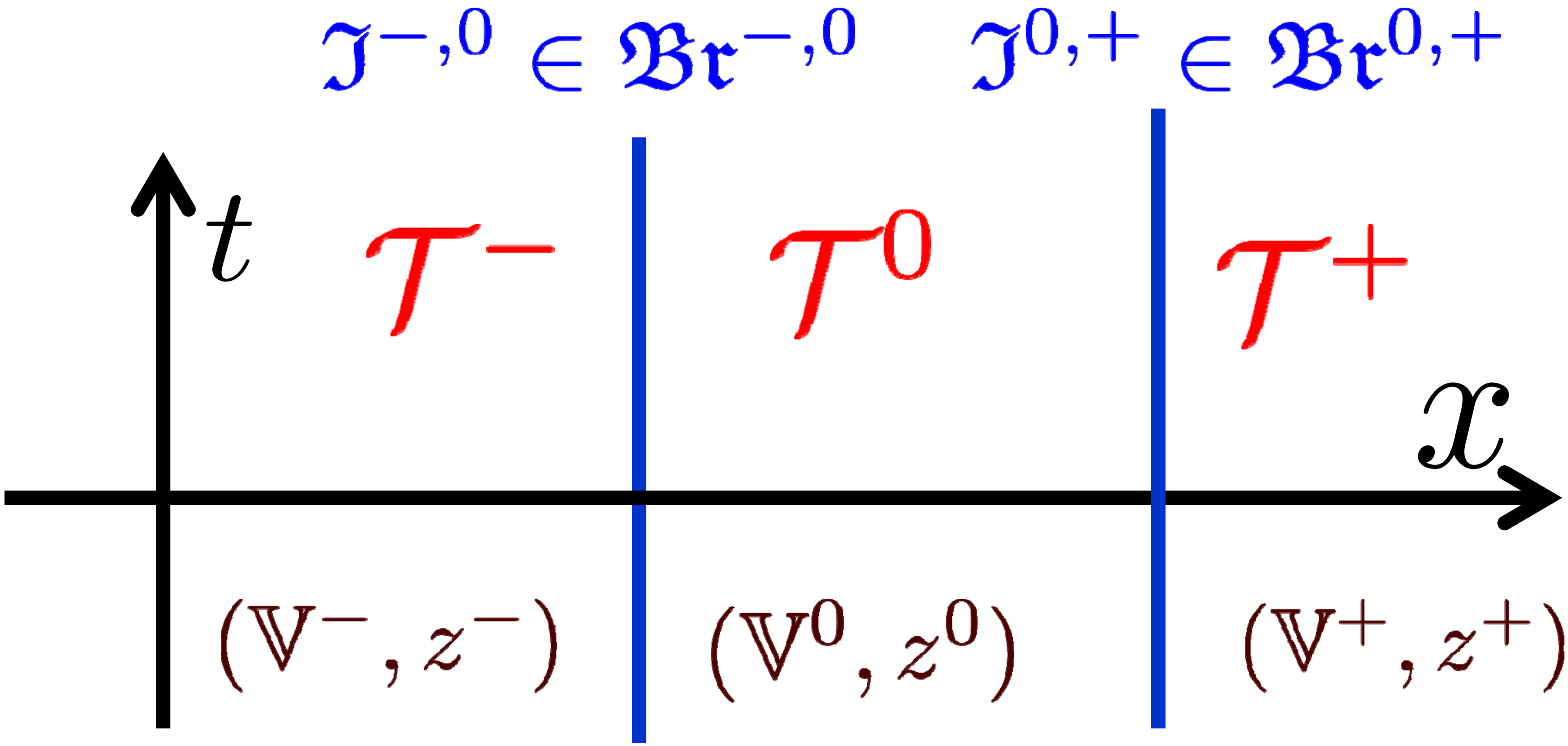
$$(i^-, j^+) \in \mathbb{V}^- \times \mathbb{V}^+$$

In fact we can define an A_∞ category of Interfaces between the two theories:

$$\mathfrak{I}^{-,+} \in \mathfrak{Br}^{-,+}$$

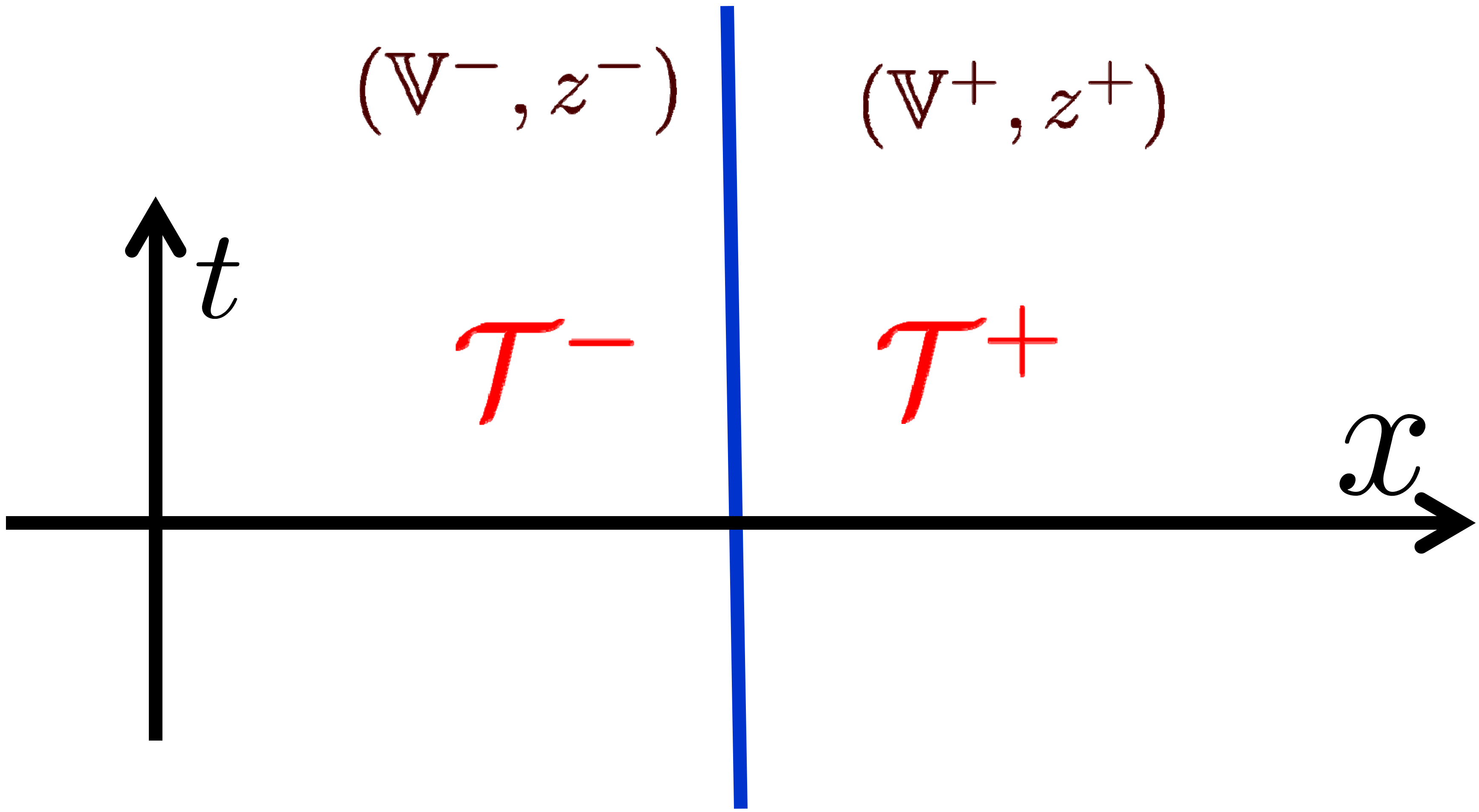
Note: If one of the Theories is trivial we simply recover the category of Branes.

