The Shapes Of Spaces and the Nuclear Force

1. **Physical Mathematics**
2. Topological Invariants Of Spaces
3. Physics: Maxwell & Yang-Mills
4. The Mathematicians Take Note
5. Topological Field Theory
6. Effective Theories & A Breakthrough
7. Settling A Debate & A Summary

Phys-i-cal Math-e-ma-tics, n.

Pronunciation: Brit. /ˈfɪzɨkl ˌmaθ(ə)ˈmatɪks / , U.S. /ˈfɪzək(ə)l ˌmæθ(ə)ˈmædɪks/

Physical mathematics is a fusion of mathematical and physical ideas, motivated by the dual, but equally central, goals of

1. Elucidating the laws of nature at their most fundamental level,

together with

2. Discovering deep mathematical truths.

Kepler

Galileo

Snapshots from the

Great Debate

over

the relation between

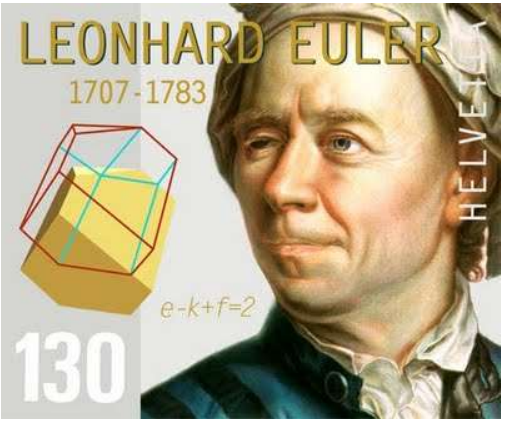
Mathematics and Physics

Newton

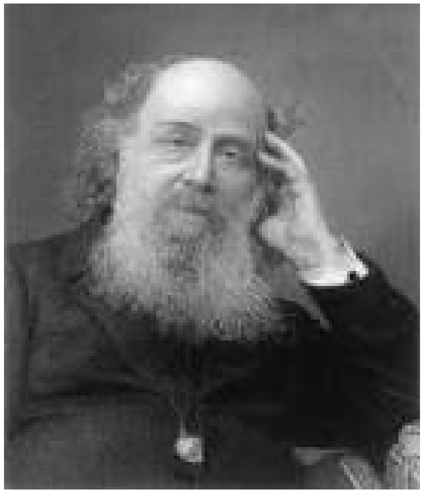
Leibniz

When did Natural Philosophers become either Physicists or Mathematicians?

Even around the turn of the 19th century …



The separation is a result of specialization and the growth of science in the 19th century.



1869: Sylvester's Challenge

A pure mathematician speaks:

of physical philosophy; the one here in print," says Professor Sylvester, "is an attempted faint adumbration of the nature of mathematical science in the abstract. What is wanting (like a fourth sphere resting on three others in contact) to build up the ideal pyramid is a discourse on the relation of the two branches (mathematics and physics) to, and their action and reaction upon, one another—a magnificent theme, with which it is to be hoped that some future president of Section A will crown the edifice, and make the tetralogy (symbolisable by $A + A'$, A, A', AA') complete."

1870: Maxwell's Answer

An undoubted physicist responds,

SECTIONAL PROCEEDINGS

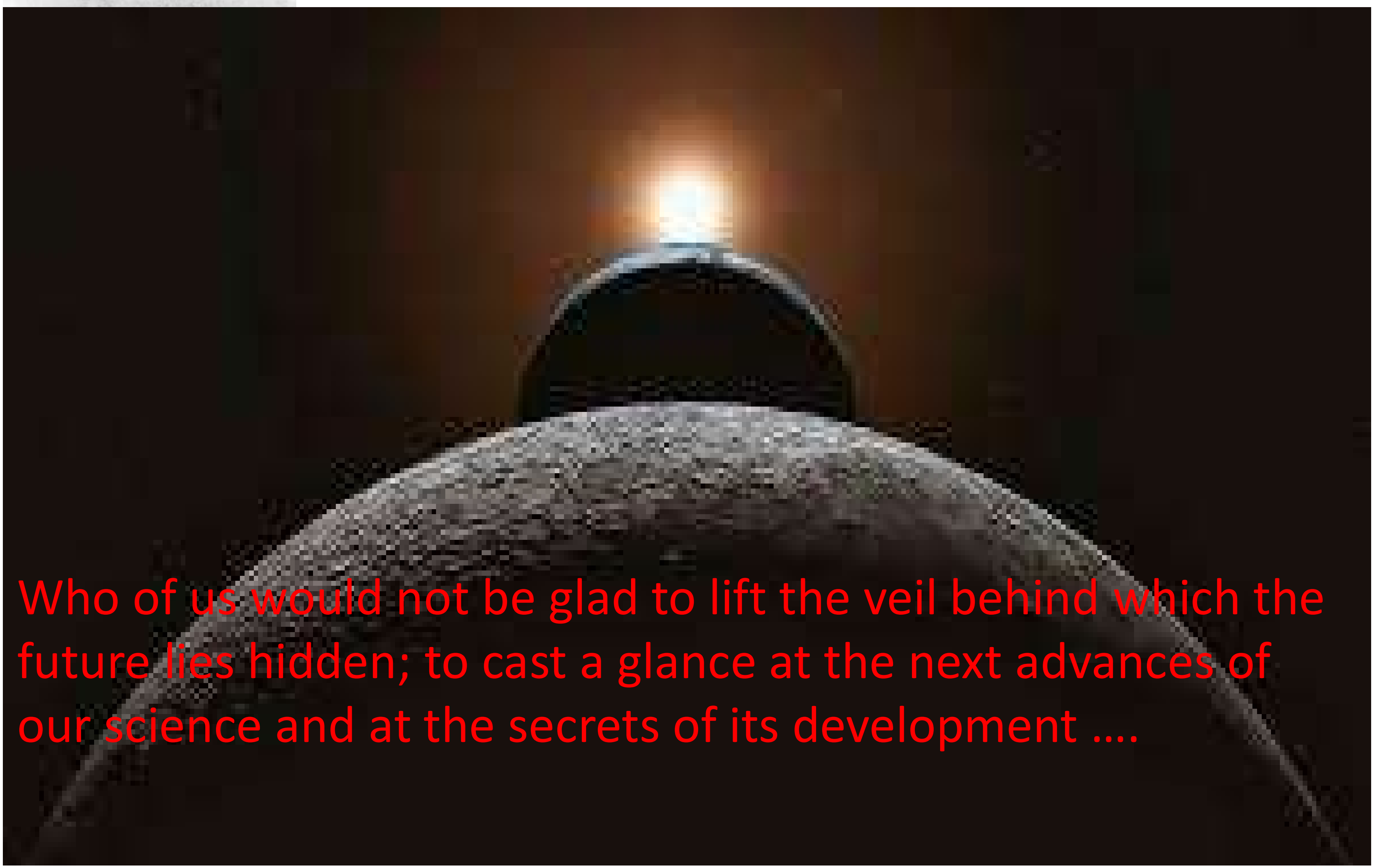
SECTION A.—*Mathematical and Physical Science.*—President, Prof. J. Clerk Maxwell, F.R.S.

The president delivered the following address :—

Maxwell recommends his somewhat-neglected dynamical theory of the electromagnetic field to the mathematical community:

phenomena must be studied in order to be appreciated. Another theory of electricity which I prefer denies action at a distance and attributes electric action to tensions and pressures in an all-pervading medium, these stresses being the same in kind with those familiar to engineers, and the medium being identical with that in which light is supposed to be propagated."

1900: The Second ICM

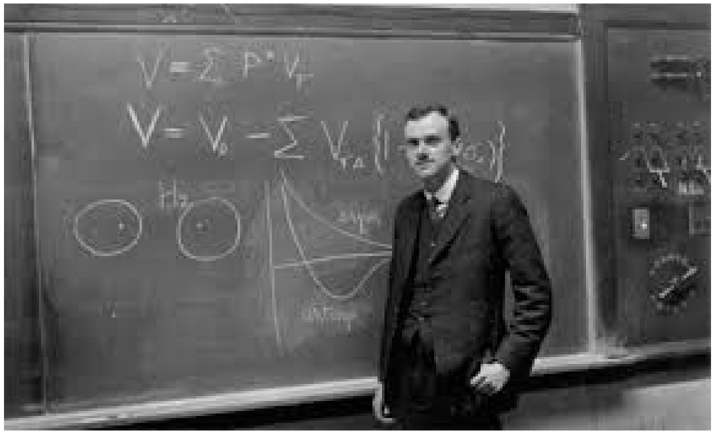


1900: Hilbert's 6th Problem

To treat […] by means of axioms, those physical sciences in which mathematics plays an important part […]

October 7, 1900: Planck's formula, leading to h.

Prerequisite: 750:502 Quantum Mechanics, or equivalent. Lorentz group; relativistic wave-equations; second quantization; global and local symmetries; QED and gauge invariance; spontaneous symmetry breaking; nonabelian gauge theories; Standard Model; Feynman diagrams; cross sections, decay rates; renormalization group.



1931: Dirac’s Paper on Monopoles

Quantised Singularities in the Electromagnetic Field

P.A.M. Dirac

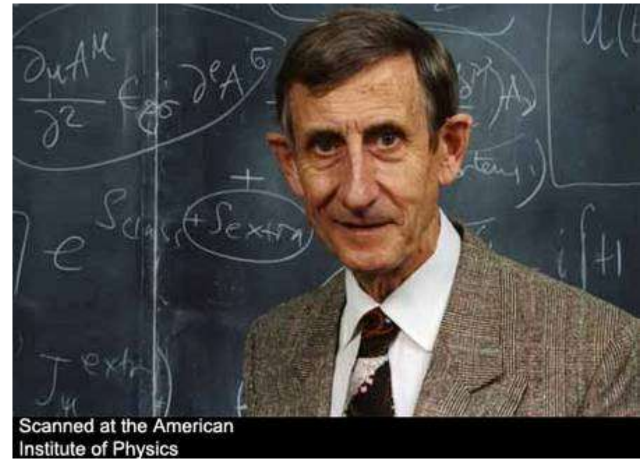
Received May 29, 1931

§ 1. Introduction

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers

for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a

1972: Dyson's Announcement



MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.
JACQUES HADAMARD

1. **Introduction.** The purpose of the Gibbs lectures is officially defined as "to enable the public and the academic community to become aware of the contribution that mathematics is making to present-day thinking and to modern civilization." This puts me in a difficult position. I happen to be a physicist who started life as a mathematician. As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce. Discussing this divorce, the

Well, I am happy to report that Mathematics and Physics have remarried!

But, the relationship has altered somewhat…

A sea change began in the 1970's …..

Some great mathematicians got interested in aspects of fundamental physics .....

While some great physicists started producing results requiring ever increasing mathematical sophistication, ...

Physical Mathematics

In the past few decades a new field has emerged
with its own distinctive character, its own aims and values,
its own standards of proof.

One of the guiding principles is certainly the discovery
of the ultimate foundations of physics.

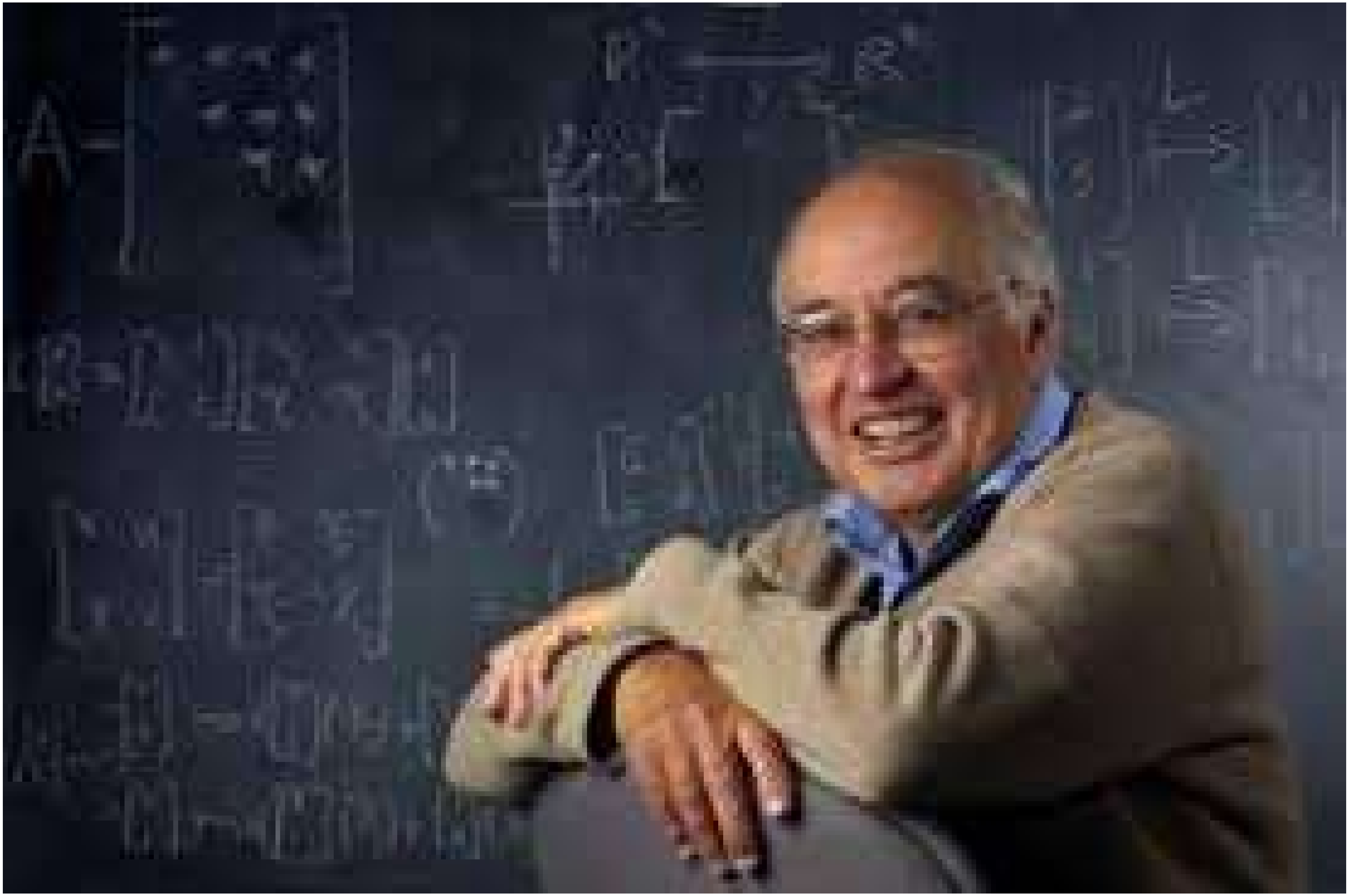
This quest has led to ever more sophisticated mathematics…

A second guiding principle is that physical insights can lead to
surprising and new results in mathematics

Such insights are a great success - just as profound and
notable as an experimental confirmation
of a theoretical prediction.

Today:

I will explain just one beautiful example of a remarkable convergence of physical and mathematical ideas.



Two Basic Questions

MATHEMATICS

What is the shape of a space?

How can we tell when two geometric objects can be deformed into each other ?

PHYSICS

What holds stuff together?

How can we describe the forces that attract and repel matter?

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MATHEMATICS

What is the shape of a space?

How can we tell when two geometric objects can be deformed into each other ?

