N=2* SYM And Four Manifold Invariants

Gregory Moore, Rutgers University

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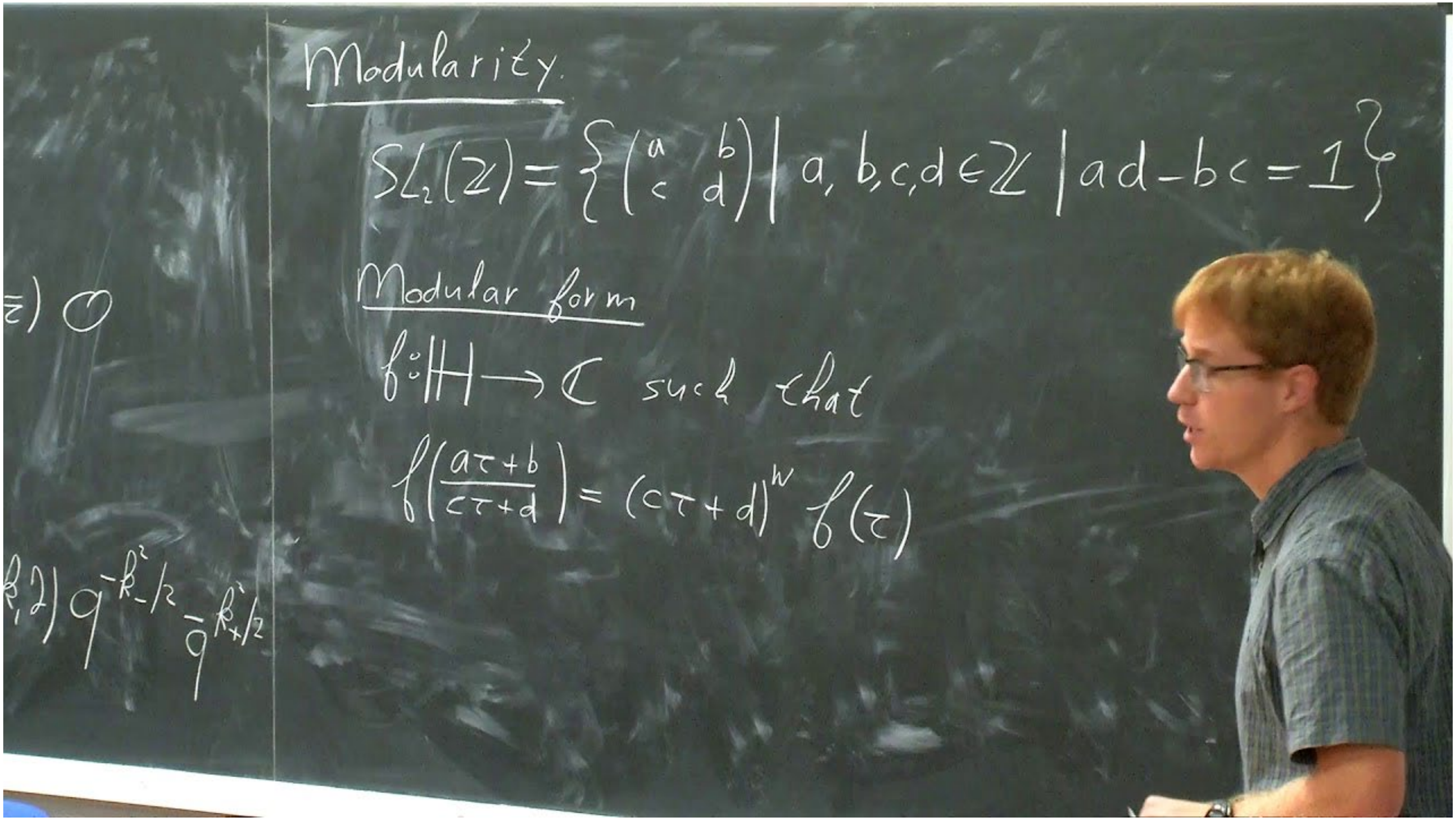
Work with JAN MANSCHOT



arXiv:2104.06492

Closely related to previous work of:

Dijkgraaf, Park, Schroers (1998)
Labastida & Lozano (1998)
Labastida & Marino (1997)
Losev, Nekrasov, Shatashvili (1997)
Moore & Witten (1997)
Vafa & Witten (1994)
Witten (1988 - 1997)

Also related: Recent work of
Göttsche, Kool, Nakajima, and Williams

1. Motivation
2. Preliminaries And Background
3. Physical Formulation Of Partition Function
4. Mathematical Formulation Of Partition Function
5. Coulomb Branch Integral: Measure & Evaluation
6. LEET Near Cusps & Explicit Results

Glorious History Of SYM & Four-Manifolds

Instantons (1975) →

Donaldson invariants (1982) →

TQFT (1988) →

Seiberg-Witten Invariants (1994)

→ Revolution of 1995

Clay Mathematics Proceedings
Volume 5, 2006

But not all questions are answered…

Will We Ever Classify Simply-Connected Smooth 4-manifolds?

Ronald J. Stern

Abstract. These notes are adapted from two talks given at the 2004 Clay Institute Summer School on *Floer homology, gauge theory, and low dimensional topology* at the Alfred Rényi Institute. We will quickly review what we do and do not know about the existence and uniqueness of smooth and symplectic structures on closed, simply-connected 4-manifolds. We will then list the techniques used to date and capture the key features common to all these techniques. We finish with some approachable questions that further explore the relationship between these techniques and whose answers may assist in future advances towards a classification scheme.