Composition of Interfaces -1



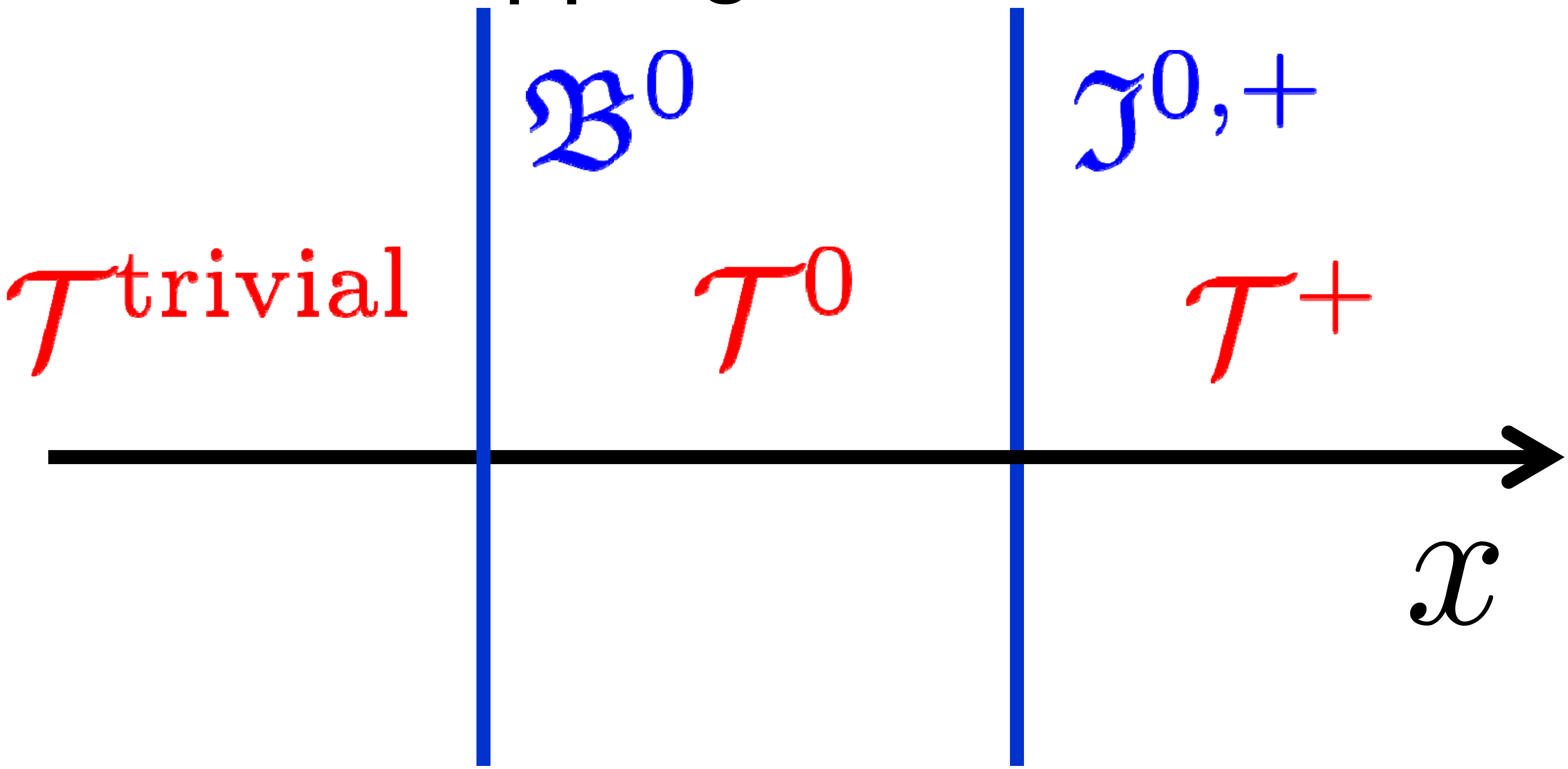
Want to define a ``multiplication" of the Interfaces…

Composition of Interfaces - 2



$$\mathfrak{I}^{-,+} = \mathfrak{I}^{-,0} \star \mathfrak{I}^{0,+} \in \mathfrak{Br}^{-,+}$$