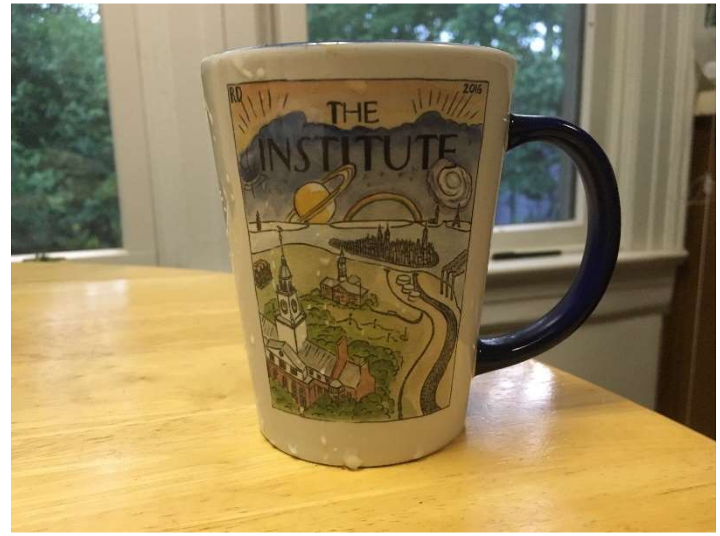
Topology

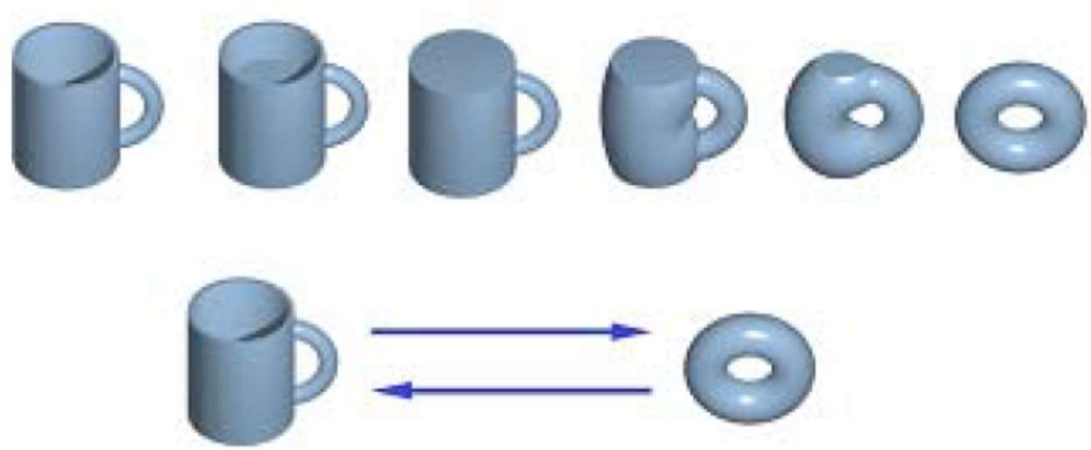
Take a geometrical object – a ``space''

You can continuously deform it :

You can squish, pull, push,

just don't cut, rip, or tear.





In topology the surface of a coffee cup and of a donut are ``***the same***.''

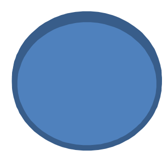
Suppose we have different spaces

– like the surface of a coffee cup, of a donut, and of a basketball –

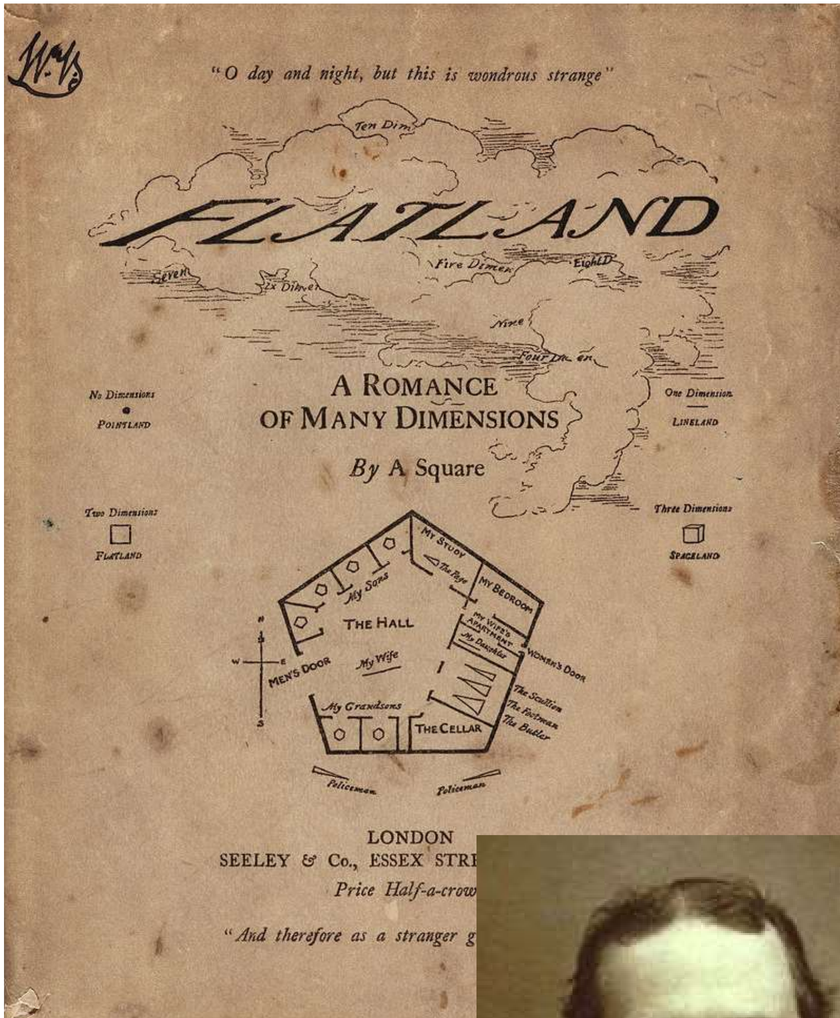
Are they ``topologically the same'' ?

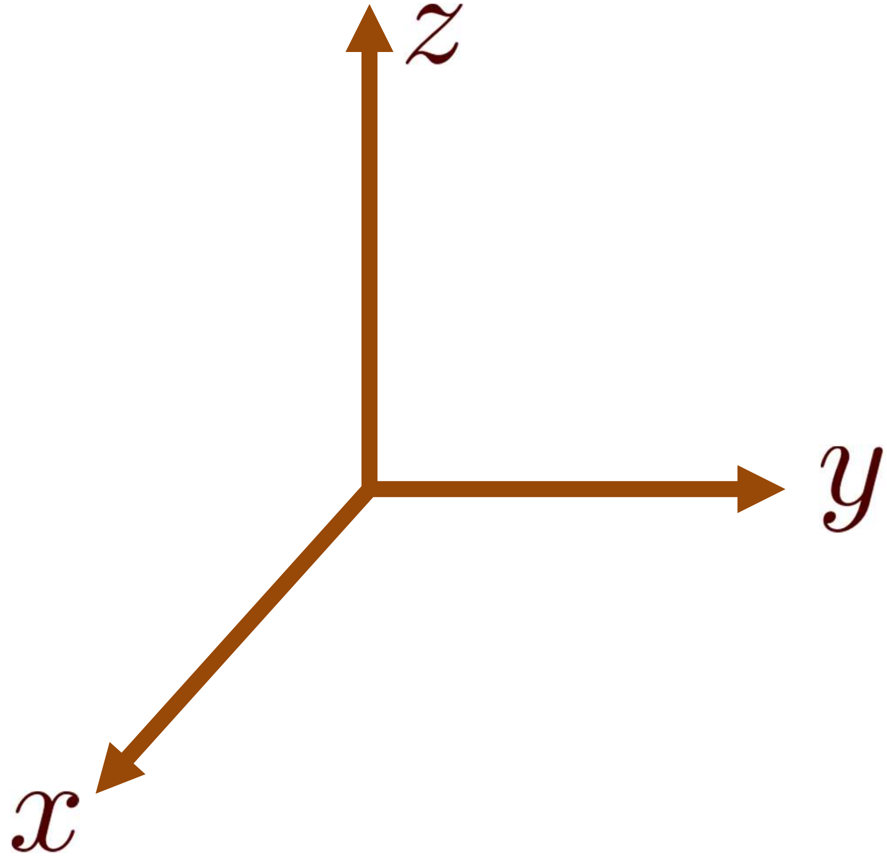
We need to find a property that does not change under deformation: ``topological invariant".

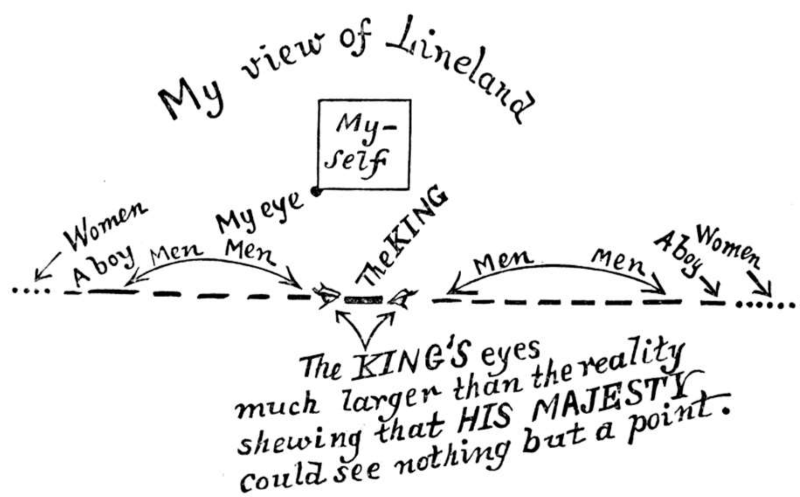
One such property is the "dimension" of a space.



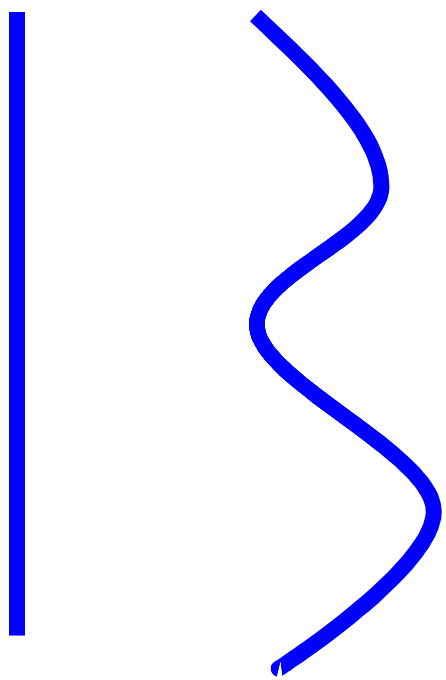
(x, y)

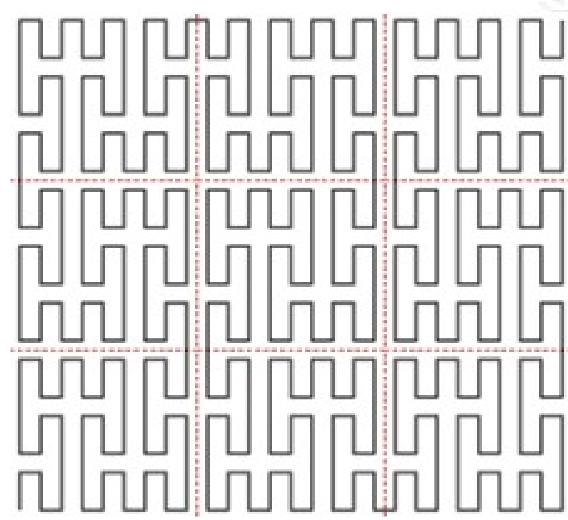

"O day and night, but this is wondrous strange"
FLATLAND
A ROMANCE
OF MANY DIMENSIONS
By A Square
No Dimensions
POINTLAND
One Dimension
LINELAND
Two Dimensions
FLATLAND
Three Dimensions
SPACELAND
MY STUDY
MY BEDROOM
My Sons
THE HALL
MEN'S DOOR
WOMEN'S DOOR
My Wife
My Grandsons
THE CELLAR
Policeman
Policeman
LONDON


z
y
x

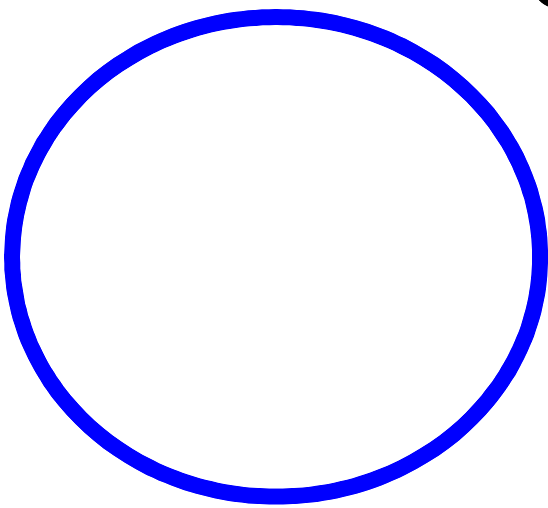

My view of Lineland
My-self
My eye
Women
A boy
Men
Men
The KING
Men
Men
A boy
Women
The KING'S eyes much larger than the reality shewing that HIS MAJESTY could see nothing but a point.

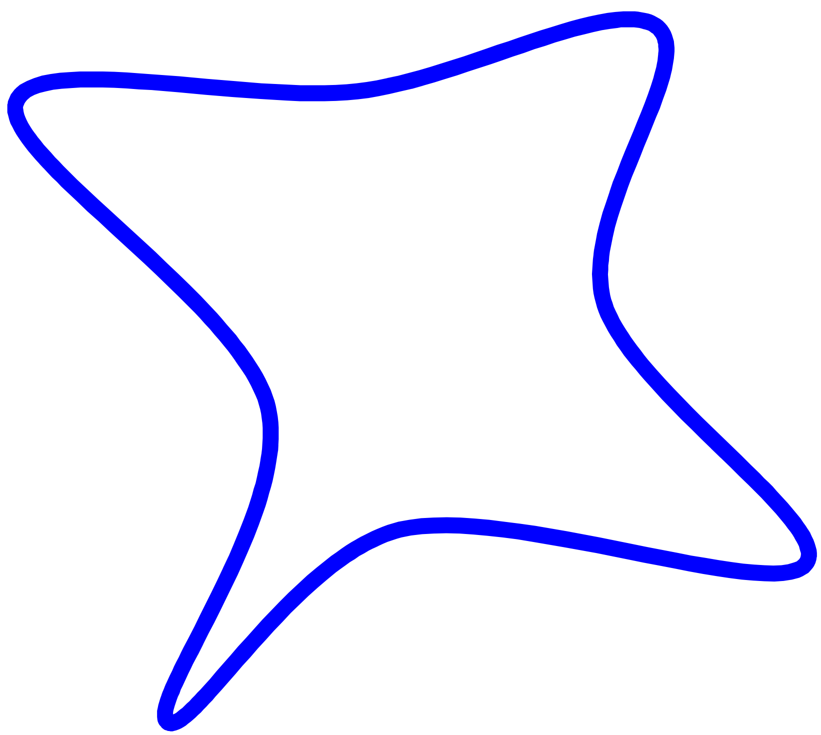
One Dimension

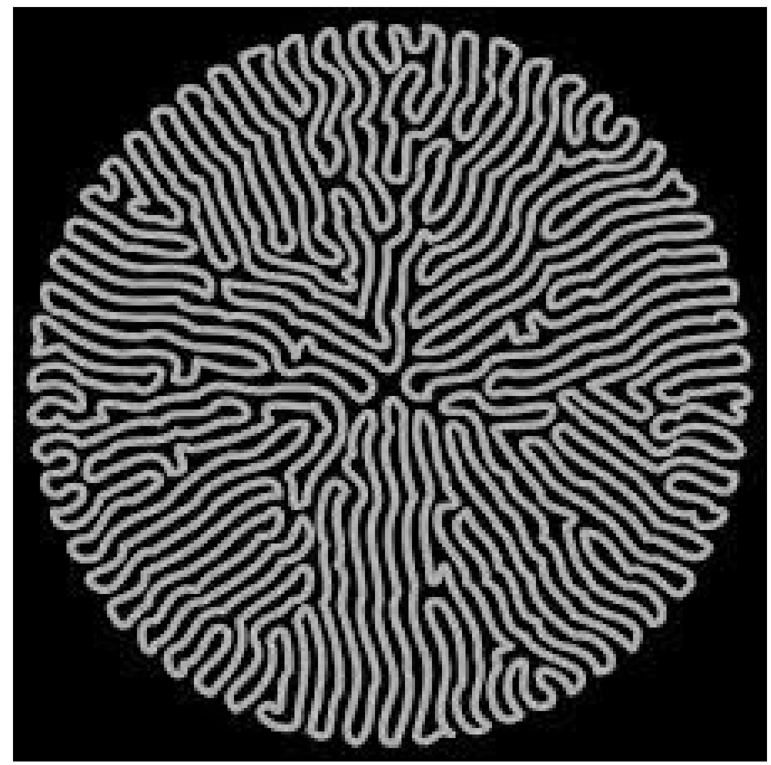




One Dimension







Topological Invariant

A ***topological invariant*** is the answer, $A(S)$, to a question you ask about a space S.