1. Introduction

The SW revolution was based on pure SU(2) N=2 SYM

The basic idea of topological twisting applies to any d=4, N=2 QFT

Is there more to learn about 4-manifolds from Susy QFT?

This Talk

Study 4-fold invariants for ``SU(2) N=2* theory''

Interpolates between Donaldson & Vafa-Witten invariants

Important lessons for several future generalizations

Key to explicit evaluation:
``Coulomb branch integral'' aka ``u-plane integral''

Automorphic forms; indefinite theta functions; mock modular forms; Jacobi Maass forms

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Four-Manifolds We Consider

X: Smooth, compact, oriented, $\partial X = \emptyset$.

For simplicity: Connected, $\pi_1(X) = 0$

We assume (as in Donaldson theory) that X admits an almost complex structure

$$\Longleftrightarrow$$

$b_2^+(X)$ *is* odd

Important: Do <u>not</u> assume X is spin

Almost Complex Structure

$T^*X \otimes \mathbb{C}$ has a basis $e^i, (e^i)^* \; i = 1,2$

Across patches: $e^i \to U^i{}_j(x)\, e^j$

$$s: U(2) \to \begin{pmatrix} Re(U) & Im(U) \\ -Im(U) & Re(U) \end{pmatrix} \in SO(4)$$

Preliminary: $Spin^c$-structure – 1/3

$$Spin^c(4) := \{(u_1, u_2) \,|\, \det(u_1) = \det(u_2)\} \subset U(2) \times U(2)$$

$$\pi: Spin^c(4) \to SO(4)$$

$$x_\mu \sigma^\mu \;\to\; u_1\, x_\mu \sigma^\mu\, u_2^{-1}$$

Spin-c structure:

Give transition functions in $Spin^c(4)$

so that $\pi(u_1, u_2) =$

$SO(4)$ transition functions of T^*X

Preliminary: $Spin^c$-structure – 2/3

$$Spin^c(4) := \{\ (u_1, u_2) | \det(u_1) = \det(u_2)\} \subset U(2) \times U(2)$$

Has two obvious 2-dimensional reps:

$2 \otimes 1$ and $1 \otimes 2$

Given a spin-c structure these define chiral spinor bundles

$$W^{\pm} \to X$$

$$c(\mathfrak{s}) := c_1(\det W^{\pm}) \in H^2(X;\mathbb{Z})$$

$$\ell = \frac{c(\mathfrak{s})^2 - 2\chi - 3\,\sigma}{8} \in \mathbb{Z}$$

An ACS $\mathcal{J}$ defines a canonical spin-c structure $\mathfrak{s}(\mathcal{J})$

$$Spin^c(4) := \{\ (u_1, u_2) |\ \det(u_1) = \det(u_2)\} \subset U(2) \times U(2)$$

$$\phi: U(2) \to Spin^c(4): \qquad u \mapsto \left(u, \begin{pmatrix} 1 & \\ & \det u \end{pmatrix}\right)$$

$$\pi \circ \phi \sim s$$

For $\mathfrak{s}(\mathcal{J})$ $\quad \ell = 0$

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SU(2) N=2* SYM

$$\underbrace{\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-}_{\text{Local Lorentz}} \oplus \mathfrak{su}(2)_R \oplus \mathfrak{g}_{gauge}$$

Hypermultiplet scalars in rep:

$$\mathcal{R} = \mathfrak{su}(2)_{gauge} \otimes \mathbb{H} \curvearrowleft SU(2)_R$$

Topological Twisting

Couple to background $SU(2)_R$ gauge field

Identify $\mathfrak{su}(2)_R$ with $\mathfrak{su}(2)_+$ in local Lorentz algebra

Hypermultiplet scalar fields become spinors under twisting

What if X is not spin?

Cure the problem by introducing an ``***ultraviolet***'' spin-c structure $\mathfrak{s}_{uv}$

So with a uv spin-c structure the hypermultiplet scalars in $N = 2^*$ -theory are spinors in W^+

For $N = 2^*$ basic topological twisting needs to be supplemented with extra data

It is not known how to twist the general d=4 N=2 theory.

Topologically Twisted Partition Function

Data needed to formulate the partition function:

$$\tau_{uv} \sim \theta + \frac{i}{g_{uv}^2} \in \mathcal{H} \qquad q_{uv} := e^{2\pi i\, \tau_{uv}}$$

$m \in \mathbb{C}$ Λ: UV scale $t := m/\Lambda$

(UV) Spin-c structure $\mathfrak{s}_{uv}$,
$c_{uv} := c(\mathfrak{s}_{uv}) \in H^2(X, \mathbb{Z})$

't Hooft flux $\nu \in H^2(X; \mathbb{Z}/2\mathbb{Z})$

….. and a metric $g_{\mu\nu}$ ….

$$T_{\mu\nu} = Q(\Lambda_{\mu\nu})$$

So metric should drop out….

$$S = \int_X \tau_{uv} \, Tr\,(F \wedge F\,) \; + \; Q(*)$$

so Z should be holomorphic in τ_{uv}….

Operators In The TQFT

Q –cohomology on depends on *homology*

$$p \in H_0(X;\mathbb{Z}) \Rightarrow p = n_1 x_1 + \cdots n_k x_k$$

$$S \in H_2(X;\mathbb{Z})$$

$$\mathcal{O}(p) = \sum_i n_i \; Tr\, \phi^2(x_i)$$

$$\mathcal{O}(S) = \int_S Tr(\phi F + \psi^2)$$

Path integral defines a ``function"

$$Z_\nu(\tau_{uv}, c_{uv}, t): H_*(X;\mathbb{Z}) \to \mathbb{C}$$

$$Z_\nu(x;\tau_{uv}, c_{uv}, t) \coloneqq \langle\, e^{\mathcal{O}(x)} \,\rangle_{\mathcal{N}=2^*}$$

We evaluate this function very explicitly and check some physical expectations.