Mapping of Branes



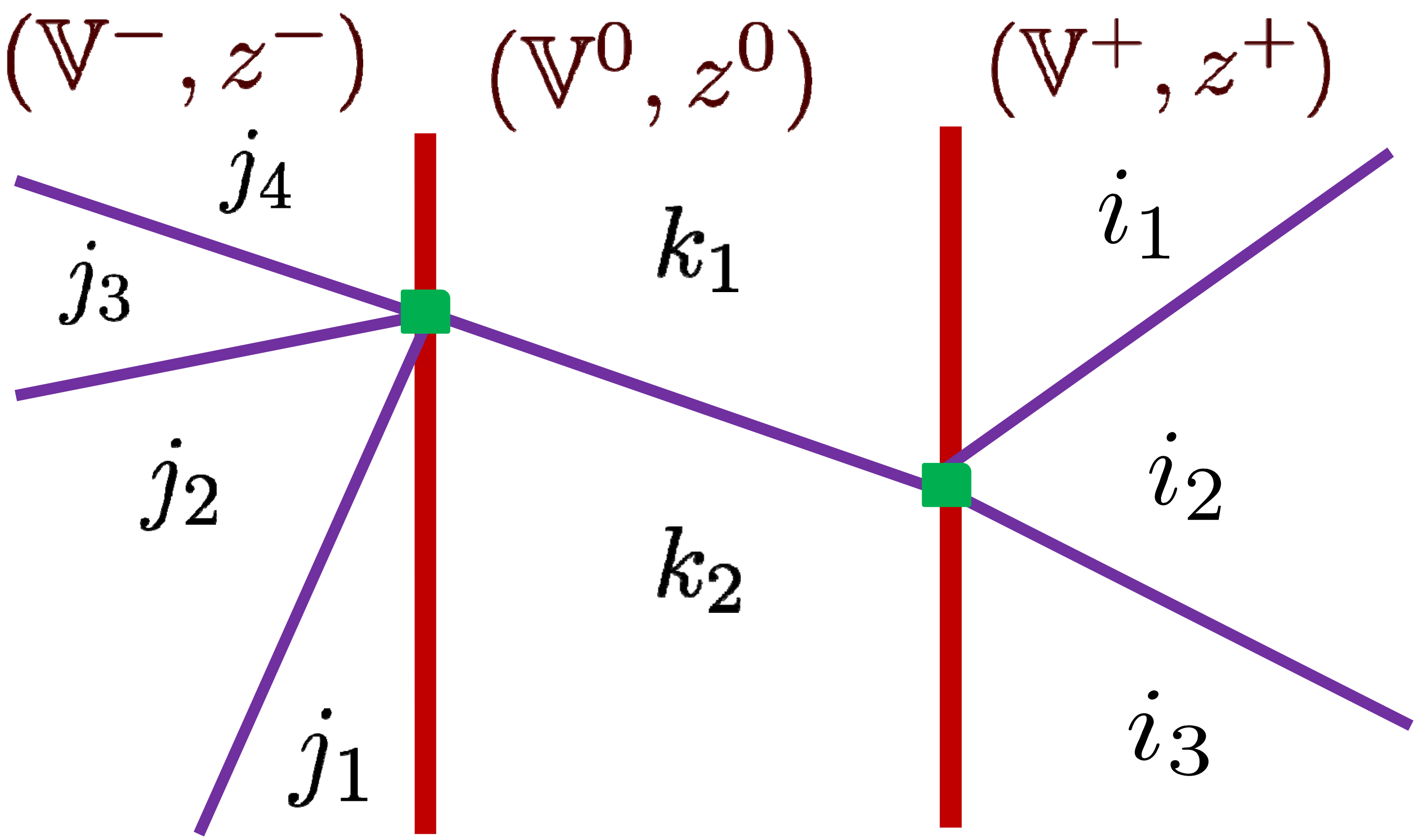
Special case: ``maps" branes in theory $\mathcal{T}^0$ to branes in theory $\mathcal{T}^+$:

$$\mathfrak{B}^0 \to \mathfrak{B}^+ := \mathfrak{B}^0 \bigstar \mathfrak{I}^{0,+}$$

Technique: Composite webs

Given data $(\mathbb{V}, z, R, K, \beta)^{-,0,+}$

Introduce a notion of ``composite webs''



Def: Composition of Interfaces

A convolution identity implies:

$$\rho(\mathfrak{t}^{-,0,+})\left[\frac{1}{1-\mathcal{B}^{-,0}}, \frac{1}{1-\mathcal{B}^{0,+}}; e^{\beta}\right]$$

Interface amplitude

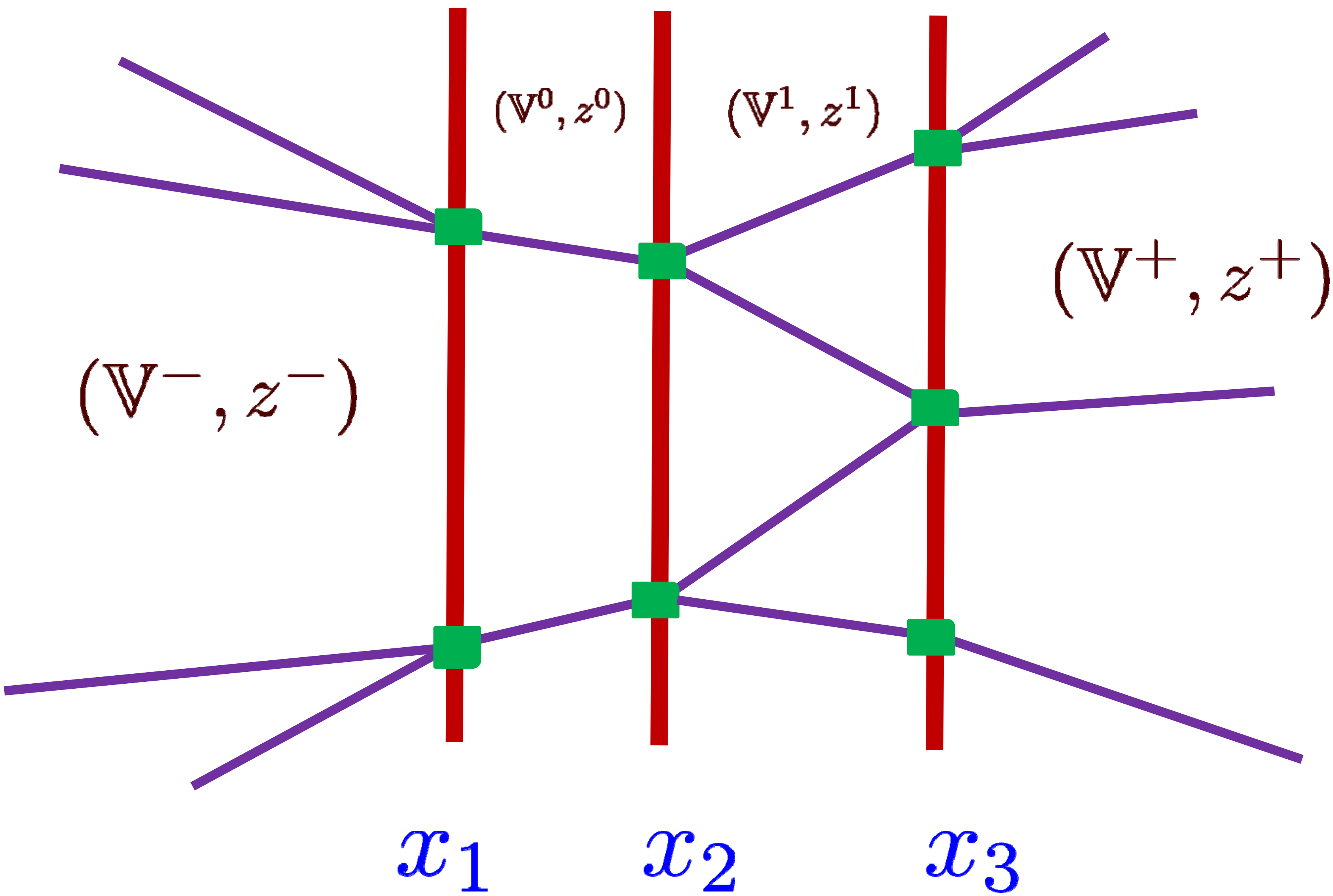
$$\mathcal{E}(\mathfrak{I}^{-,0} \bigstar \mathfrak{I}^{0,+}) = \oplus_{j^0} \mathcal{E}(\mathfrak{I}^{-,0})_{i^- j^0} \otimes \mathcal{E}(\mathfrak{I}^{0,+})_{j^0 k^+}$$