If two spaces S_1 and S_2 can be deformed into each other, i.e. if two spaces are topologically equivalent.

Then, the answers to the question must be the same: $A(S_1) = A(S_2)$

A One-Dimensional Topological Invariant

Question:

Take a little walk: Do you get back home ?

(always walk forward)

If $A(S) = \text{Yes}$

Then S is topologically a circle

If $A(S) = \text{No}$

Then S is topologically a line

A Complete Topological Invariant

Topological invariant: A question so that,

If S_1 can be deformed to S_2 ***then***:
the answers are the same $A(S_1) = A(S_2)$

Dimension is a topological invariant –
But it doesn't distinguish two different spaces:
Circle and Line are both one dimensional

A ***complete topological invariant*** is a question so that

If $A(S_1) = A(S_2)$ ***then*** S_1 can be deformed to S_2

Two Dimensions



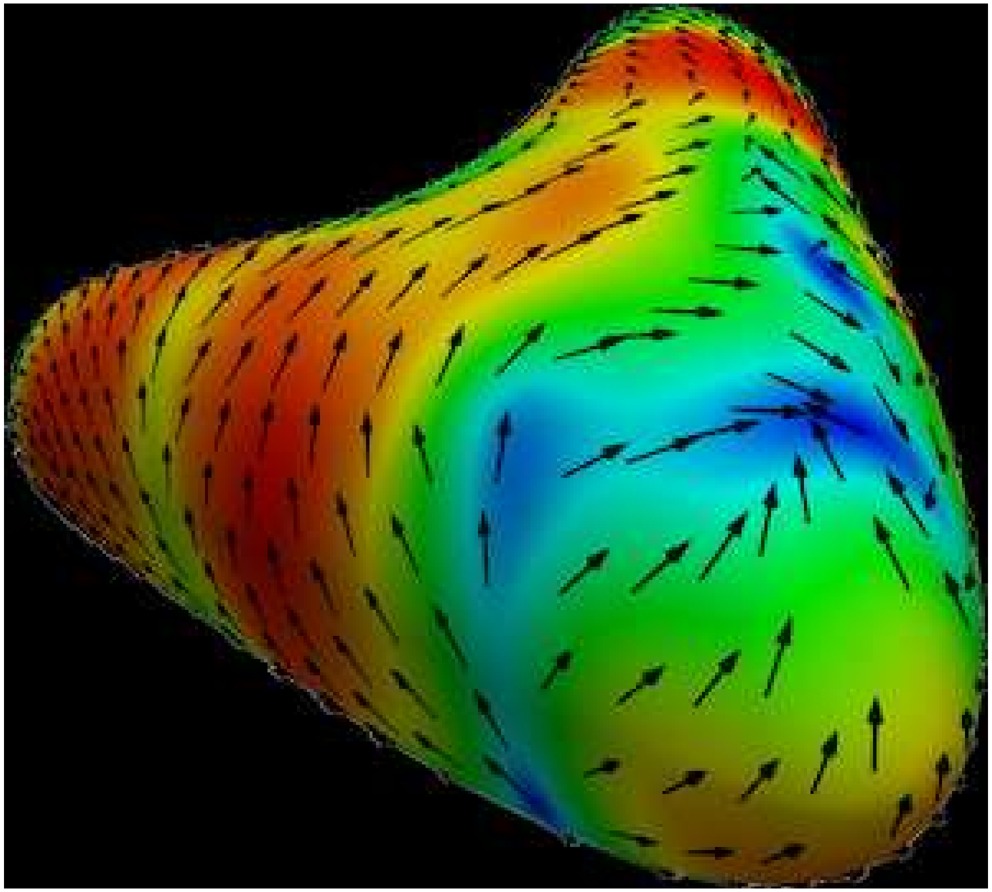

etc.

Measuring Topology By Counting

Complete topological invariant: Count holes

More than one way to count …

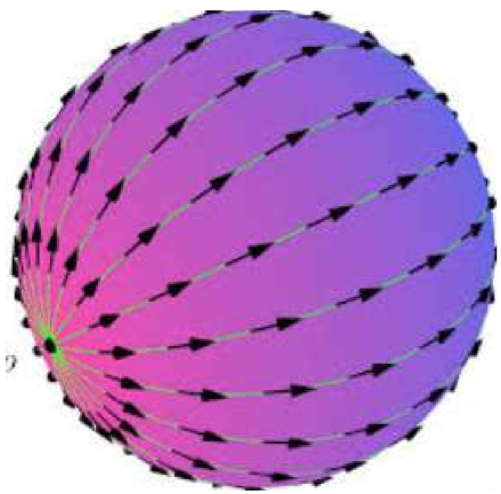
If water is flowing on the surface:

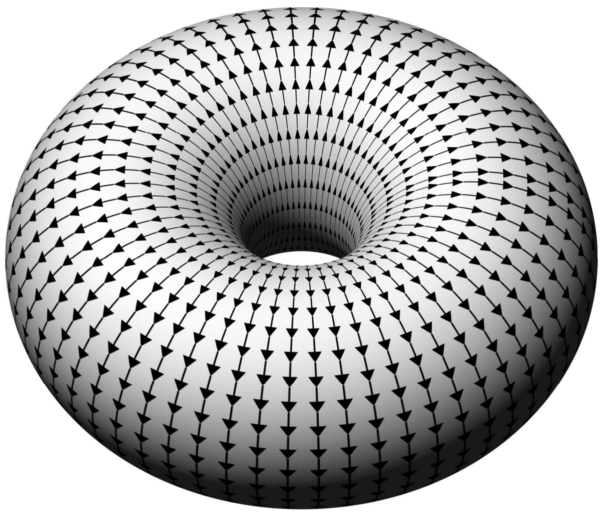


Then you COUNT (adding with ± 1) the drains and spigots for the water:

This is also a complete topological invariant.

That sum will be $2 - 2 \times (\# \, holes)$

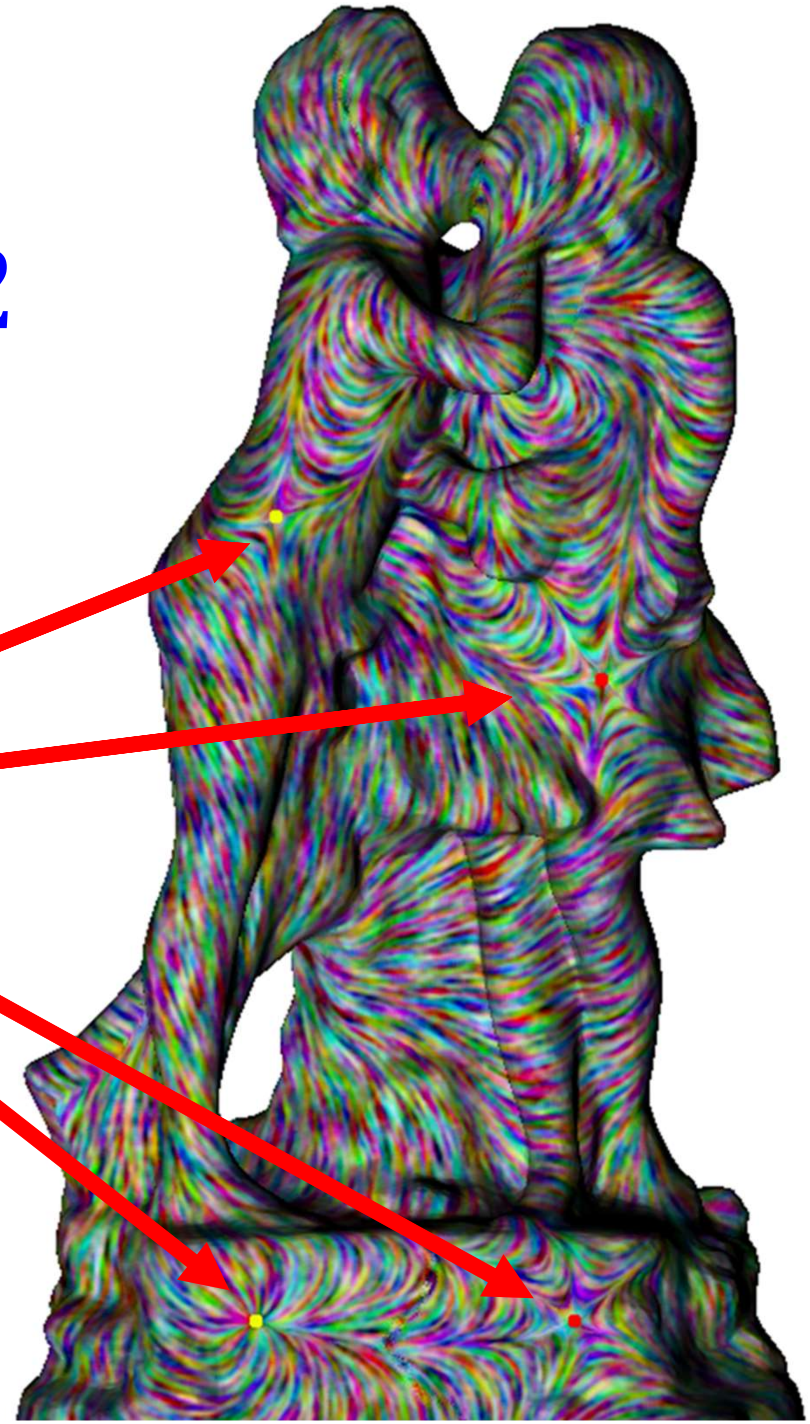




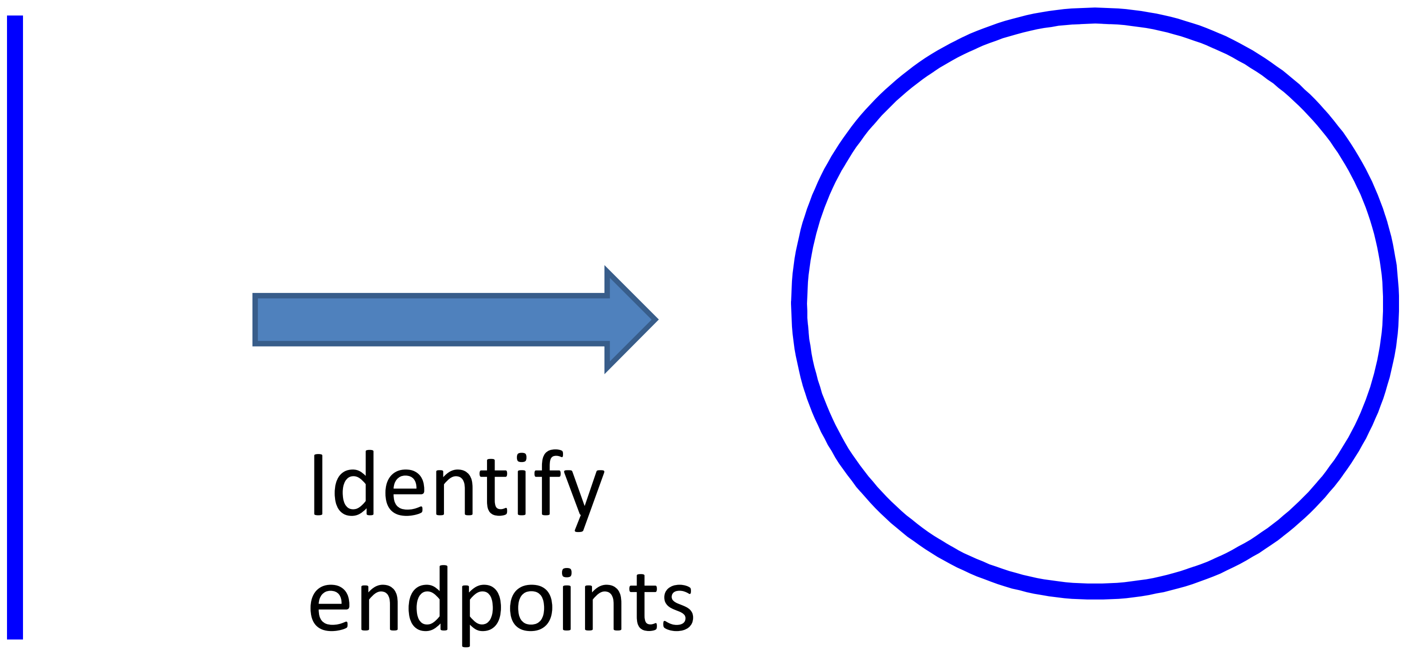
$2 = 2 - 2 \times 0$ $\qquad$ $0 = 2 - 2 \times 1$