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4. **Mathematical Formulation Of Partition Function**

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Mathematical Formulation Of The Invariants

Principal $SO(3)$ bundle $P \to X$

$A \in \mathcal{A}(P)$ $\quad M \in \Gamma(W^+ \otimes adP \otimes \mathbb{C})$

$W^+ \to X$: Positive chirality rank two bundle associated to uv spin-c structure $\mathfrak{s}_{uv}$

Q –fixed point equations

$$F^+ + [M, \bar{M}] = 0 \qquad \boldsymbol{\gamma} \cdot \boldsymbol{D} M = 0$$

``adjoint SW equations''

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

$\mathcal{M}_{Q,k,\nu}$: Component of moduli ``space'' of solutions to nonabelian monopole equations

$$w_2(P) = \nu$$

$$k(P) = -\frac{1}{8\pi^2}\int Tr(F \wedge F)$$

Math Definition Of Partition Function

$$Z_\nu(x;\tau_{uv},c_{uv},t) \coloneqq \langle\, e^{\mathcal{O}(x)} \rangle_{\mathcal{N}=2^*}$$

$$= \sum_{k\geq 0} q_{uv}^k \int_{\mathcal{M}_{Q,k,\nu}} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}};t)$$

$$\mu: H_*(X,\mathbb{Z}) \to H^{4-*}\left(\mathcal{M}_{Q,k,\nu};\mathbb{Q}\right)$$

$\mathcal{E}_{\mathfrak{s}}$: Obstruction bundle for elliptic complex

Q-symmetry: Path integral $\to$ $\int_{\mathcal{M}_{Q,k,\nu}} \cdots$

Index Computations

$$v \dim \mathcal{M}_{Q,k} = \dim G \frac{c_{uv}^2 - (2\chi + 3\,\sigma)}{4} = 2\ell \dim G$$

N.B. Independent of instanton number k !

$$\dim \mathcal{M}_{inst,k} = 8k - \frac{3}{2}(\chi + \sigma)$$

$$Index\ \boldsymbol{\gamma} \cdot \boldsymbol{D} = -8k + \frac{3}{8}(c_{uv}^2 - \sigma)$$

$\Rightarrow$ Correlation functions on $H_*(X)$ infinite q_{uv} - series,
even with x=0

$U(1)_b$ Symmetry

$$F^+ + [M, \bar{M}] = 0 \quad \boldsymbol{\gamma} \cdot \boldsymbol{D} M = 0$$

$$U(1)_b \; : \; M \to e^{i\,\theta} M$$

$U(1)_b$ acts on the moduli space $\mathcal{M}_{Q,k}$ of these eqs.

$$\mathcal{O}(x) \to \mu(x) \in \; H^*_{U(1)_b}(\mathcal{M}_{Q,k})$$

$t = \frac{m}{\Lambda}$: $U(1)_b$ equivariant parameter

[Labastida-Marino; Losev-Shatashvili-Nekrasov]

$U(1)_b$ Localization

$$\sum_{k\geq 0} q_{uv}^{k} \int_{\mathcal{M}_{Q,k,\nu}} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}}; t)$$

$$F^{+} + [M, \bar{M}] = 0 \qquad \boldsymbol{\gamma} \cdot \boldsymbol{D} M = 0$$

Fixed point set for $M \to e^{i\theta} M$ has TWO branches

Branch 1: $\mathcal{M}_{inst,k,\nu}$: $M = 0 \;\&\; F^{+} = 0$

Branch 2: $\mathcal{M}_{ab}$: $M \sim \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

$U(1)_b$ Localization

$$\int_{\mathcal{M}_{Q,k,\nu}} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}}; t)$$

$$= \int_{\mathcal{M}_{inst,k,\nu}} e^{\mu(x)} Eul(\tilde{\mathcal{E}}_{\mathfrak{s}}; t) + \int_{\mathcal{M}_{ab}} e^{\mu(x)} Eul(\tilde{\mathcal{E}}_{\mathfrak{s}}; t)$$

First focus on the instanton contribution.

$t \to 0, \infty$ Limits Of Instanton Contribution

$$\sum_{k\geq 0} q_{uv}^{k} \int_{\mathcal{M}_{inst,k,v}} e^{\mu(x)} Eul(\mathcal{E}_{\mathfrak{s}}; t)$$

$$Eul(\mathcal{E}_{\mathfrak{s}}; t) = \prod_{i} (x_i + t) \quad = t^{-Index(D)} \sum_{n} \frac{c_n(\mathcal{E}_{\mathfrak{s}})}{t^n}$$

Leading term for $m \to \infty$: $c_0(\mathcal{E}_{\mathfrak{s}})$= 1

$\Rightarrow$ Donaldson invariants

Leading term for $m \to 0$: $c_{top}(\mathcal{E}_{\mathfrak{s}})$

$\mathfrak{s}_{uv} = \mathfrak{s}(\mathcal{J})$: $\mathcal{E}_{\mathfrak{s}} \cong T^*\mathcal{M}_k \Rightarrow$ ``Euler character of $\mathcal{M}_{inst,k}$ ''

Relation To Vafa-Witten Invariants-1/2

VW invariants compute the
``Euler character of $\mathcal{M}_{inst,k}$''

and they are S-duality covariant….

Our instanton contribution also computes the
Euler character (for $\mathfrak{s}_{uv} = \mathfrak{s}(\mathcal{J})$ and $m \to 0,$)
and together with the $\mathcal{M}_{ab}$ contribution is
S-duality covariant.