$$\mathfrak{Br}^{-,0} \times \mathfrak{Br}^{0,+} \to \mathfrak{Br}^{-,+}$$

Physically: An OPE of susy Interfaces

Theorem: The product is an A_∞ bifunctor

Associativity?



Homotopy Equivalence

(Standard homological algebra)

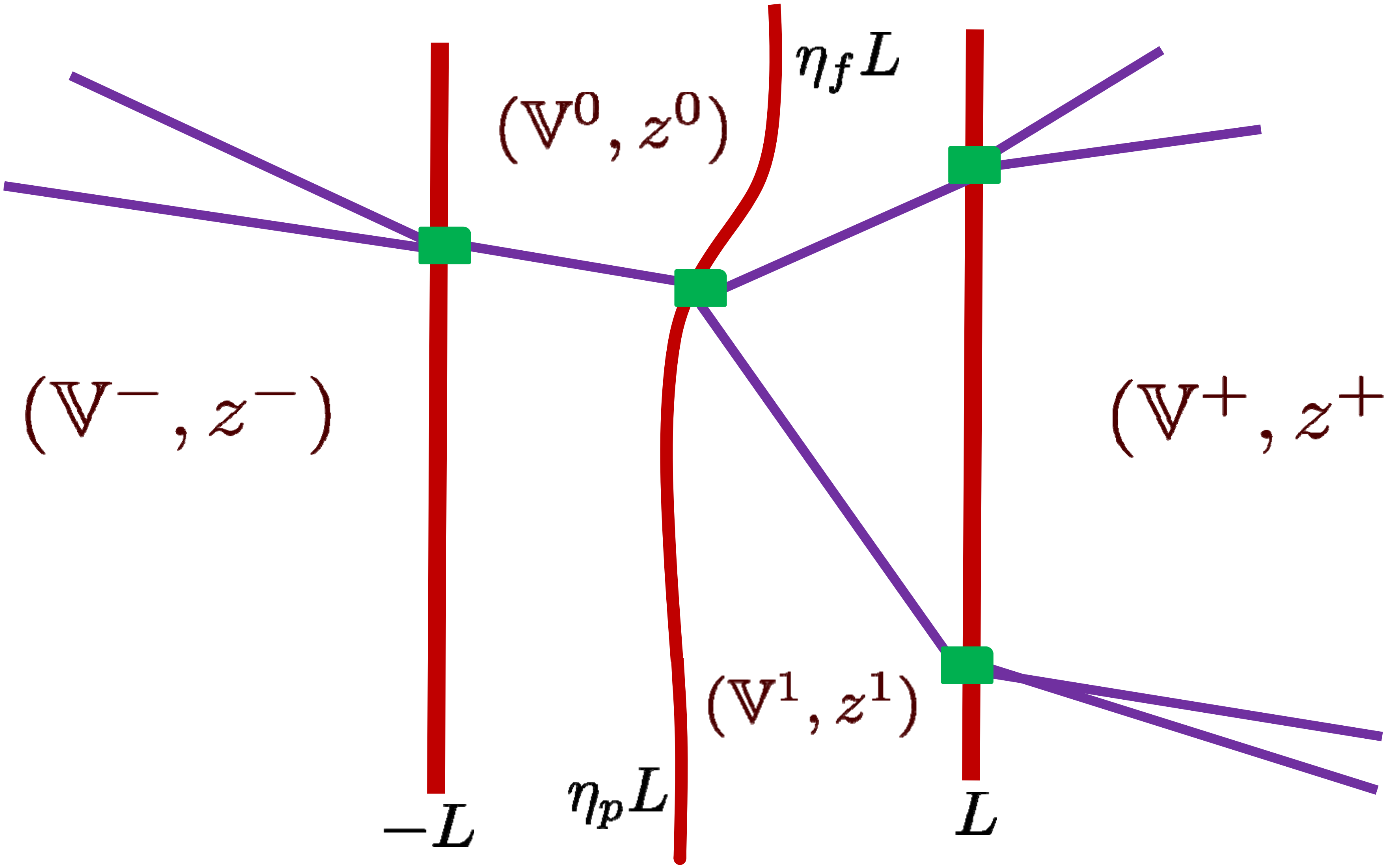
$$\delta_1, \delta_2 \in \mathrm{Hom}(\mathfrak{I}, \mathfrak{I}')$$

$$\delta_1 \sim \delta_2 \quad \Longleftrightarrow \quad \delta_1 - \delta_2 = M_1(\delta_3)$$

$$\mathfrak{I} \sim \mathfrak{I}' \quad \Longleftrightarrow \quad \begin{array}{l} M_2(\delta', \delta) \sim \mathrm{Id} \\ M_2(\delta, \delta') \sim \mathrm{Id} \end{array}$$

$$\mathfrak{Br}^{-,0} \times \mathfrak{Br}^{0,1} \times \mathfrak{Br}^{1,+} \to \mathfrak{Br}^{-,+}$$