$$-2 = 2 - 2 \times 2$$

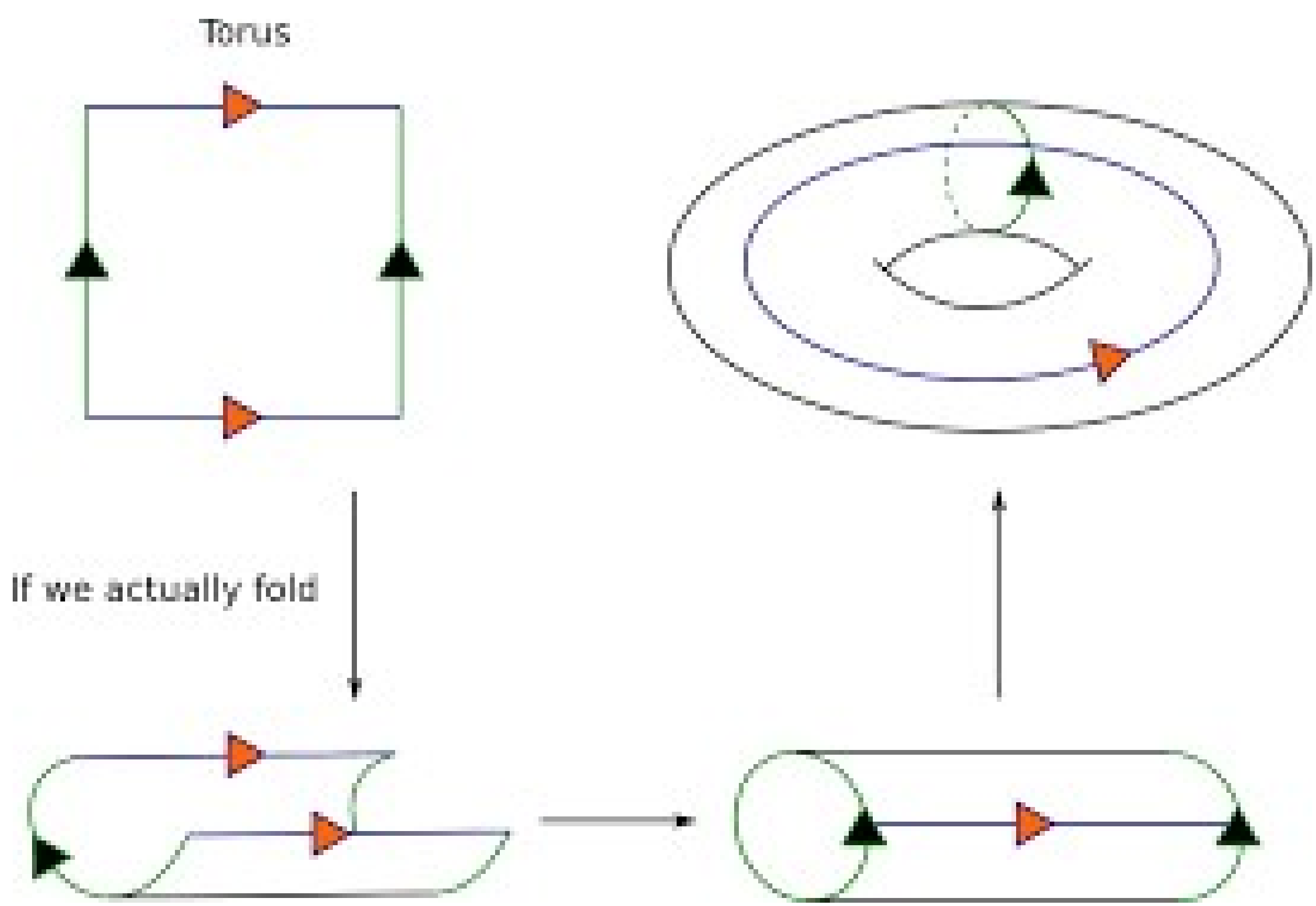
SUM OVER
DRAINS &
SPIGOTS = -2



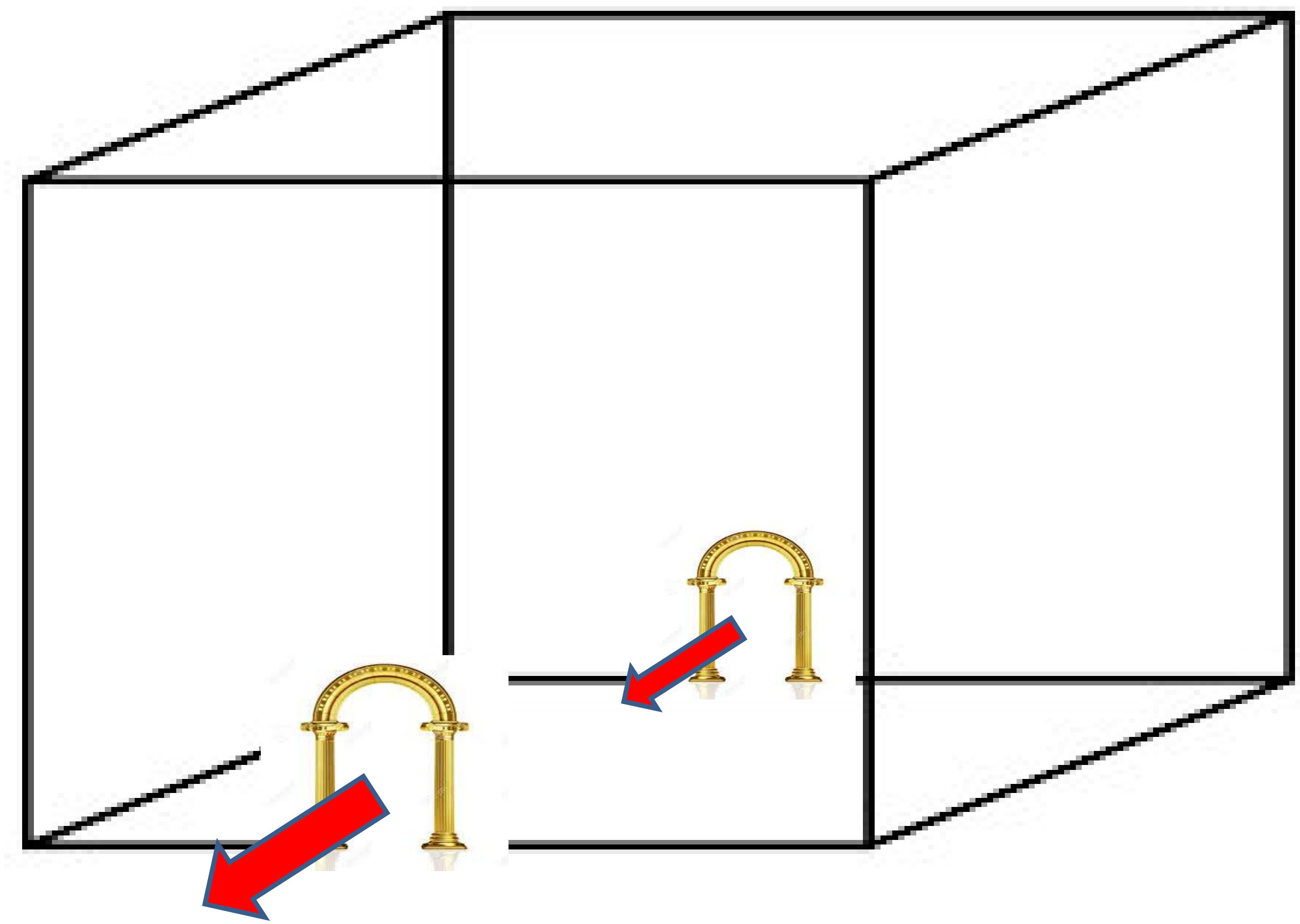
Making Spaces Abstractly: Gluing



Making Spaces Abstractly: Gluing

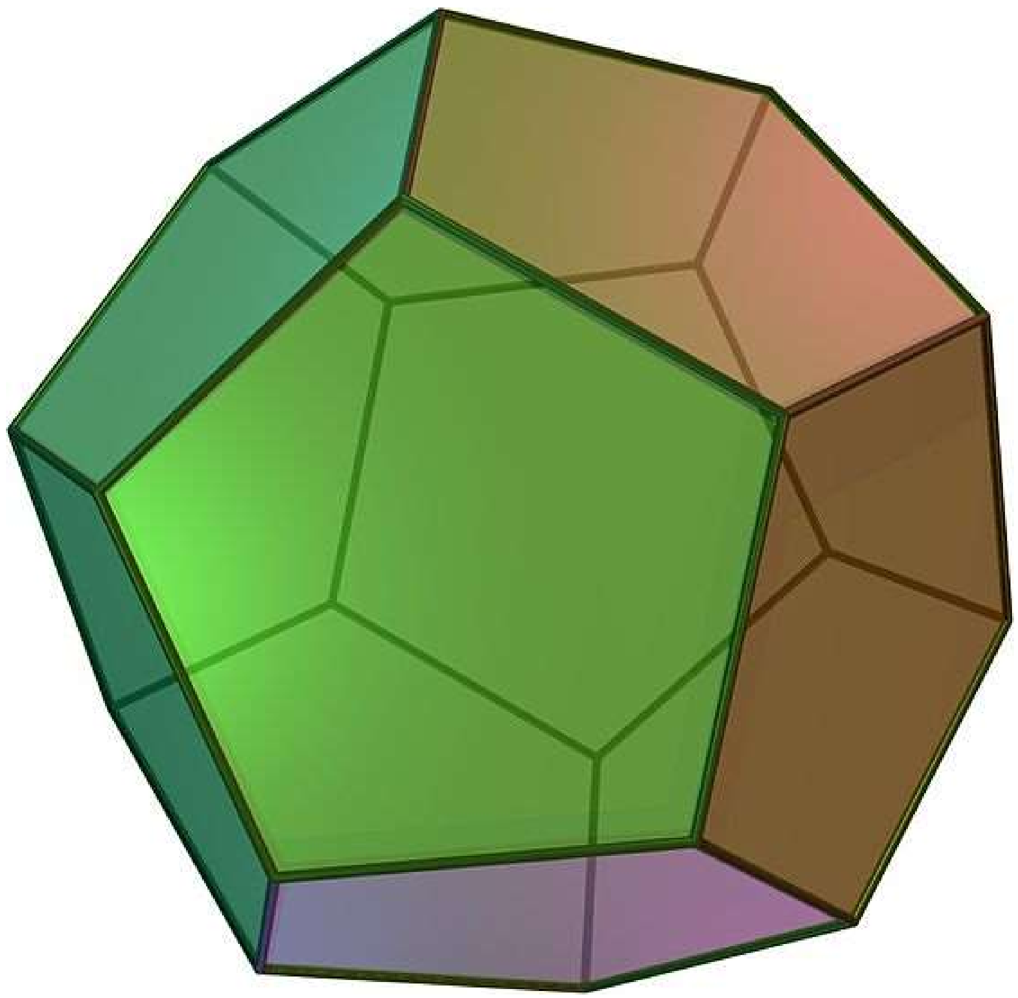


Three Dimensions



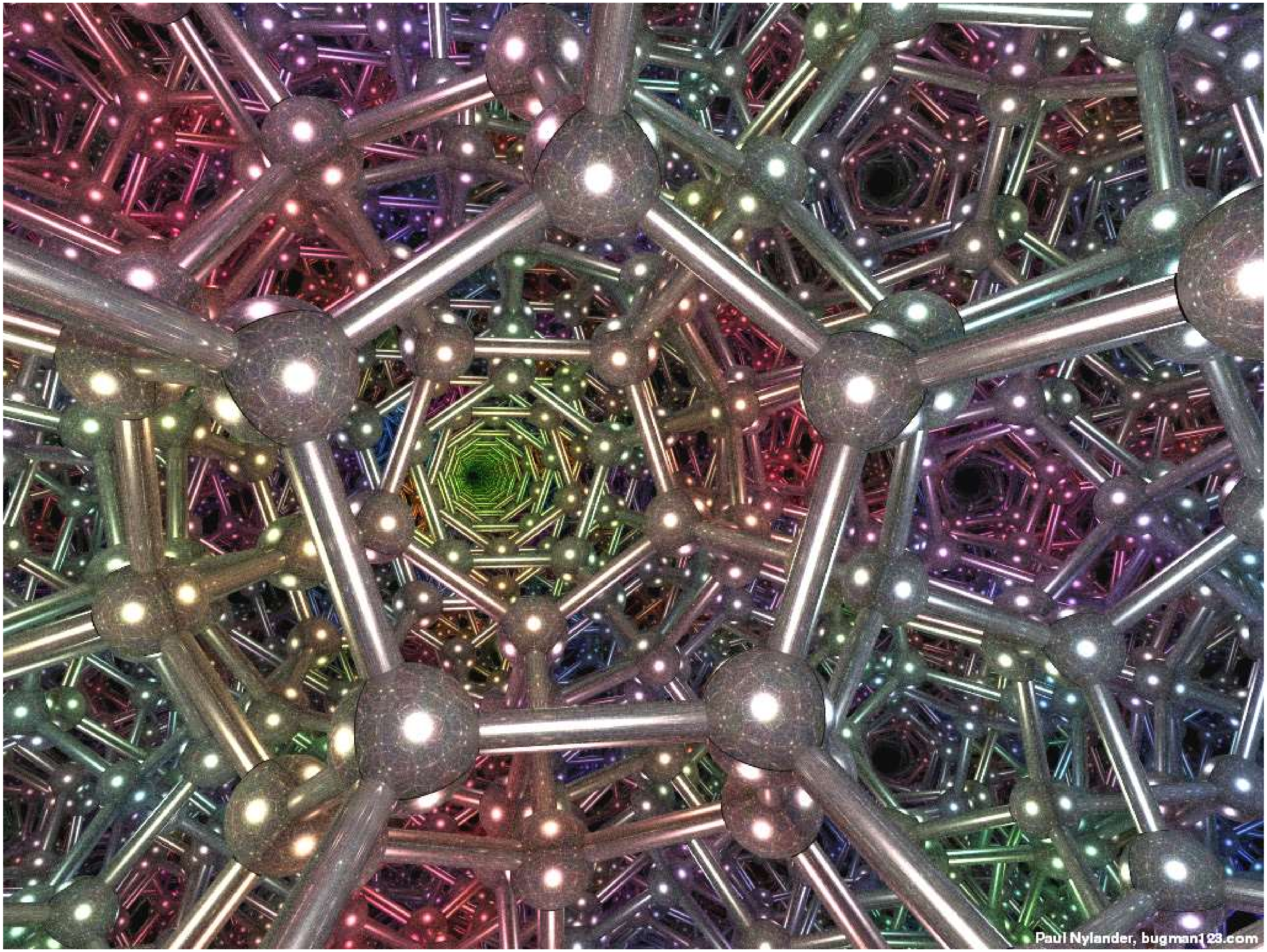
Identify opposite sides: Gives a space with no boundary.

Three Dimensions: Things Get Harder



Identify opposite sides: Gives a space with no boundary.

This Is What You See



More Than Three Dimensions?

Makes mathematical sense.

Any point has an address, and the address just has more numbers – one for each dimension

$$(x_1, x_2, x_3, x_4, \dots)$$

Topological Classification In Higher Dimensions

There is an  think clearly
about thes ensions

Homo sapiens topologensis

This animal can find topological invariants of higher dimensional spaces…

You might expect that as the number of dimensions increases, the problem gets harder… but…

there is a big surprise …..

Four Dimensions Is The Hardest !

There are many unanswered questions.

One thing we know for sure is that the world of possible four-dimensional spaces is really wild.

We do not know anything even close to a complete topological invariant.

Clay Mathematics Proceedings
Volume 5, 2006

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

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PHYSICS

What holds stuff together?

How can we describe the forces that attract and repel matter?

What forces are there?