Natural guess: we get VW invariants.

In cases where we can compare (such as projective surfaces) $\lim_{t \to 0} Z_\nu$ does indeed reproduce the Vafa-Witten invariants.

This is surprising since the DW and VW twists are very different.

The Q-fixed point equations are different, but can be viewed as deformation equivalent. (long story…)

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The partition function on a compact manifold will equal a sum over all the vacua.

In particular we need to integrate over the Coulomb branch.

Coulomb Branch Integral

In principle defined for general class S theory.

$$Z_{\nu}^{CB} = \int_{\mathcal{B}} du \, d\bar{u} \, \mathcal{H} \, \Psi$$

$\mathcal{H}$ is ***holomorphic*** and ***metric-independent***

Ψ: ***NOT holomorphic*** and ***metric- DEPENDENT***
``indefinite theta function''

$\mathcal{B}$: Base of a Hitchin system

Today: $u \in \mathbb{C} \cong \mathcal{B}$ will be identified with a modular curve

5 Coulomb Branch Integral: Measure & Evaluation

5a Seiberg-Witten Review

5b Formulating The Measure And Integral

5c Evaluation Using Mock This & That

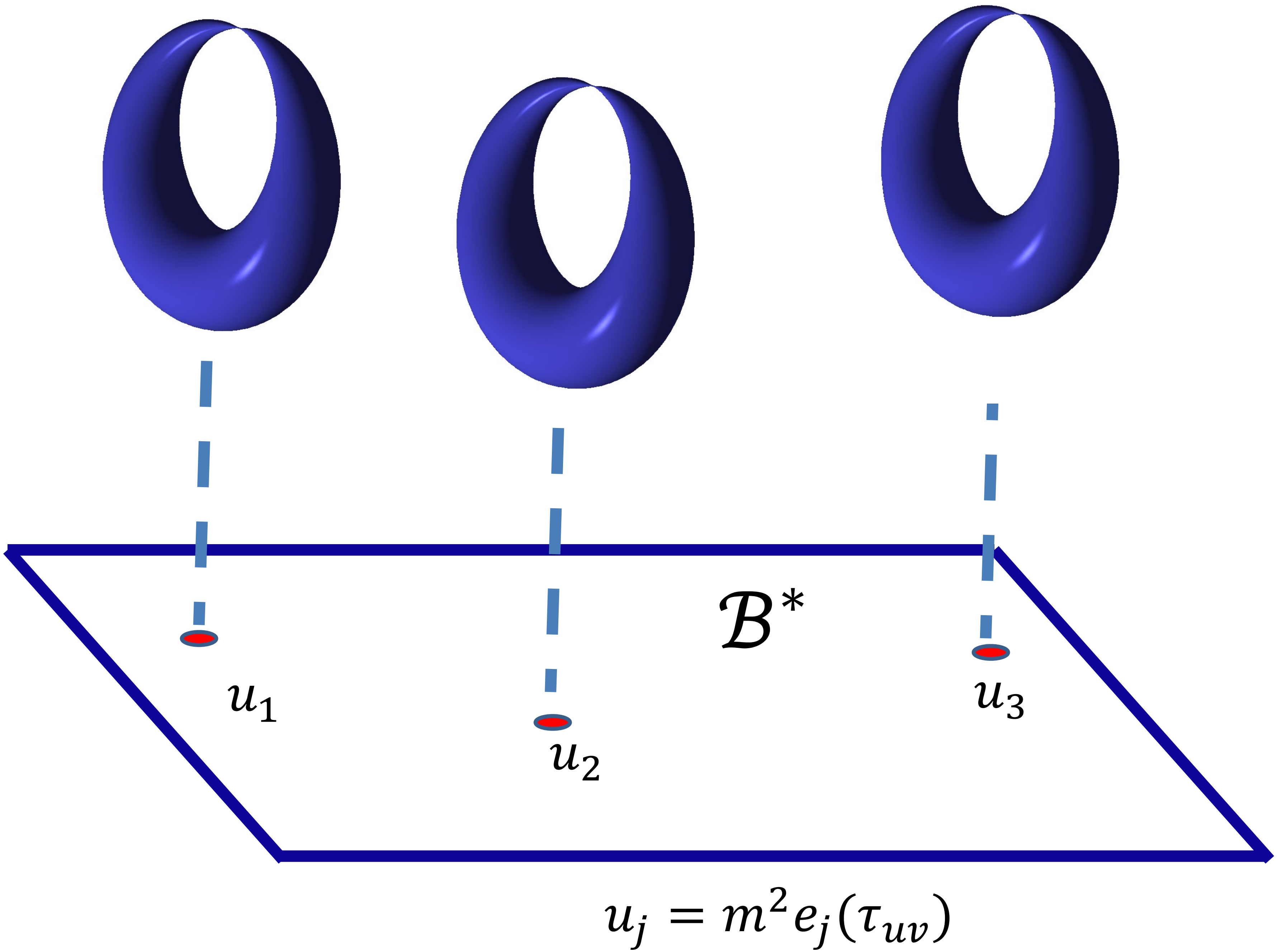
Seiberg-Witten Review – 1/6

$$E_u \quad y^2 = \prod_{i=1}^{3}(x - \alpha_i) \qquad \alpha_i = u\, e_i(\tau_{uv}) + m^2 e_i(\tau_{uv})^2$$

$e_i(\tau_{uv})$ half-periods of $E_{\tau_{uv}} = \mathbb{C}/(\mathbb{Z} + \tau_{uv}\mathbb{Z})$

Together with $\lambda \in \Omega^{1,0}(E_u)$ s.t. $\frac{d\lambda}{du} = \frac{dx}{y}$