Product is associative up to homotopy equivalence



Webology: Deformation type, taut element, convolution identity, …

An A_∞ 2-category

Objects, or 0-cells are Theories:

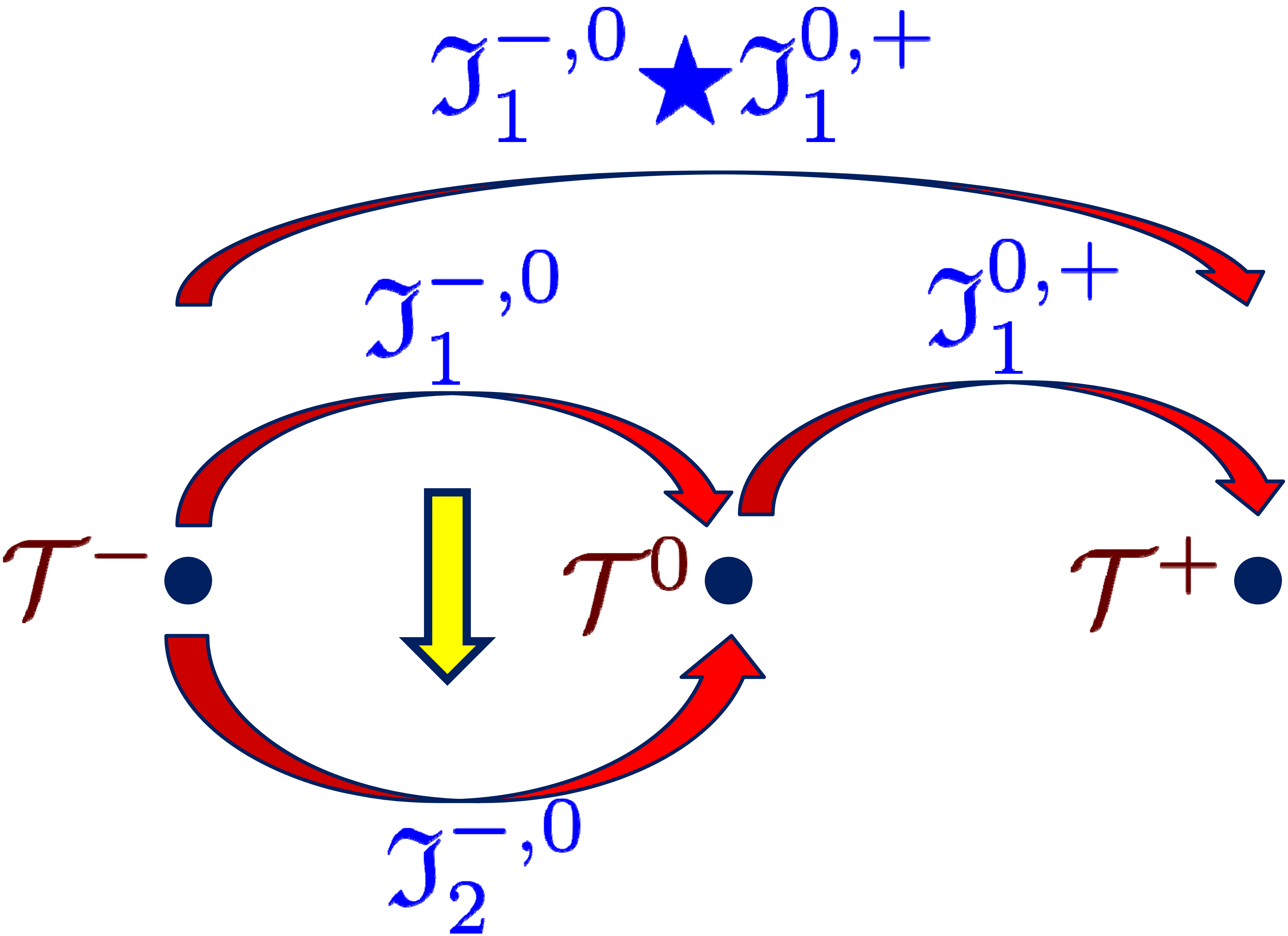
$$\mathcal{T} = (\mathbb{V}, z, R, K, \beta)$$

1-Morphisms, or 1-cells are objects in the category of Interfaces:

$$\mathfrak{I} \in \mathfrak{Br}(\mathcal{T}^{-}, \mathcal{T}^{+})$$

2-Morphisms, or 2-cells are morphisms in the category of Interfaces:

$$\delta \in \mathrm{Hop}(\mathfrak{I}_1^{-,+}, \mathfrak{I}_2^{-,+})$$



$\mathfrak{I}_1^{-,0} \bigstar \mathfrak{I}_1^{0,+}$
$\mathfrak{I}_1^{-,0}$
$\mathfrak{I}_1^{0,+}$
$\mathcal{T}^-$
$\mathcal{T}^0$
$\mathcal{T}^+$
$\mathfrak{I}_2^{-,0}$

Parallel Transport of Categories

For any continuous path:

$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$

we ***want*** to associate an A∞ functor:

$$\mathbb{F}[\wp] : \mathfrak{Br}(\mathcal{T}^{\rm in}) \to \mathfrak{Br}(\mathcal{T}^{\rm out})$$

$$\mathbb{F}[\wp_1 \circ \wp_2] \cong \mathbb{F}[\wp_1] \circ \mathbb{F}[\wp_2]$$

$$\wp \sim \wp' \quad \Longrightarrow \quad \tau : \mathbb{F}[\wp] \cong \mathbb{F}[\wp']$$

Interface-Induced Transport

Idea is to induce it via a suitable Interface:

$$\mathbb{F}[\wp] : \mathfrak{B}^{\mathrm{in}} \to \mathfrak{B}^{\mathrm{in}} \bigstar \mathfrak{I}^{\mathrm{in,out}}$$

But how do we construct the Interface?