Gravity

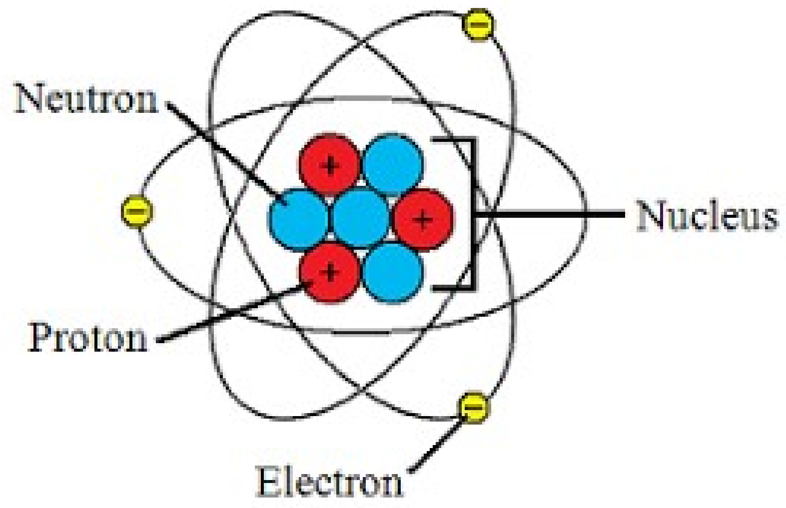
Electricity & Magnetism

Nuclear Force

Protons have positive electric charge

It's cozy in there:

$$r = 10^{-15} m.$$

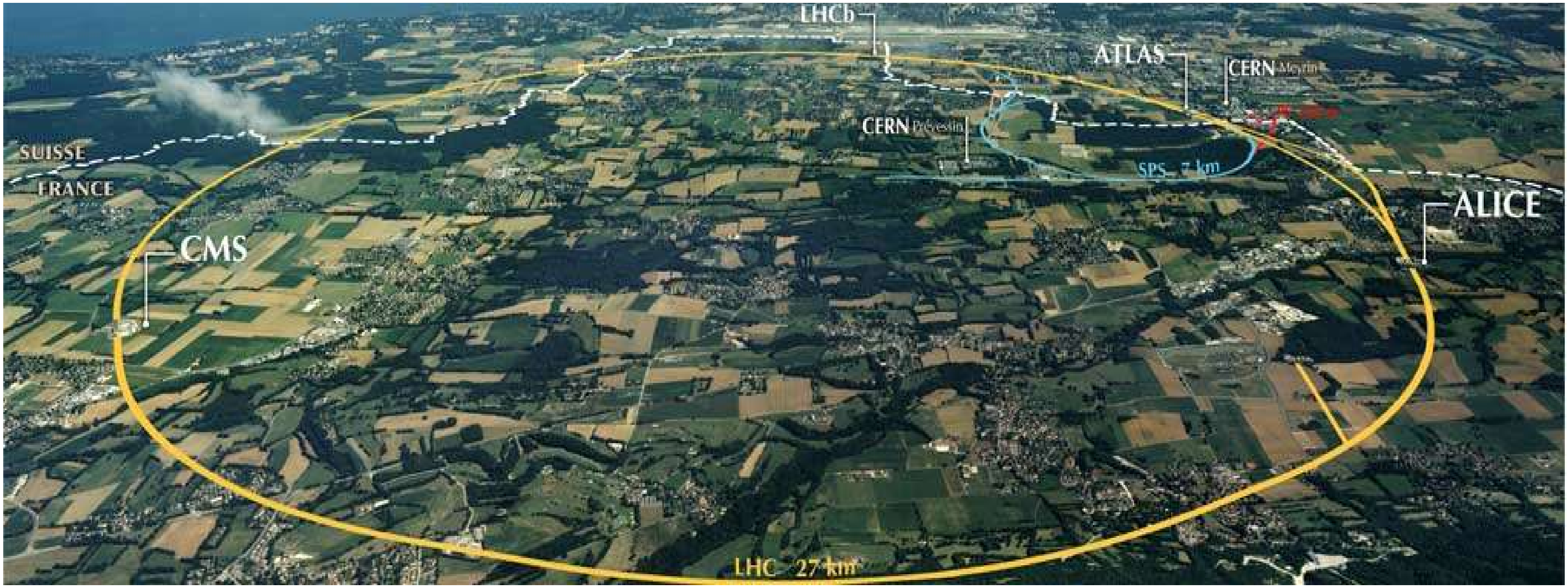


Electrical repulsion between two protons at this distance produces an acceleration .... $\sim 10^{28}\ g$

Fastest roller coaster $\sim 6\ g$

Fighter pilots $\sim 9\ g$

The strong force is very subtle – it has been studied with particle accelerators for decades - up to the present day…



Large Hadron Collider at CERN

Mathematical Description Of Forces

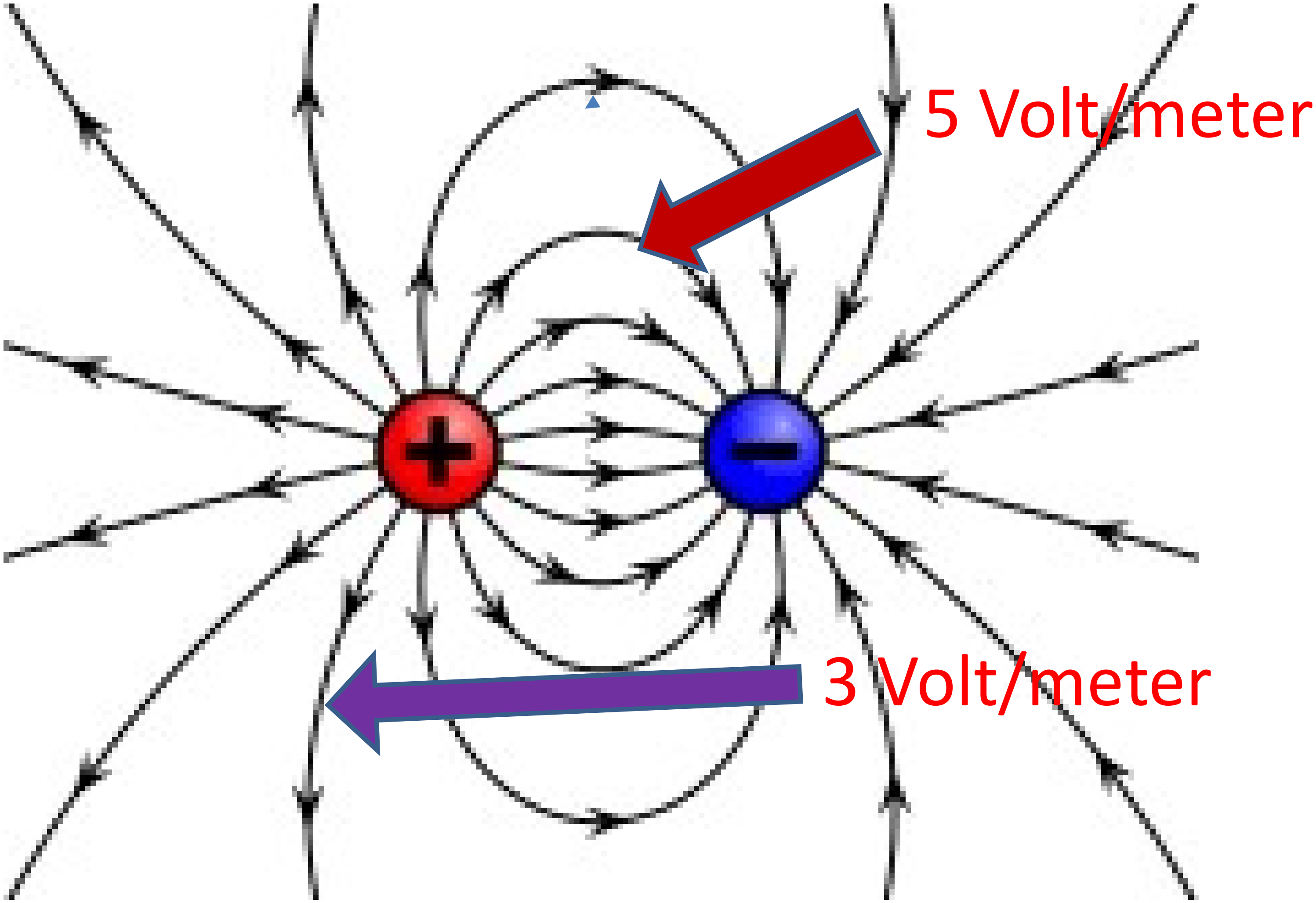
Michael Faraday's concept of a ``field''

Charged particles create fields

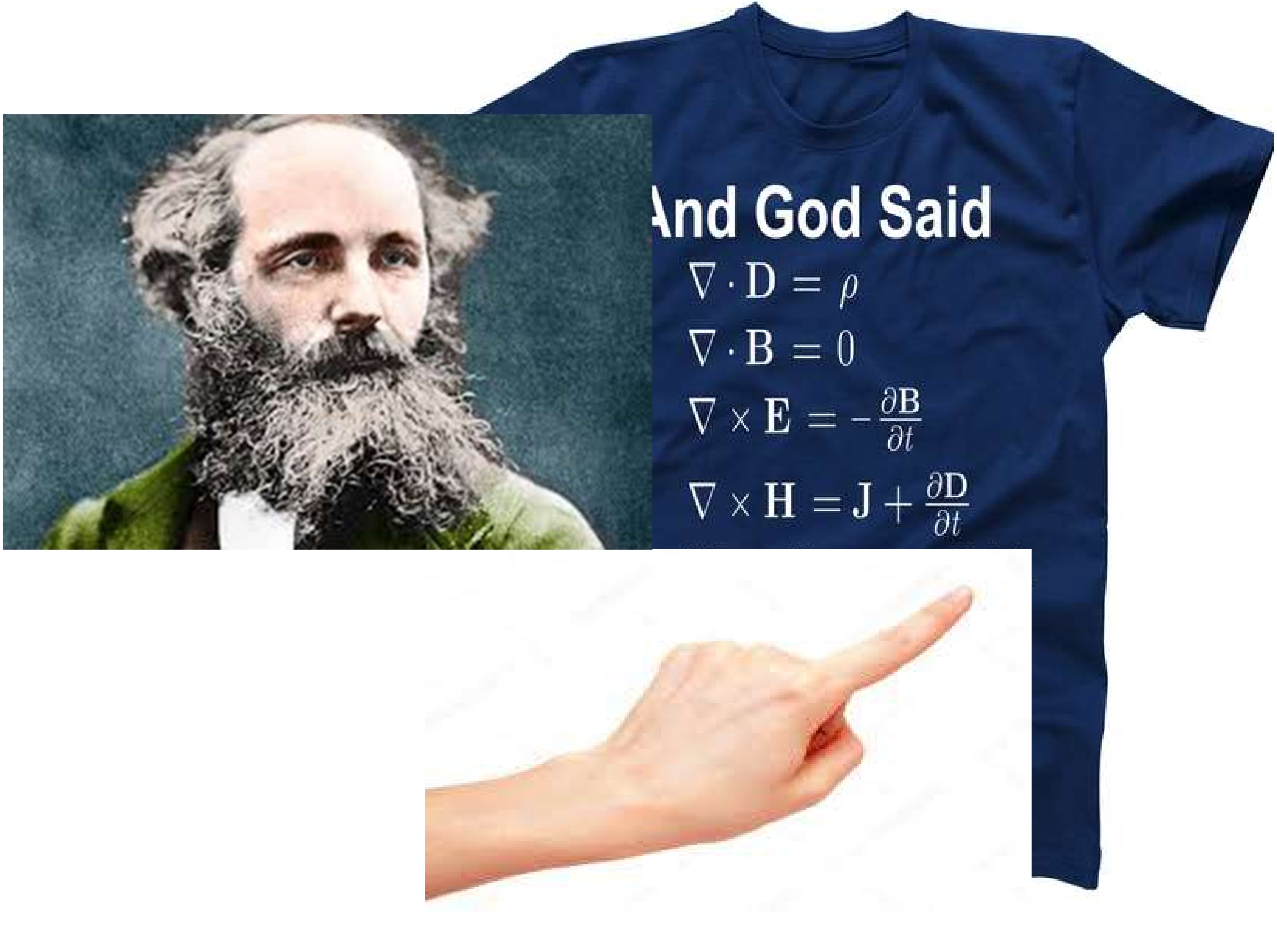
Fields move charged particles

So we can ``see'' a field by looking at how ``test particles'' of in the presence of the field.

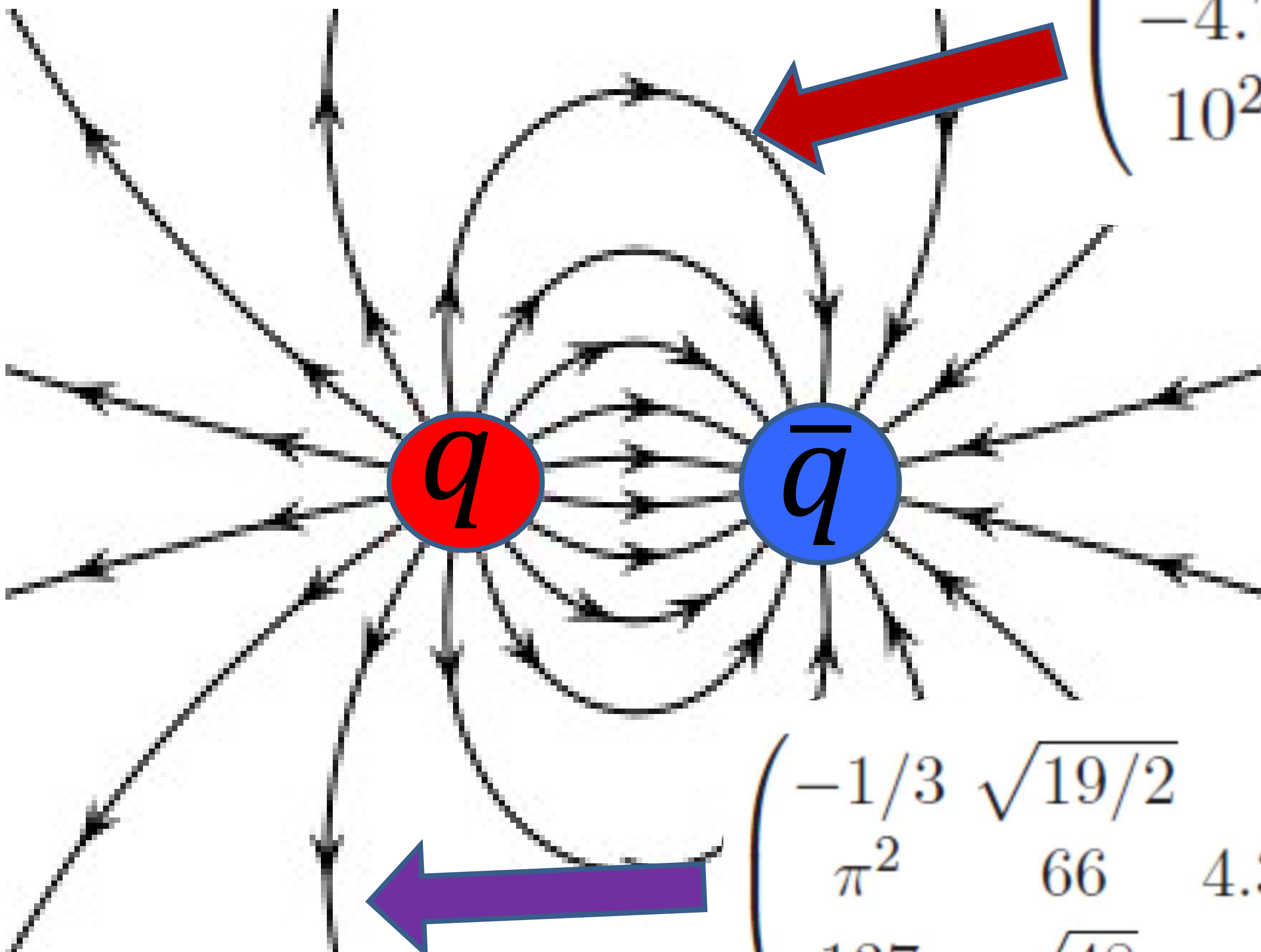
Electric & Magnetic Forces



Physicists use equations to find the fields:
That's called ***field theory***.