Discriminant $\sim \eta^{24}(\tau_{uv}) \prod_{i=1}^{3}\left(u - m^2 e_i(\tau_{uv})\right)^2$


$\mathcal{B}^*$
u_1
u_2
u_3
$u_j = m^2 e_j(\tau_{uv})$

Special Geometry

$H_1(E_u;\mathbb{Z})$: Fibers of a local system over $\mathcal{B}^*$

Definition: A ``duality frame'' is a local choice of A, B –cycles

Periods of λ define homomorphism $Z_u: H_1(E_u;\mathbb{Z}) \to \mathbb{C}$

$$a(u) := \oint_A \lambda \qquad a_D(u) := \oint_B \lambda$$

Fact: There is a locally holomorphic function $\mathcal{F}(a)$

$$a_D = \frac{d\mathcal{F}}{da}$$

$$\frac{da}{du} = \oint_A \frac{dx}{y} \qquad \frac{da_D}{du} = \oint_B \frac{dx}{y} \qquad \tau = \frac{da_D}{da} = \frac{d^2\mathcal{F}}{da^2}$$

N.B.

$\tau(u, m, \tau_{uv})$ should not be confused with τ_{uv}

$$\lim_{m\to 0} \tau(u, m, \tau_{uv}) = \tau_{uv} \qquad \lim_{u\to \infty} \tau(u, m, \tau_{uv}) = \tau_{uv}$$

Weak Coupling Prepotential

$u \to \infty$: $\exists$ Canonical duality frame (``weak coupling'') :

$$\mathcal{F}(a,m) = \frac{1}{2}\tau_{uv}a^2 + $$

$$+\, m^2 \left(\log\left(\frac{2a}{m}\right) - \frac{3}{4} + \frac{3}{2}\log\left(\frac{m}{\Lambda}\right) \right)$$

$$+a^2 \sum_{n=2}^{\infty} f_n(\tau_{uv}) \left(\frac{m}{a}\right)^{2n}$$

$f_n(\tau_{uv})$: polynomials:

E_2, E_4, E_6 wt $= 2n - 2$

[Minhahan, Nemeschansky, Warner; Dhoker, Phong]

Λ – dependence: New, and important for our story.

Derived using Nekrasov partition function in

[Manschot, Moore, Xinyu Zhang 2019]

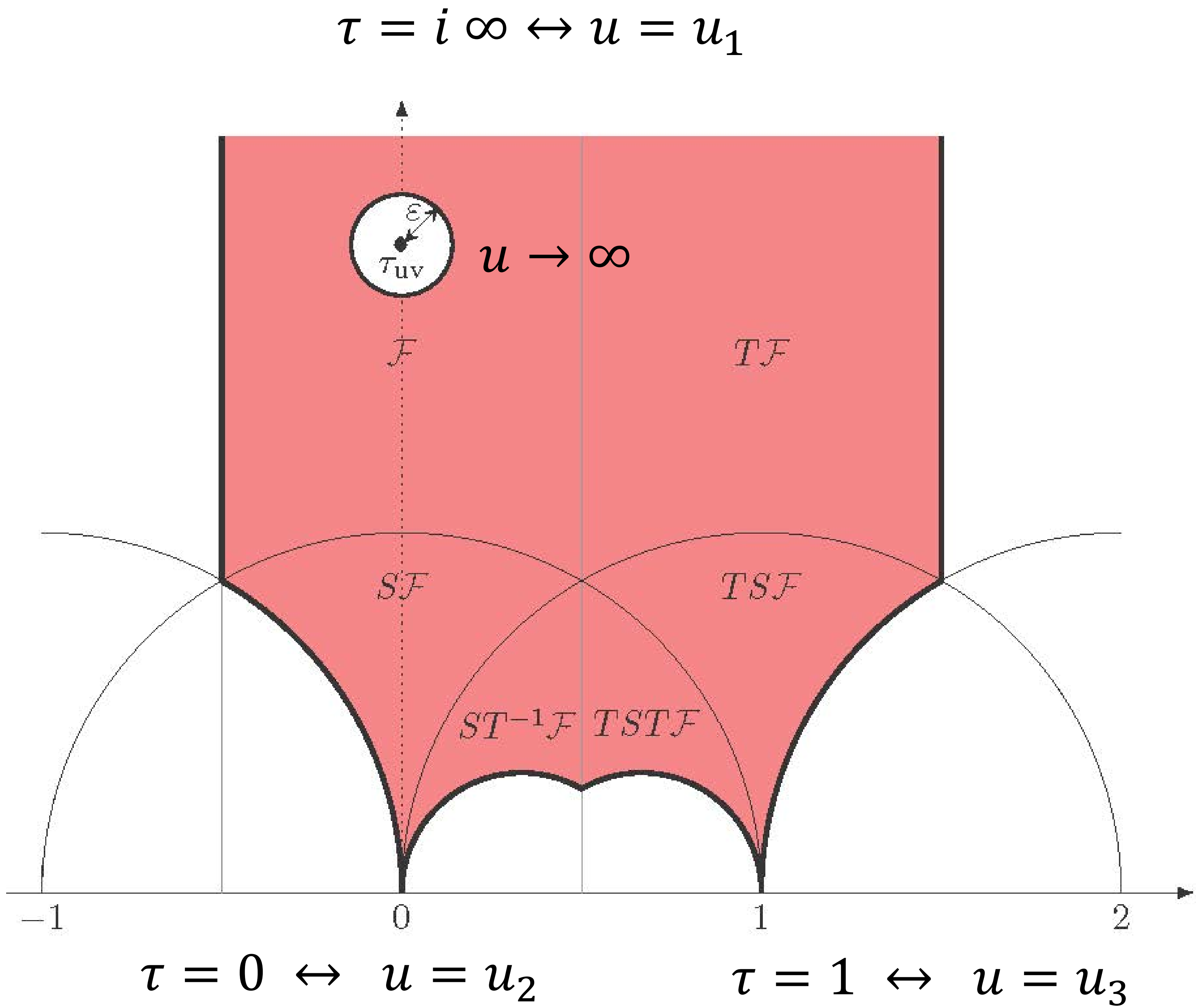
Modular Parametrization

Remarkably: One can invert these equations and express periods as bimodular forms in τ, τ_{uv}