Example: Spinning Weights

$$z_i(x) = e^{\mathrm{i}\vartheta(x)} z_i$$

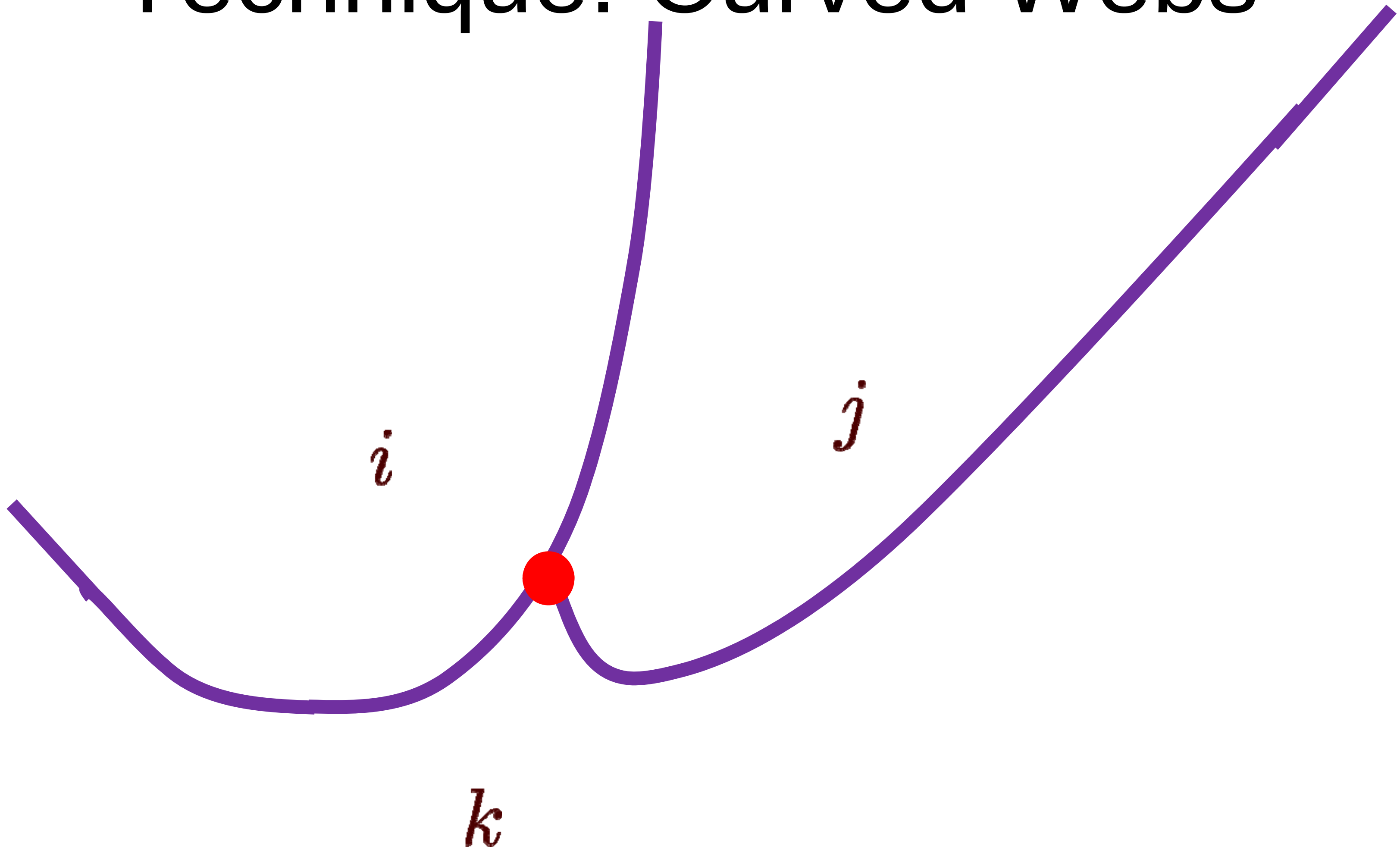
$$(\mathbb{V}, R, K, \beta) \quad \text{constant}$$

We can construct explicitly: $\mathfrak{I}[\vartheta(x)]$

$$\vartheta_1(x) \sim \vartheta_2(x) \Longrightarrow \mathfrak{I}[\vartheta_1(x)] \sim \mathfrak{I}[\vartheta_2(x)]$$

$$\mathfrak{I}[\vartheta_1 \circ \vartheta_2] \sim \mathfrak{I}[\vartheta_1] \bigstar \mathfrak{I}[\vartheta_2]$$