And God Said
$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Nuclear Force



$$\begin{pmatrix} 2.3 & 0.001 & -\pi \\ -4.78 & 1.06 & \sqrt{2} \\ 10^{28} & -\sqrt{5} & 16 \end{pmatrix}$$

Volt/meter

$$\begin{pmatrix} -1/3 & \sqrt{19/2} & 24 \\ \pi^2 & 66 & 4.32901 \\ 137 & \sqrt{48} & -44.67 \end{pmatrix}$$

Volt/meter

$$3 \times 5 = 15$$

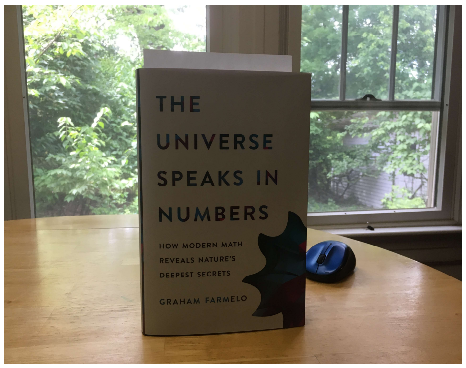
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$5 \times 3 = 15$$

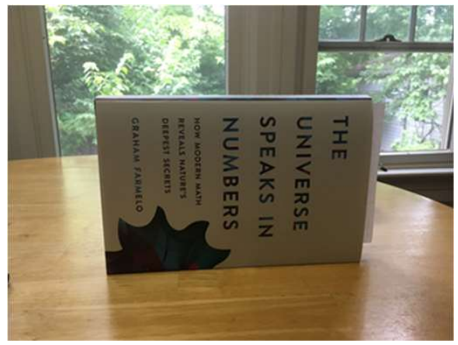
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Commutative vs. Noncommutative

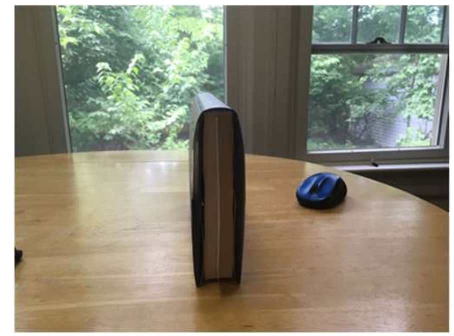
Abelian vs. Nonabelian

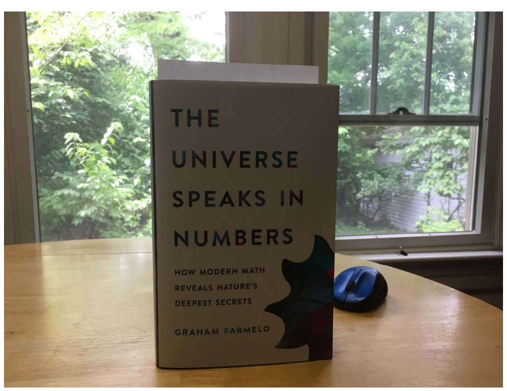


¼ Turn
x-axis



¼ Turn
z-axis

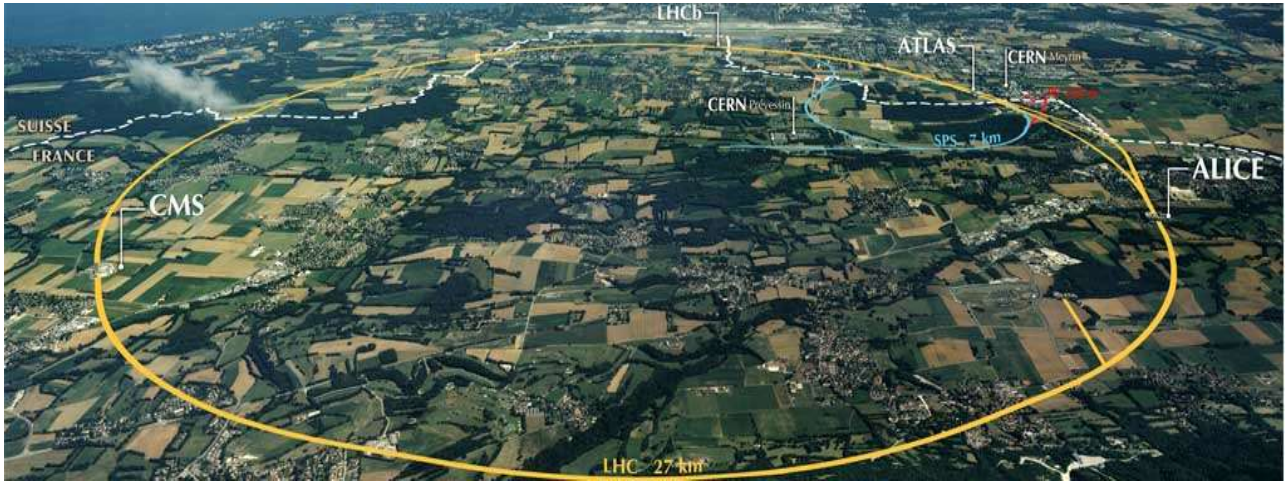




¼ Turn
z-axis

¼ Turn
x-axis

Abelian field theory



Nonabelian field theory

Yang-Mills Equations

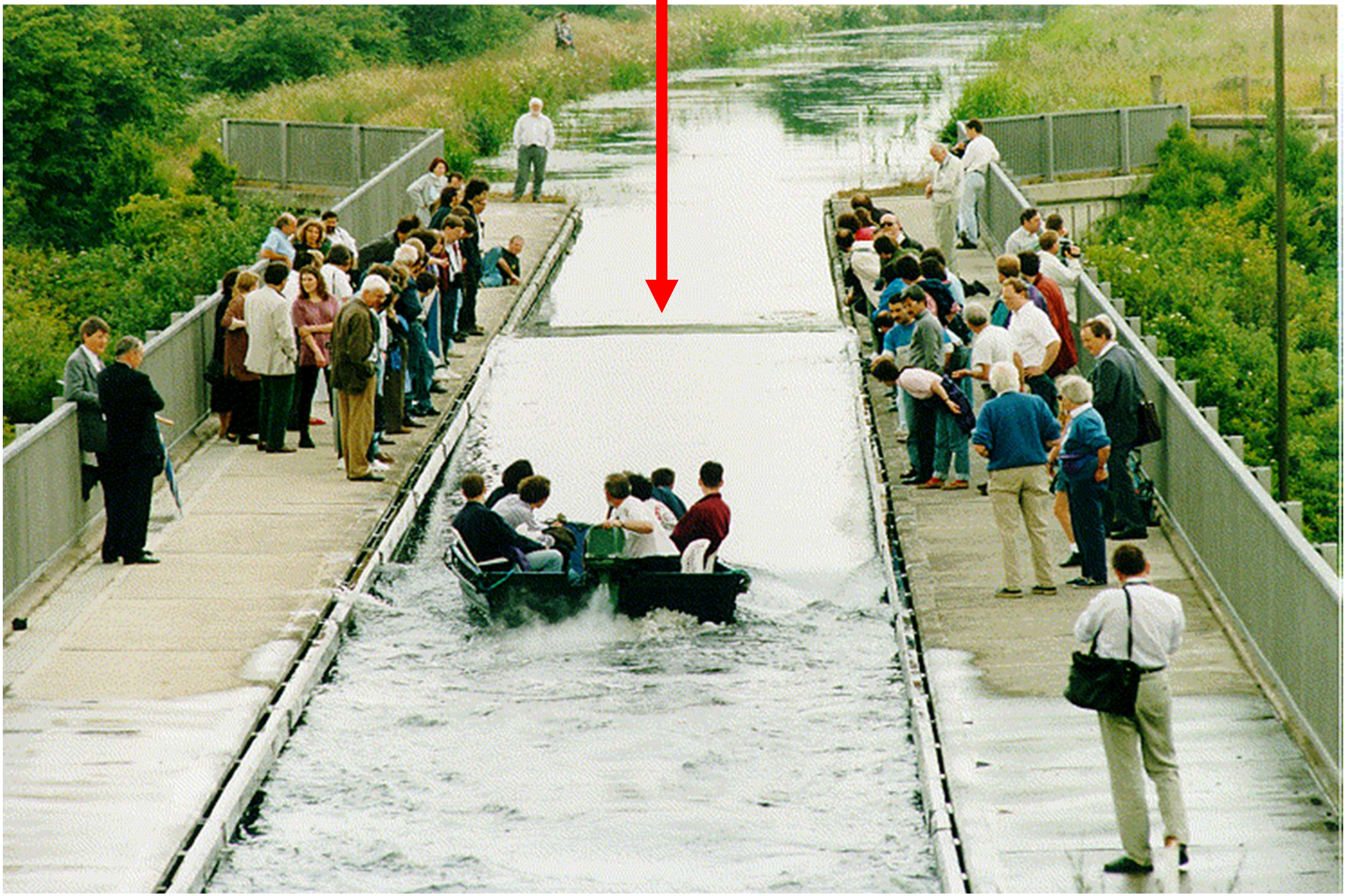
Equations that govern the nuclear force field were first written at BNL & IAS, 1954

These generalize Maxwell's equations.

Noncommutative light.

But they have another kind of solution ...

Soliton



Instantons = Solitons Of YM Equations

Instantons are localized in space AND ``time''.

So they are localized in all four dimensions, including ``time'' – hence the name.

1975: A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu.S. Tyupkin

Meanwhile, back at the ranch

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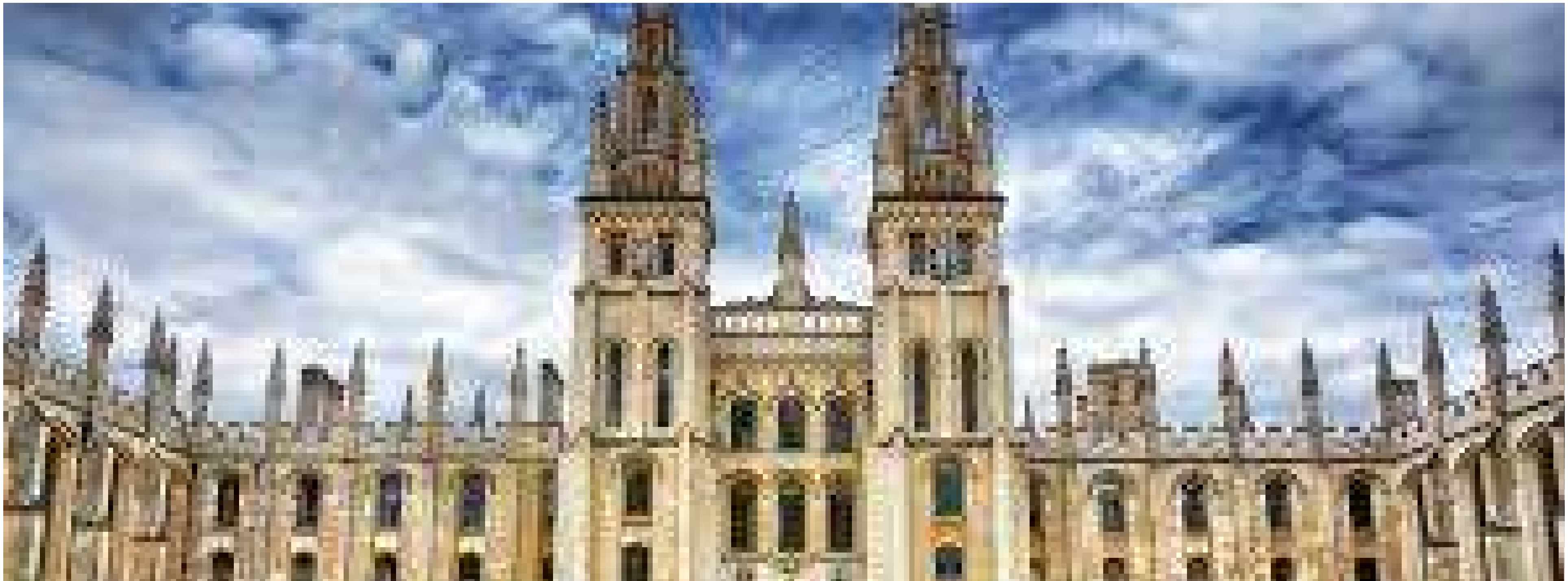


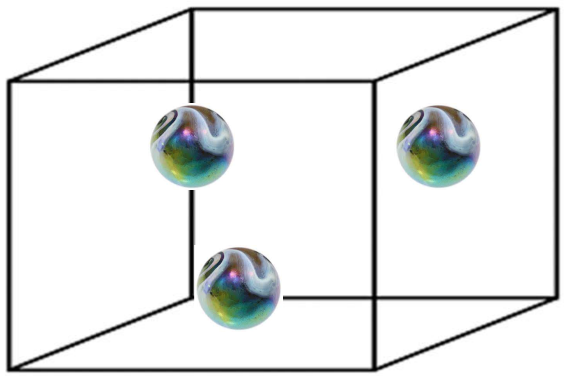
Donaldson Invariants

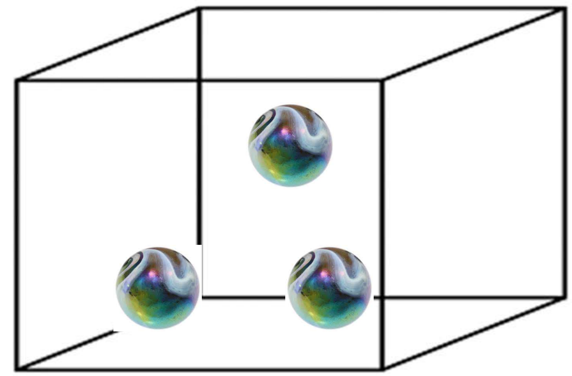
Let's COUNT the number of ways of putting instantons into a four-dimensional space

Instantons are localized objects (in all four dimensions) and they can only "fit" in a compact four-dimensional space in special ways.

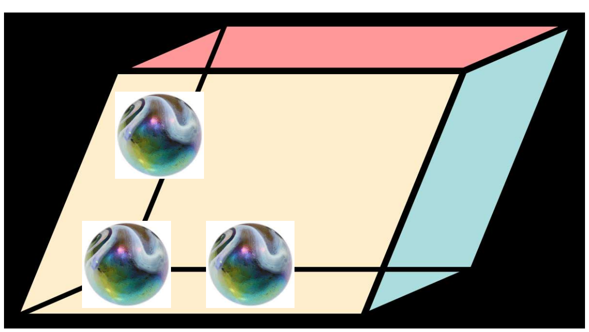
Big surprise: This is a ***topological invariant*** of the space !!!

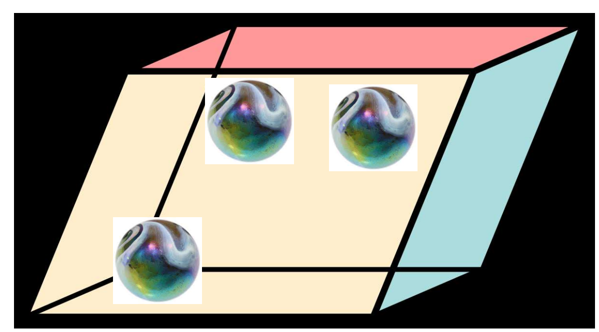
Cartoon Of Donaldson Invariants





``Donaldson invariant’’ for 3 instantons = 2





``Donaldson invariant’’ for 3 instantons = 2

Using this new (1984) topological invariant Donaldson could prove dramatic new results:

Example: There are exotic four-dimensional spaces where you can't do calculus

Donaldson invariants are extremely hard to compute and interpret

It took a lot of effort to compute a few special examples….