$$m^2 \left(\frac{da}{du}\right)^2 = \frac{\vartheta_4^4(\tau)\vartheta_3^4(\tau_{uv}) - \vartheta_3^4(\tau)\vartheta_4^4(\tau_{uv})}{\eta^6(\tau_{uv})}$$

$$m^{-2}\, u(\tau, \tau_{uv}) = \frac{e_1^2(\tau_{uv})\, e_{23}(\tau) + cycl.}{e_1(\tau_{uv}) e_{23}(\tau) + cycl}$$

$$\mathcal{B} \cong \mathcal{H}/\Gamma(2) \quad \cong \mathcal{F}\big(\Gamma(2)\big)$$



$\tau = i\infty \leftrightarrow u = u_1$
ε
$\tau_{\rm uv}$
$u \to \infty$
$\mathcal{F}$
$T\mathcal{F}$
$S\mathcal{F}$
$TS\mathcal{F}$
$ST^{-1}\mathcal{F}$
$TST\mathcal{F}$
-1
0
1
2
$\tau = 0 \leftrightarrow u = u_2$
$\tau = 1 \leftrightarrow u = u_3$

5 Coulomb Branch Integral: Measure & Evaluation

5a Seiberg-Witten Review

5b Formulating The Measure And Integral

5c Evaluation Using Mock This & That

Coulomb Branch Measure

$$Z_{\nu}^{CB} = \int_{\mathcal{F}(\Gamma(2))} \Omega$$

$$\Omega = d\tau \wedge d\bar{\tau} \;\; \mathcal{H} \; \Psi$$

Begin with Maxwell partition function Ψ

$$\Psi \sim \sum_{fluxes} e^{-S_{class.}}$$

Frame dependent.

Not holomorphic.

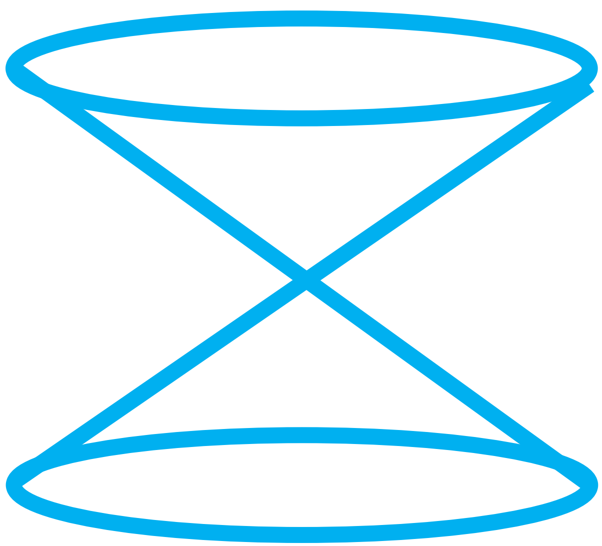
Metric dependent.

The ``Period Point'' J

$$b_2^+ > 1 \Rightarrow Z_\nu^{CB} = 0$$

$$b_2^+ = 1 \quad Z_\nu^{CB} \neq 0$$

$H^2(X;\mathbb{R})$



$*J = J$

$J^2 = 1$

$J \in$ Forward Light Cone

Maxwell Partition Function

$$\Psi^J \sim \sum_{fluxes} e^{-\int \bar{\tau}(u) f_+^2 + \tau(u)\, f_-^2}$$

Sum over the first Chern class

$$\lambda \in 2L + \bar{\nu} \ , \quad L = H^2(X;\mathbb{Z})$$

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\bar{\nu}} \partial_{\bar{\tau}} E_\lambda^J \ q^{-\frac{1}{4}\lambda^2} e^{\pi i\, \lambda \cdot z}$$

$$z = c_{uv}\, v(\tau, \tau_{uv}) + S \frac{du}{da} \qquad v := \frac{d^2 \mathcal{F}}{da\,dm}$$

Maxwell Partition Function

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \; e^{\pi i\, \lambda \cdot z}$$

$$z = c_{uv}\, v(\tau, \tau_{uv}) + S\,\frac{du}{da} \qquad v := \frac{d^2\mathcal{F}}{dadm}$$

$$E_\lambda^J = Erf(x_\lambda) \qquad Erf(x) := \int_0^x e^{-\pi t^2} dt$$

$$x_\lambda = \sqrt{Im\,\tau}\left(\lambda + \frac{Im\,z}{Im\,\tau}\right) \cdot J$$

Maxwell Coupling To $\mathfrak{s}_{uv}$

$$\sim \exp\left(\int_X v\, F_b^+ f^+ + \bar{v} F_b^- f^- \right)$$

$$v := \frac{d^2 \mathcal{F}}{da\, dm}$$

~~Folklore~~

Nonholomorphic!