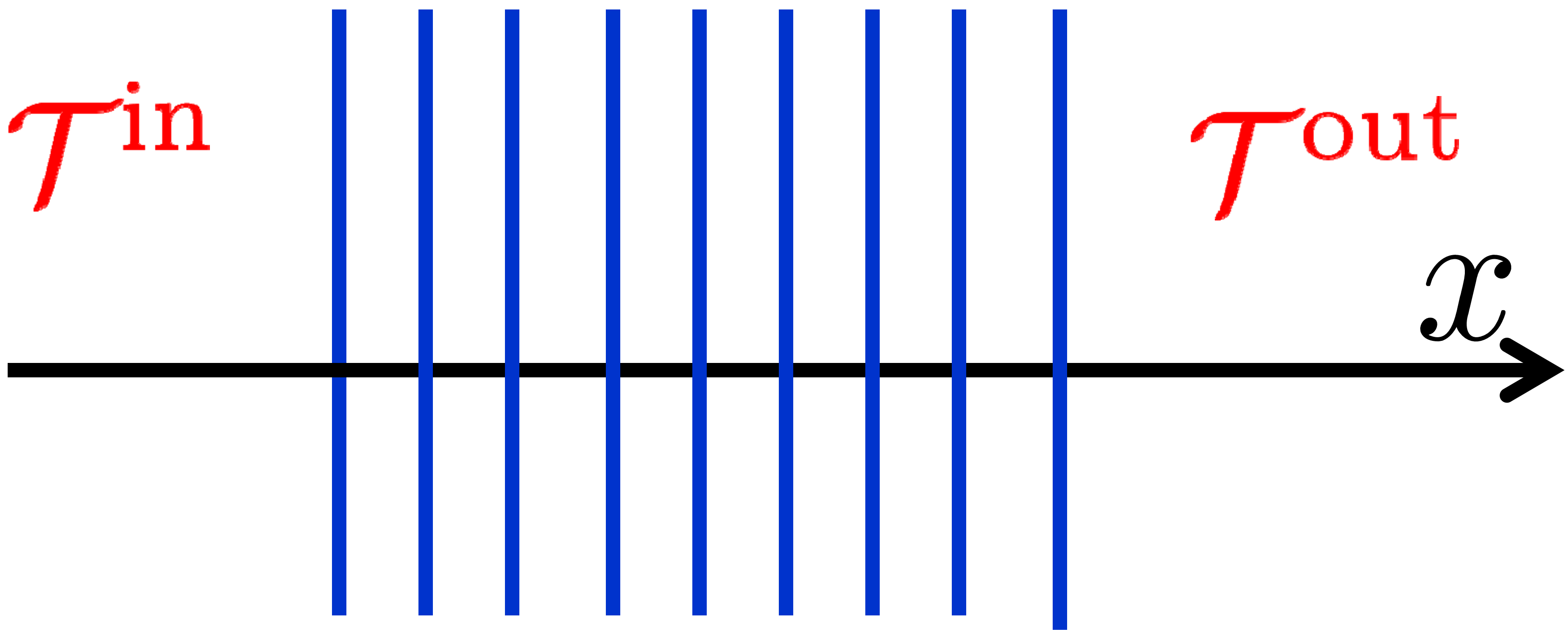
Technique: Curved Webs



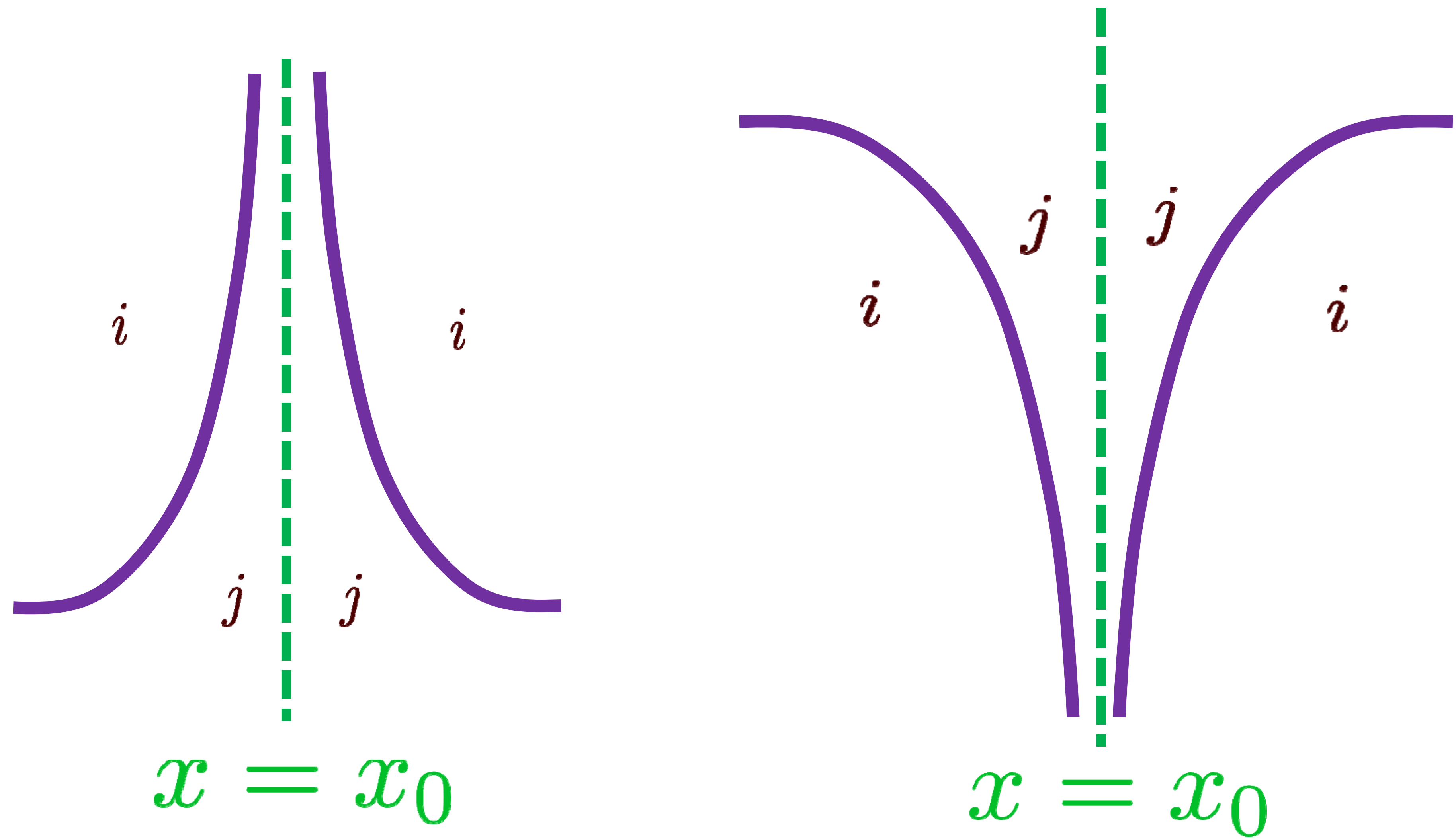
Webology: Deformation type, taut element, convolution identity, …

Reduction to Elementary Interfaces:

The Interface is trivial except as some special “binding points”



$$\mathfrak{I}^{\rm in,out} := \mathfrak{I}_1 \star \cdots \star \mathfrak{I}_n$$



Future stable   Past stable

CP-Factors for $\mathfrak{I}[\vartheta(x)]$

$$\oplus_{j,j' \in \mathbb{V}} \mathcal{E}_{j,j'} e_{j,j'} =$$

$$= \bigotimes_{i \neq j} \bigotimes_{x_0 \in \curlyvee_{ij} \cup \curlywedge_{ij}} S_{ij}(x_0)$$

$$S_{ij}(x_0) = \mathbb{Z}\mathbf{1} + R_{ij} e_{ij}$$ Future stable

$$S_{ij}(x_0) = \mathbb{Z}\mathbf{1} + R^{*}_{ji} e_{ij}$$ Past stable

In this way we categorify the ``detour rules'' of the nonabelianization map of spectral network theory.

General Case?