Mathematicians hit a wall…

(although Peter Kronheimer & Tom Mrowka were closing in)

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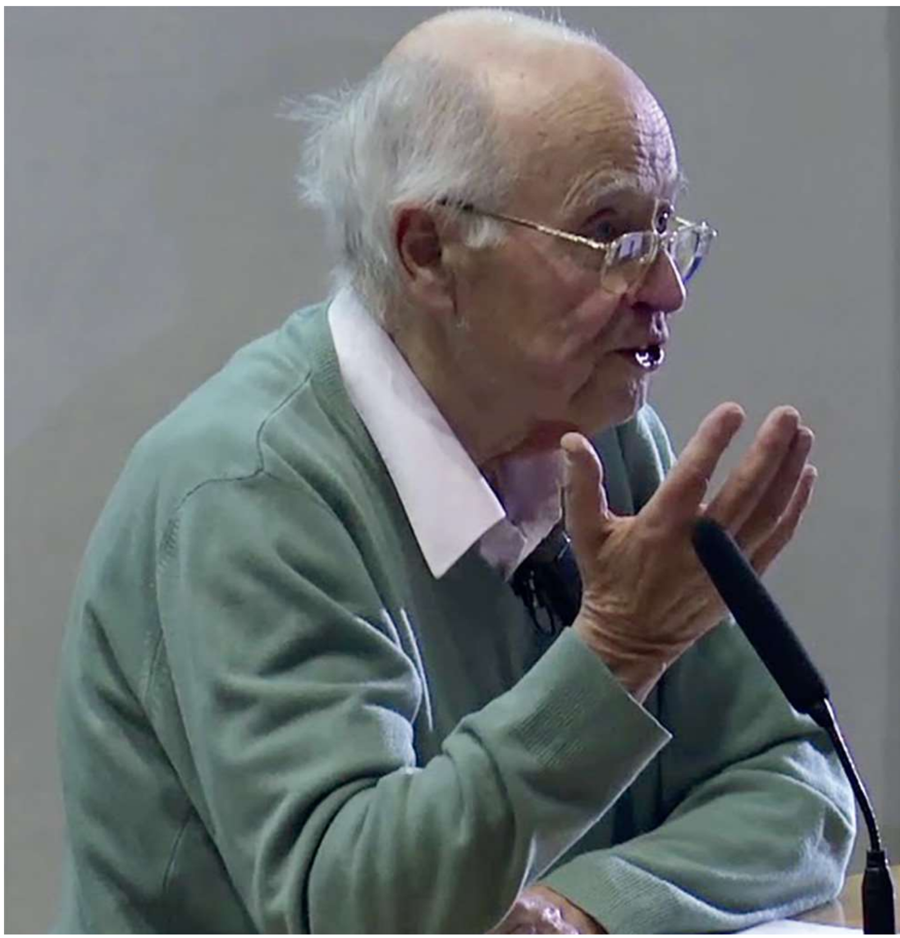
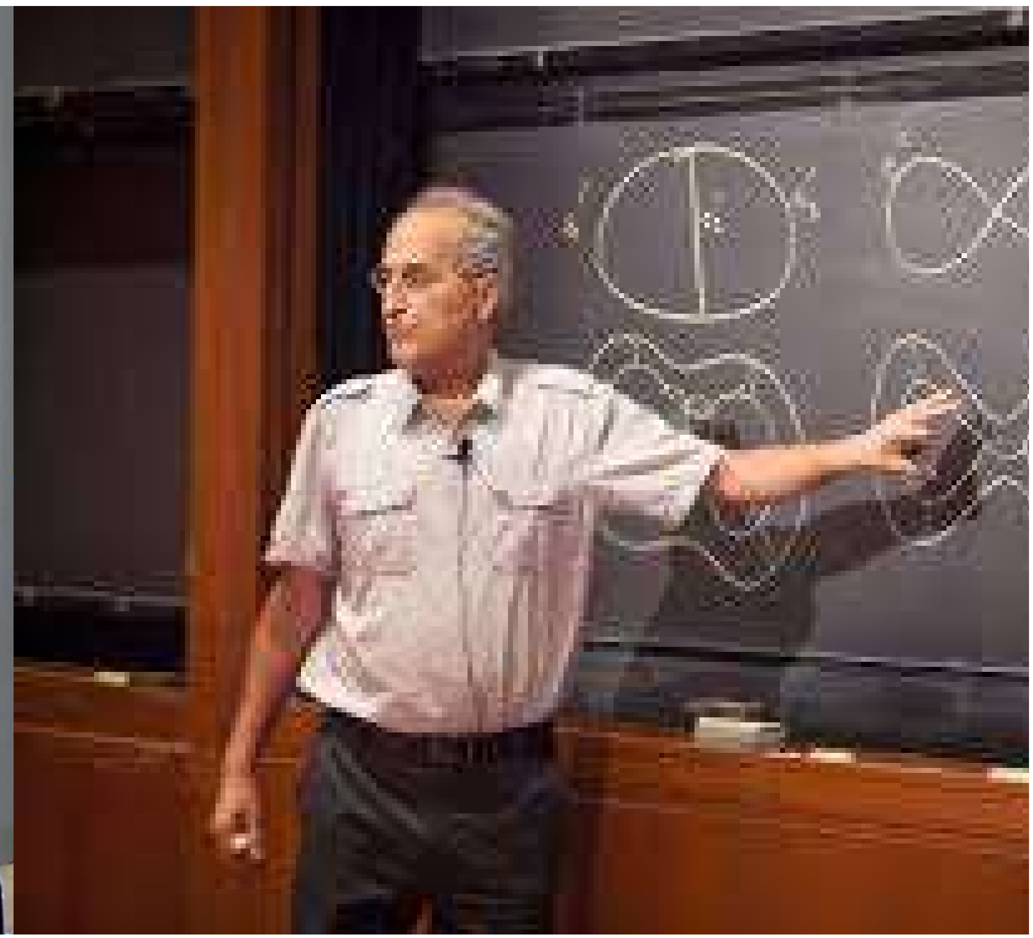
A Turning Point: Atiyah's Questions (1987)

What is ***<u>the physical interpretation</u>*** of the Donaldson & Jones invariants?

Witten's Answers

Donaldson and Jones invariants can be computed within the framework of a Yang-Mills field theory[1]

[1] *Technically: A certain generalization of a Yang-Mills theory with a different set of quarks and electrons from what we see in nature*

Commun. Math. Phys. 117, 353–386 (1988)

Communications in
Mathematical
Physics

Topological Quantum Field Theory

Edward Witten*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. A twisted version of four dimensional supersymmetric gauge theory is formulated. The model, which refines a nonrelativistic treatment by Atiyah, appears to underlie many recent developments in topology of low dimensional manifolds; the Donaldson polynomial invariants of four manifolds and the Floer groups of three manifolds appear naturally. The model may also be interesting from a physical viewpoint; it is in a sense a generally covariant quantum field theory, albeit one in which general covariance is unbroken, there are no gravitons, and the only excitations are topological.

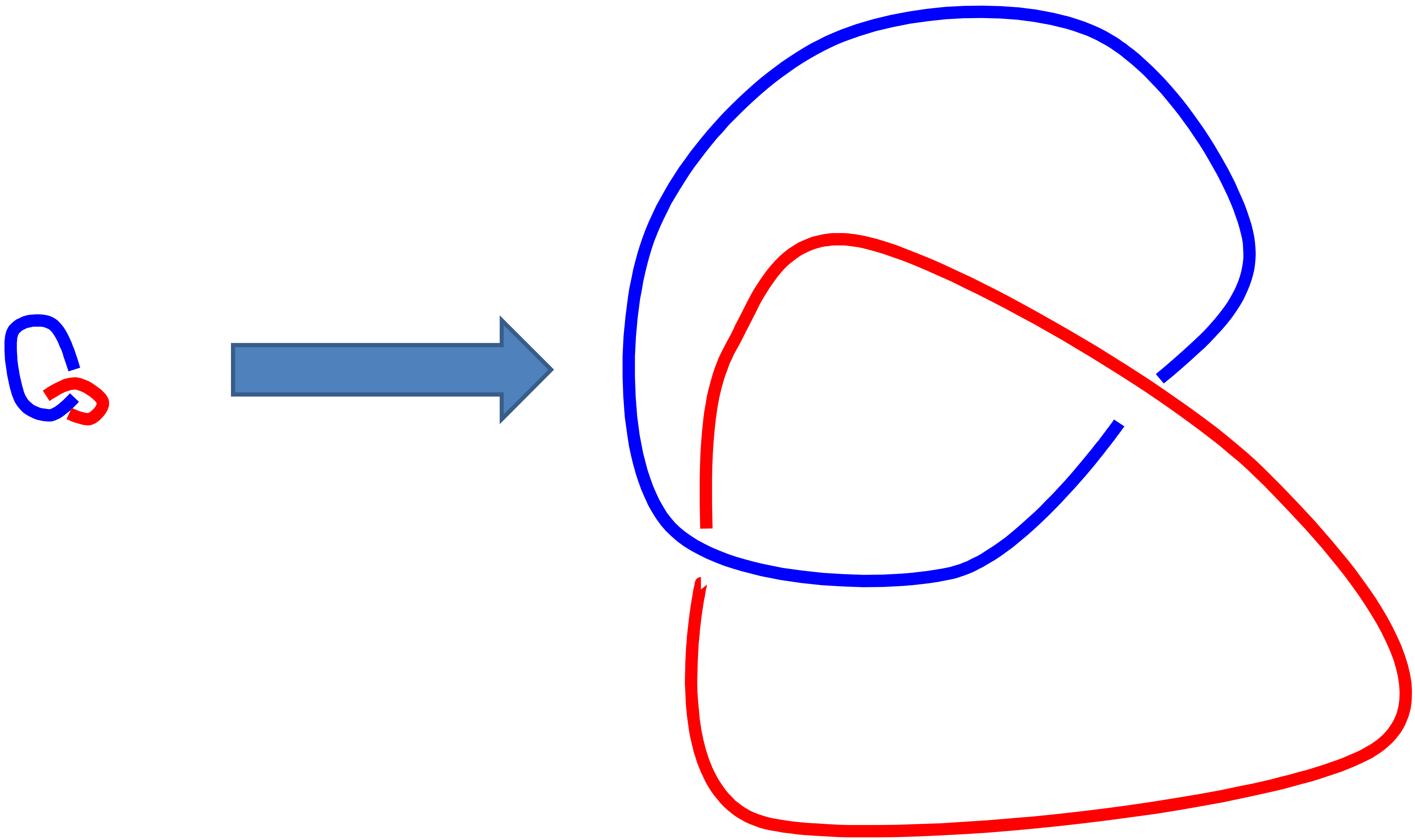
Topological Field Theory

Vast simplification of physics:
Length scales and time scales do not matter

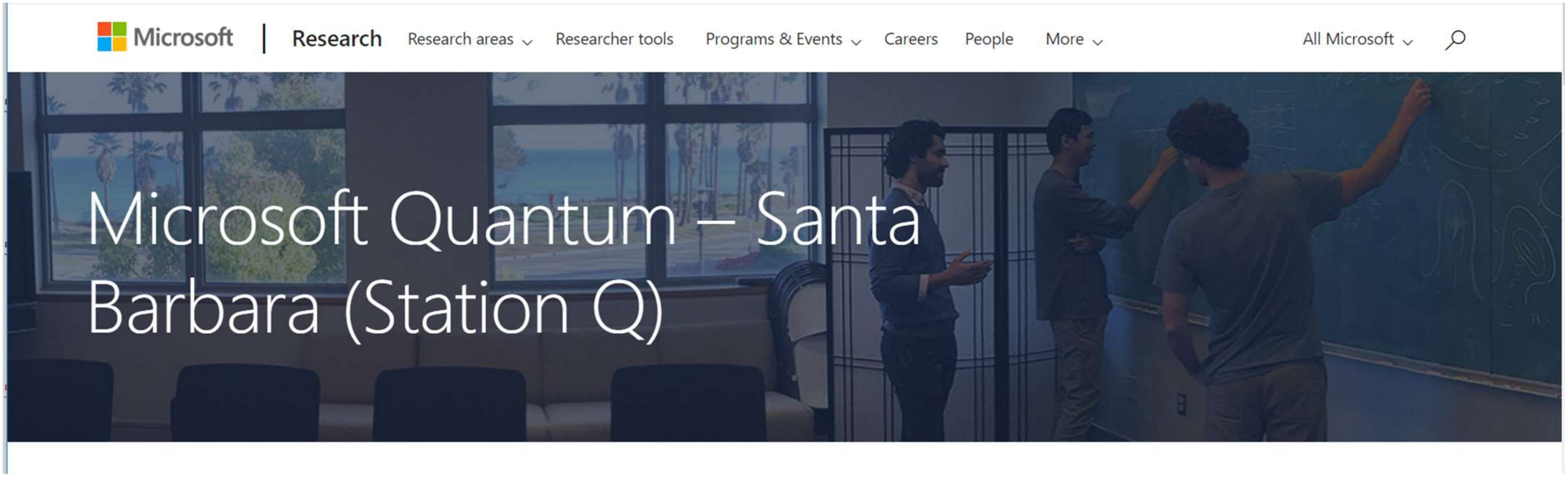
No difference between yesterday, last week, 100 years ago, 1 billion years ago.

No difference between your commute to work, and your extra-galactic vacation trip.

What does matter is topology!



Huge impact in both physics and math:
Thousands of papers
Fresh woods & pastures new …



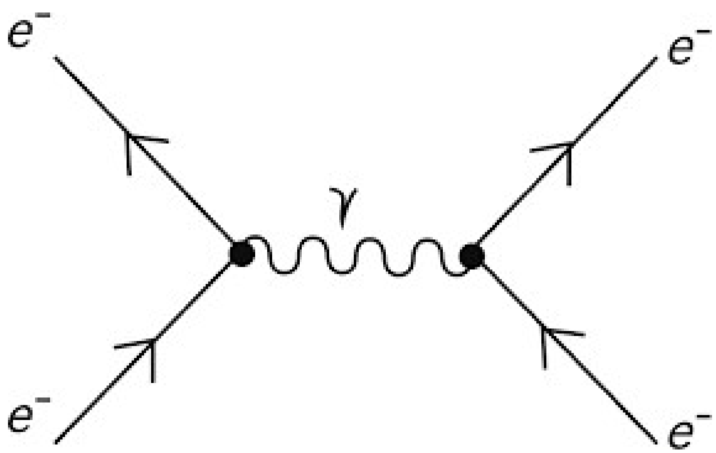
It might even be of practical use. It is being employed by Microsoft's Station Q as a road to quantum computation.

Quantum Field Theory

Computing Donaldson invariants a la Witten requires computing probabilities in a quantum Yang-Mills theory

How hard can that be?

Abelian: Maxwell's theory: Hard, but solvable - Feynman, Schwinger, Tomonaga (1946-1949)



Nonabelian: $MUCH^n$ harder: Not solved yet

Recall: Witten reformulated the Donaldson
invariants as probabilities
for certain events in Yang-Mills theories
in four-dimensional spaces.

But computing probabilities in
Yang-Mills theories is extremely difficult

So we seem to have exchanged
one hard problem for another…..

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Effective Theories

Topology is unaffected by distance –
by scaling things up to bigger and bigger sizes.