$\Rightarrow$ 6d derivation will involve $\mathbb{C}^*$ –valued quadratic refinement

Remarkable Equation For $v(\tau, \tau_{uv})$

$$v := \frac{d^2\mathcal{F}}{da\,dm} = (a_D - a\tau)/m$$

$$\frac{\vartheta_2(v, 2\tau)}{\vartheta_3(v, 2\tau)} = \frac{\vartheta_2(0,2\tau_{uv})}{\vartheta_3(0,2\tau_{uv})}$$

Determines bimodular $v(\tau, \tau_{uv})$

Holomorphic Part Of Measure

$$\mathcal{H}_{bare} = A_1^{\sigma} A_2^{\chi} A_3^{c_{uv}^2}$$

Include observables:

$$\mathcal{H} = \mathcal{H}_{bare} A_4^{p} A_5^{c_{uv}\cdot S} A_6^{S^2}$$

Depend on duality frame –

- but the local system has nontrivial monodromy.

Local Topological Interactions

$$A_1^8 = \prod_i (u - u_i) =$$

$$(2m)^6 \frac{\eta(\tau_{uv})^{24}\eta(\tau)^{12}}{(\vartheta_4(\tau)^4\vartheta_3(\tau_{uv})^4 - \vartheta_3(\tau)^4\vartheta_4(\tau_{uv})^4)^3}$$

$$A_2^{-4} = \frac{\vartheta_4^4(\tau)\vartheta_3^4(\tau_{uv}) - \vartheta_3^4(\tau)\vartheta_4^4(\tau_{uv})}{m^2\eta^6(\tau_{uv})}$$

$$A_3 \;\coloneqq \exp\left(-2\,\pi\,i\,\frac{d^2\mathcal{F}}{dm^2}\right) = \left(\frac{\Lambda}{m}\right)^{\frac{3}{2}} \frac{\vartheta_1(2\tau, 2v)}{\vartheta_2^2(\tau_{uv})\vartheta_4(2\tau)}$$

With all these ingredients we can now check that the CB <u>measure</u> is indeed monodromy invariant and hence well-defined. (Nontrivial!)

The measure on the Coulomb branch is physical and must be single-valued

Even though several couplings in the LEET are multi-valued.

Even though there is no gobal duality frame.

Interesting constraint on low-energy couplings

What about <u>defining</u> the <u>integral</u> of the measure?

$$u \to u_j$$

$$\mathcal{H} \to q_j^{-\frac{\ell}{2}} F(\tau_{uv}) \left(1 + \mathcal{O}(q_j)\right)$$

$$u \to \infty \text{ i.e. } \tau \to \tau_{uv}$$

$$(\tau - \tau_{uv})^{\ell - \frac{3}{2}} \sum_{\lambda} \cdots e^{-\frac{m}{\Lambda}(\tau - \tau_{uv})^{-\frac{1}{2}} S \cdot \lambda}$$

Do the phase integral first.
(as in string theory)

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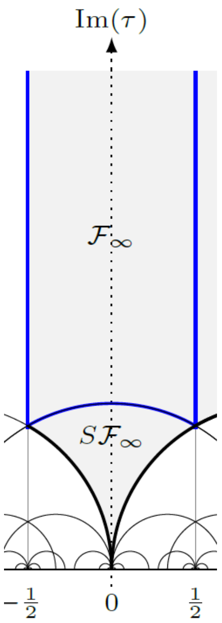
Relation To Mock Modular Forms -1.1

Z_ν^{CB} : A sum of integrals of the form

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, (Im\, \tau)^{-s} f(\tau, \bar{\tau})$$

Support of c is bounded below

$$f(\tau, \bar{\tau}) = \sum_{m-n\in\mathbb{Z}} c(m,n) q^m \bar{q}^n$$



Strategy: Find $\hat{h}(\tau, \bar{\tau})$ such that

$$\partial_{\bar{\tau}} \hat{h} = (Im\, \tau)^{-s} f(\tau, \bar{\tau})$$

$\hat{h}\,(\tau, \bar{\tau})$ is modular of weight (2,0)

Relation To Mock Modular Forms – 1.2

We choose an explicit solution

$$\partial_{\bar{\tau}} R = (Im\tau)^{-s} f(\tau, \bar{\tau})$$

vanishing exponentially fast at $Im\tau \to \infty$

R is not modular, but it's failure to be modular must be holomorphic.

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) + R$$

$h(\tau)$: mock modular form

$$h(\tau) = \sum_{m \in \mathbb{Z}} d(m) q^m \qquad q = e^{2\pi i \tau}$$

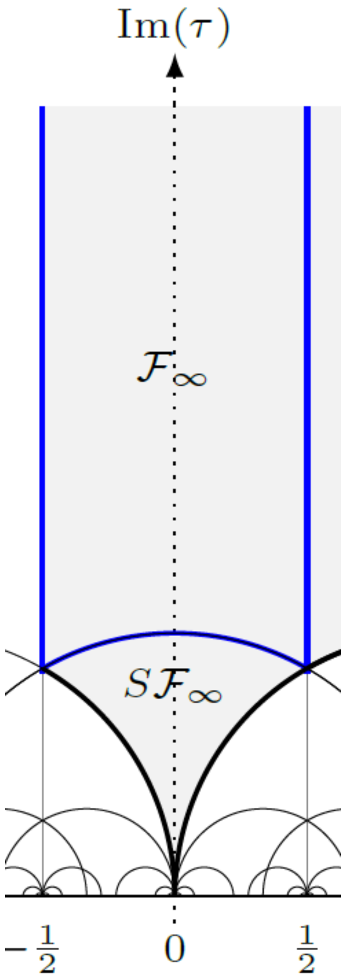
Doing The Integral

$$I_f = \int_{\mathcal{F}_\infty} d\tau d\bar{\tau}\, y^{-s}\, f(\tau,\bar{\tau})$$

$$\partial_{\bar{\tau}}\hat{h} = y^{-s}\, f(\tau,\bar{\tau})$$

$$I_f = d(0)$$

$$h(\tau) = \sum_{m\in\mathbb{Z}} d(m) q^m$$



Note: $d(0)$ undetermined by diffeq but fixed by the modular properties: Subtle!