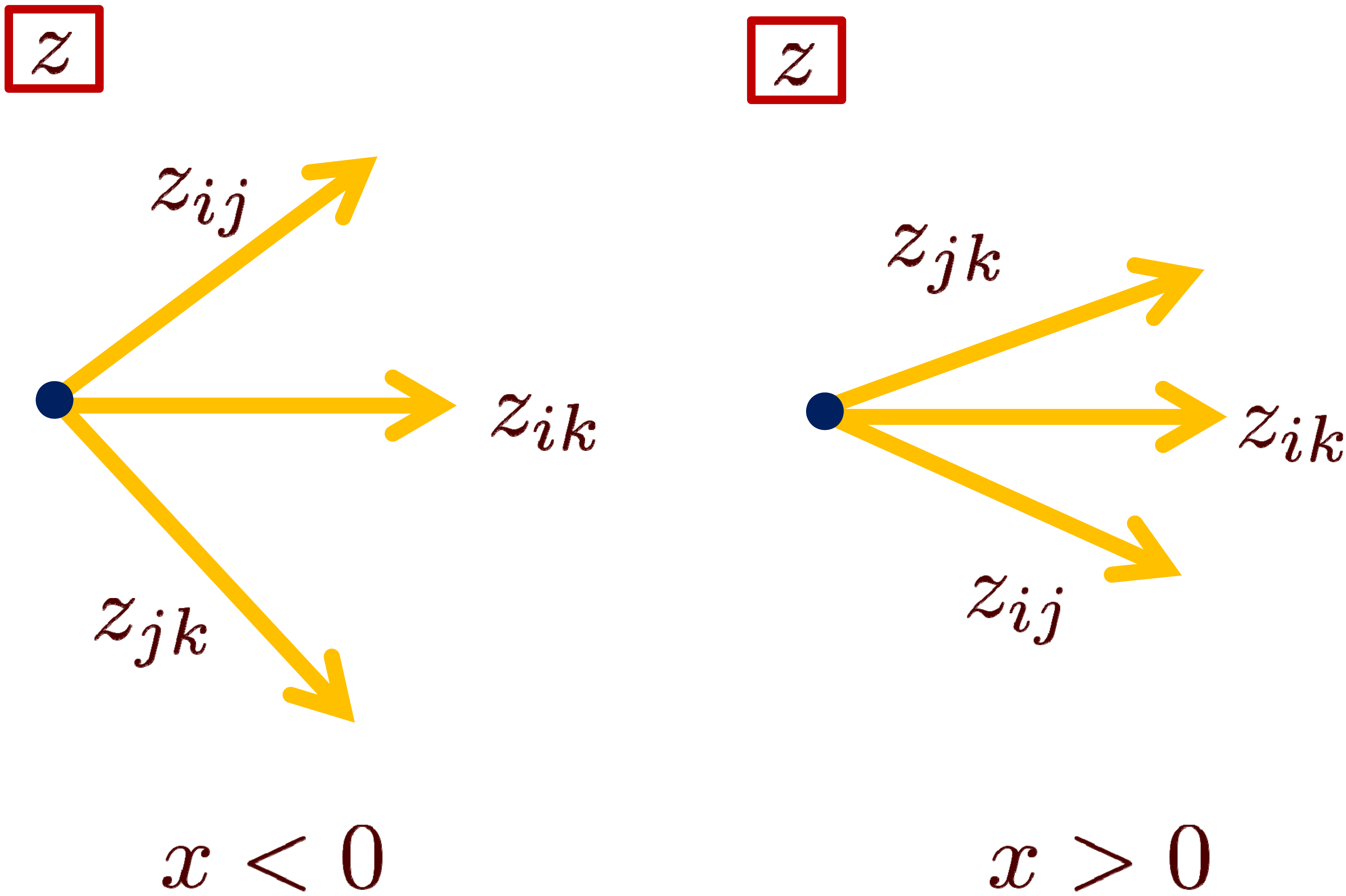
$$\wp(x) = (\mathbb{V}, z, R, K, \beta)(x)$$

To *continuous* $\wp$ we ***want*** to associate an A_∞ functor

$$\mathbb{F}[\wp] : \mathfrak{Br}(\mathcal{T}^{\mathrm{in}}) \to \mathfrak{Br}(\mathcal{T}^{\mathrm{out}})$$

etc.

You can't do that for arbitrary $\wp(x)$!



z
z_{ij}
z_{ik}
z_{jk}
$x < 0$
z
z_{jk}
z_{ik}
z_{ij}
$x > 0$

Categorified Cecotti-Vafa Wall-Crossing

We cannot construct $\mathbb{F}[\wp]$ keeping β and R_{ij} constant!

Existence of suitable Interfaces needed for flat transport of Brane categories implies that the web representation jumps discontinuously:

$$R_{ik}^{\mathrm{out}} - R_{ik}^{\mathrm{in}} =$$

$$(R_{ij}^{+} - R_{ij}^{-}) \otimes \left(R_{jk}^{+} - R_{jk}^{-}\right)$$

Categorified Wall-Crossing

In general: the existence of suitable wall-crossing Interfaces needed to construct a flat parallel transport F[$\wp$] demands that for certain paths of vacuum weights the web representations (and interior amplitude) must jump discontinuously.

Moreover, the existence of wall-crossing interfaces constrains how these data must jump.

Outline

- Introduction, Motivation, & Results
- Morse theory and LG models: The SQM approach
- Boosted solitons and ζ-webs
- Webs and their representations: L_∞
- Half-plane webs & Branes: A_∞
- Interfaces & Parallel Transport of Brane Categories
- Summary & Outlook

Summary

1. Motivated by 1+1 QFT we constructed a web-based formalism

2. This naturally leads to L_∞ and A_∞ structures.

3. It gives a natural framework to discuss Brane categories and Interfaces and the 2-category structure

4. There is a notion of flat parallel transport of Brane categories. The existence of such a transport implies categorified wall-crossing formulae

Other Things We Did

1. Detailed examples ($\mathbb{Z}_N$ symmetric theories)

2. There are several interesting generalizations of the web-based formalism, not alluded to here. (Example: Colliding critical points.)

3. The web-based formalism also allows one to discuss bulk and boundary local operators in the TFT.

4. Applications to knot homology

Outlook

We need a better physical interpretation of the interaction amplitudes β_l

The generalization of the categorified 2d-4d wall-crossing formula remains to be understood. (WIP: with Tudor Dimofte)