In Yang-Mills theory, when one asks questions about events at larger and larger scales the answers can simplify

The answers can be the ***SAME*** as answers to analogous questions in a ***DIFFERENT***, but simpler, theory

Effective Theories At Long Distance

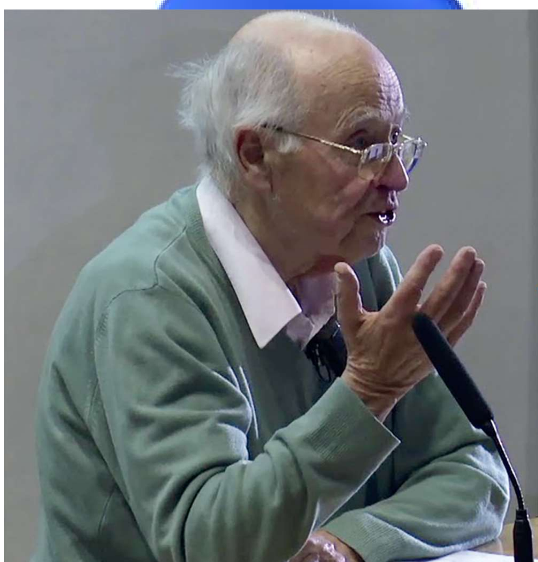
10^{-15} *meters*

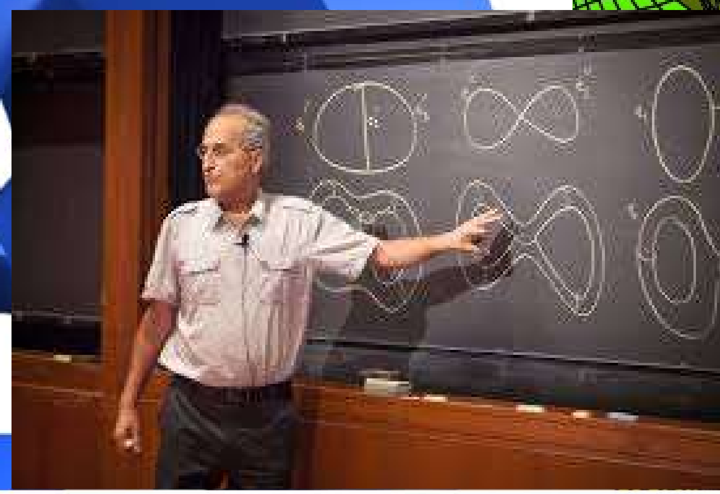
LEMMA (3.3). *If* $k > \frac{3}{2}(1+a)$ *then*:

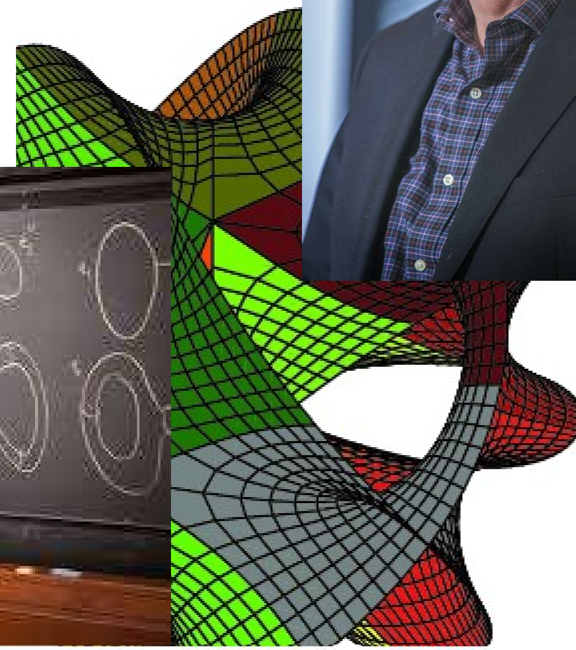
(i) $Q_k(V_1, \ldots V_d; g)$ *depends only on the homology classes* $\alpha_1, \ldots, \alpha_d$ *and defines a symmetric multilinear map*:

$$q_k^{(g)}: H_2(X) \times \ldots H_2(X) \to Z,$$

$$\bar{M}_k = \bar{M}_{k,X}(g) \subset M_k \cup (M_{k-1} \times X) \cup (M_{k-2} \times S^2(X)) \cup \ldots .$$



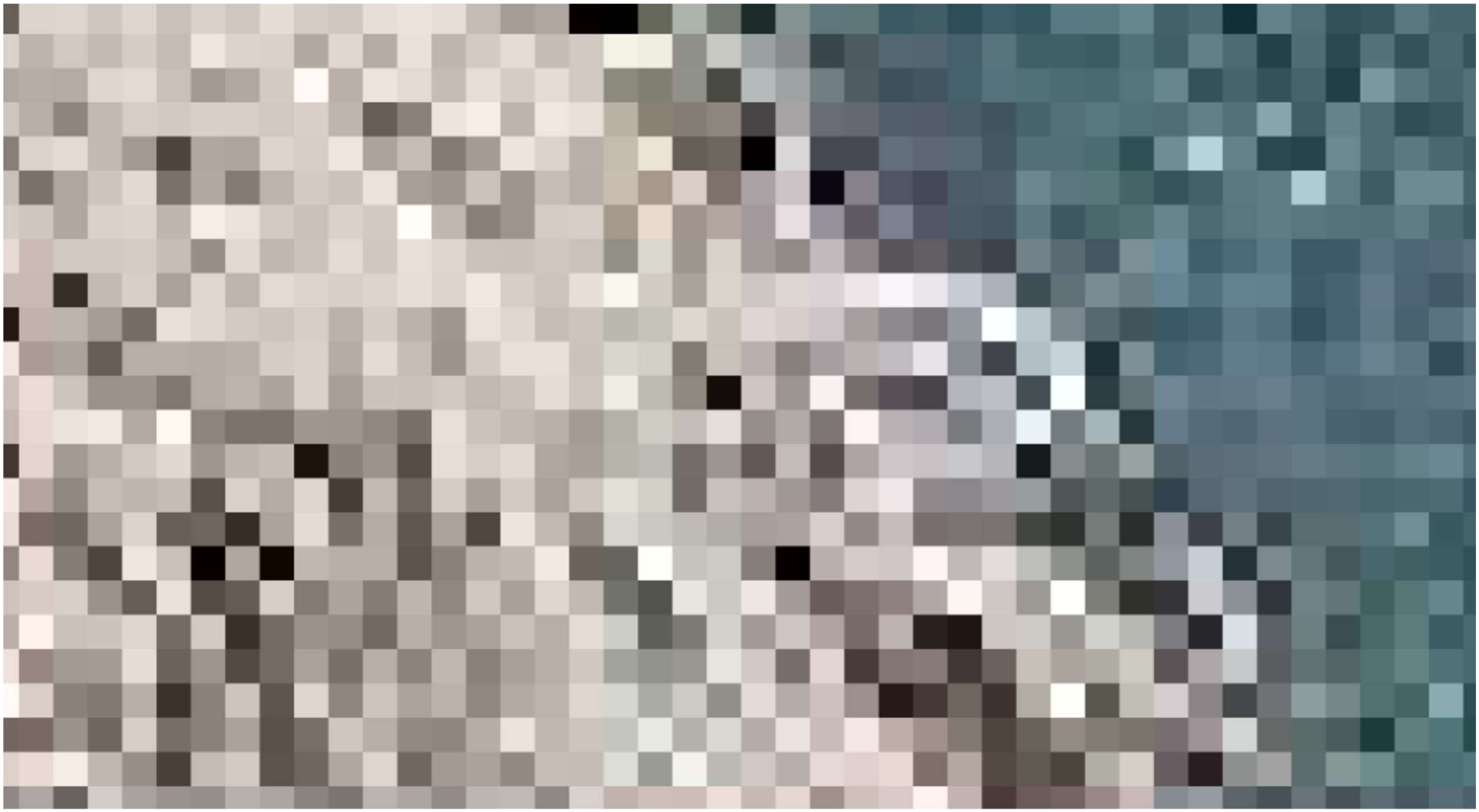




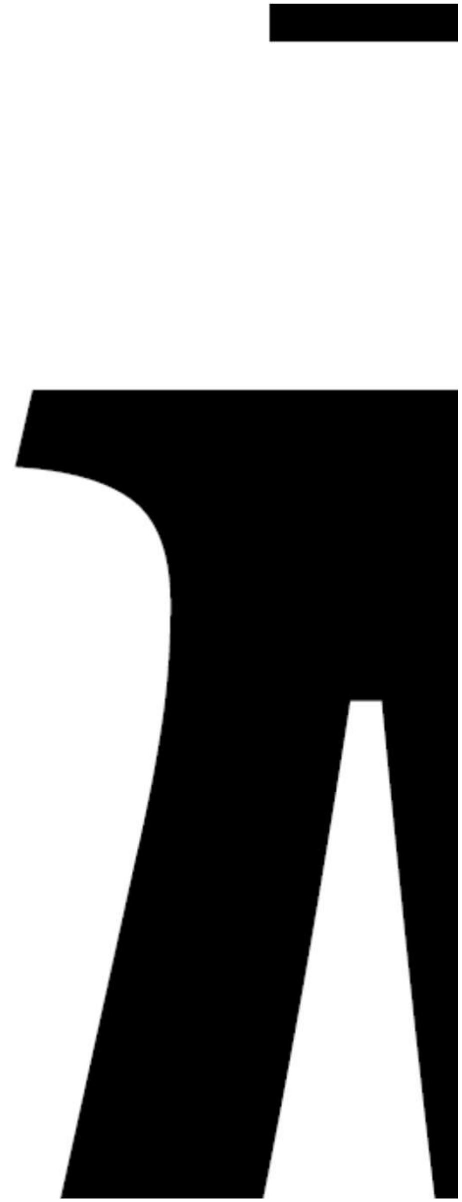
$$\mathscr{L} = \int_M d^4x \sqrt{g}\, \mathrm{Tr}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\phi D_\alpha D^\alpha \lambda - i\eta D_\alpha \psi^\alpha + iD_\alpha \psi_\beta \cdot \chi^{\alpha\beta} - \frac{i}{8}\phi[\chi_{\alpha\beta}, \chi^{\alpha\beta}] - \frac{i}{2}\lambda[\psi_\alpha, \psi^\alpha] - \frac{i}{2}\phi[\eta, \eta] - \frac{1}{8}[\phi, \lambda]^2\right].$$

$$\delta\mathscr{L} = \tfrac{1}{2}\int_M \sqrt{g}\,\delta g^{\alpha\beta} T_{\alpha\beta}\,.$$

$10\ meters$



10^{500} $meters$

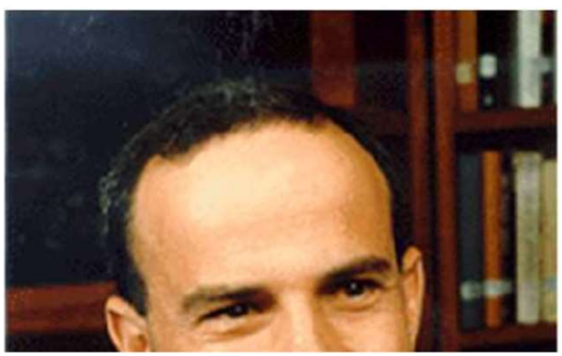


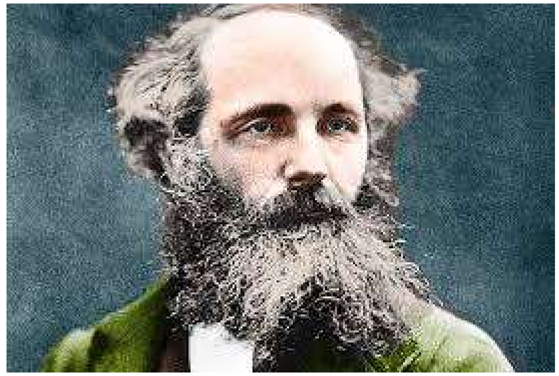
$10^{wtf}\ meters$

$$= i$$

$10^{wtff}\ meters$

$+ \quad = \quad \bar{M}$





$$F^+ = \bar{M} M$$

$$\mathrm{DM} = 0$$

Seiberg-Witten Paper

Seiberg & Witten (1994)

Viewed from afar: The YM theory used by Witten simplifies dramatically:

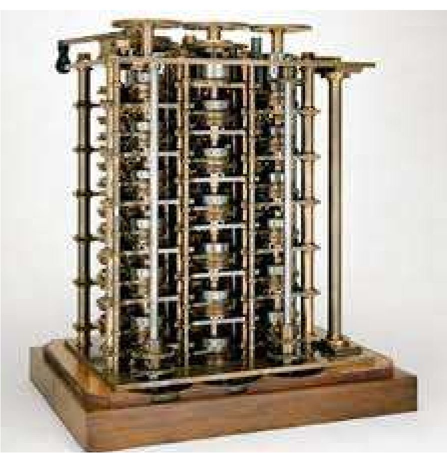
It is a field theory –
based on an *ABELIAN* group

Much easier to work with

Seiberg-Witten equations have soliton-like solutions: ``vortices in a superconductor’’

Seiberg-Witten topological invariants: Count these `` vortices’’ in our 4-dimensional space

The Donaldson invariants can be written in terms of the Seiberg-Witten invariants: They carry the same information about the four-dimensional space



Much easier
to compute!


MIDNIGHT
All Your Holiday Shopping Done Before The Sun Comes Up

Mad Dash After A Breakthrough

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY

In the last three months of 1994 a remarkable thing happened: this research area was turned on its head by the introduction of a new kind of differential-geometric equation by Seiberg and Witten: in the space of a few weeks long-standing problems were solved, new and unexpected results were found, along with simpler new proofs of existing ones, and new vistas for research opened up.

opments, which are due to various mathematicians, notably Kronheimer, Mrowka, Morgan , Stern and Taubes, building on the seminal work of Seiberg [S] and Seiberg and Witten [SW]. It is written as an attempt to take stock of the progress stemming

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Two Basic Questions

MATHEMATICS

What is the shape of a space?

How can we tell when two geometric objects can be deformed into each other ?

PHYSICS

What holds stuff together?

How can we describe the forces that attract and repel matter?

S

...ED IN BULLETIN OF THE
...ICAN MATHEMATICAL SOCIETY
...ume 30, Number 2, April 1994, Pages 178-207

RESPONSES TO "THEORETICAL MATHEMATICS: TOWARD A CULTURAL SYNTHESIS OF MATHEMATICS AND THEORETICAL PHYSICS", BY A. JAFFE AND F. QUINN

...ULTURAL
...L PHYSICS

Michael Atiyah et al.

Michael Atiyah
The Master's Lodge
Trinity College
Cambridge CB2 1TQ
England, U.K.

...PEARED IN BULLETIN O...
...MERICAN MATHEMATI...
...Volume 29, Number 1...

"TH...
SYNTH...

I find myself agreeing with much of the detail of the Jaffe-Quinn argument, especially the importance of distinguishing between results based on rigorous proofs and those which have a heuristic basis. Overall, however, I rebel against their general tone and attitude which appears too authoritarian.

My fundamental objection is that Jaffe and Quinn present a sanitized view of mathematics which condemns the subject to an arthritic old age. They see an inexorable increase in standards of rigour and are embarrassed by earlier periods of sloppy reasoning. But if mathematics is to rejuvenate itself and brea...

Abstract. Is ...
physics and mat...
norms discourage...
there can be benefi...
Serious caution is r...
...obvious damag...
...rk that should


Shape of a space?
Strong force ?


Topological Invariants!
Yang-Mills
instantons!


Donaldson invariants
(instanton counting)
Topological Field Theory


Long distance
effective theory?


Seiberg-Witten Theory !


Seiberg-Witten invariants!

Clay Mathematics Proceedings
Volume 5, 2006

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction


NOT
That's all Folks!