Evaluation Of CB Integral ?

$$Z_\nu^{CB} = \int_{\mathcal{F}(\Gamma(2))} \Omega \qquad \Omega = d\tau \wedge d\bar{\tau} \ \mathcal{H} \ \Psi_\nu^J$$

$$\Psi_\nu^J = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_\lambda^J \ q^{-\frac{1}{4}\lambda^2} \ e^{-2\pi i \, \lambda \cdot z}$$

$$z = c_{uv} \, v(\tau, \tau_{uv}) + S \, \frac{du}{da}$$

$$\Omega = d\,\Lambda \qquad \Lambda = d\tau \, \mathcal{H} \ \hat{G} \qquad \Psi_\nu^J = \partial_{\bar{\tau}} \, \hat{G}$$

Evaluation Of CB Integral ?

$$\Psi_{\nu}^{J} = \sum_{\lambda \in 2L+\nu} \partial_{\bar{\tau}} E_{\lambda}^{J} \; q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \, \lambda \cdot z}$$

$$\Psi_{\nu}^{J} = \partial_{\bar{\tau}} \hat{G}$$

$$\hat{G} = \sum_{\lambda \in 2L+\nu} E_{\lambda}^{J} \; q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \lambda \cdot z}$$

??? NO!!! $\lim_{|\lambda_{+}| \to +\infty} E_{\lambda}^{J} = \pm 1$

Evaluating Difference Of CB Integrals

$$\Psi^{J_1} - \Psi^{J_2} = \partial_{\bar{\tau}}\, \hat{G}^{J_1,J_2}$$

$$\widehat{G^{J_1,J_2}} = \sum_{\lambda \in 2L+\nu} E_\lambda^{J_1,J_2} q^{-\frac{1}{4}\lambda^2} e^{-2\pi i \lambda \cdot z}$$

$$E_\lambda^{J_1,J_2} = \ Erf\left(x_\lambda^{J_1}\right) - Erf\left(x_\lambda^{J_2}\right)$$

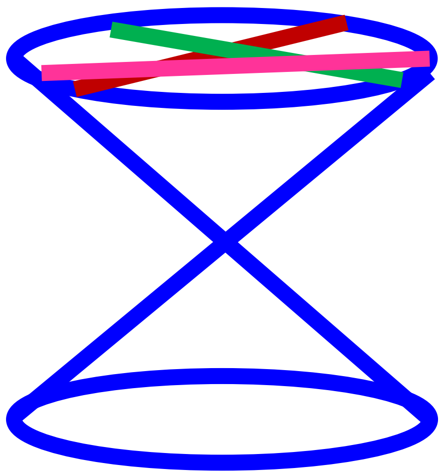
Converges nicely!

$\Rightarrow$ Can use this to evaluate the difference

$Z_\nu^{CB,J_1} - Z_\nu^{CB,J_2}$ by a sum of residues.

Metric & Holomorphic Anomaly

Wall crossing involves modular functions



For the boundary at $u \to \infty$ the modular parameter $\tau \to \tau_{uv}$. This leads to ***continuous*** metric dependence.

Closely related: Nonholomorphic in τ_{uv}

$$\frac{\partial}{\partial \bar{\tau}_{\mathrm{uv}}} \mathrm{Z}_{\nu}^{\mathrm{CB}} = \mathrm{y}^{-\frac{3}{2}} \eta^{-2\chi} \sum_{\lambda} K[\lambda_+, \lambda_-] \overline{q_{uv}}^{\lambda_+^2} \mathrm{q}_{uv}^{-\lambda_-^2}$$

The Special Period Point

For any manifold with $b_2^+ = 1$

$\exists$ special J_0 such that $\Psi_\nu^{J_0}$ factorizes:

$$\Psi_\nu^{J_0} = f_\nu \; \Theta_{L_-}(\tau, z)$$

$$f_\nu = \sum_{\lambda \in 2\mathbb{Z}+\nu} \partial_{\bar{\tau}} E_\lambda^J \; q^{-\frac{1}{4}\lambda^2} \, e^{-2\pi i \, \lambda\cdot z}$$

Measure As A Total Derivative

$$\Omega = d\,\Lambda \qquad \Lambda = d\tau\,\mathcal{H}\;\hat{G}$$

Where we can write $\hat{G}$ explicitly so that Λ is:

1. Well-defined
2. Nonsingular away from $\tau \in \{0, 1, i\infty, \tau_{uv}\}$
3. Good q_i expansion near cusps

Harmonic Jacobi-Maass Forms

These conditions determine $\hat{G}$ uniquely.

Modular completion of an Appel-Lerche sum

$$F(\tau,z) \sim \frac{e^{-2\pi i\, z}}{\vartheta_4(2\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n^2-\frac{1}{4}}}{1+e^{4\pi i\, z} q^{2n-1}}$$

$$z = c_{uv}\, v(\tau,\tau_{uv}) + S\,\frac{du}{da}(\tau,\tau_{uv})$$

Technical Comment On Poles

Λ must be nonsingular away from $\tau \in \{0,1, i\,\infty, \tau_{uv}\}$

$F(\tau, z)$ has poles for

$$z = \frac{1}{4}(a + b\,\tau) \qquad a = 1\, mod\, 2 \qquad b = 2\;\; mod\, 4$$

But for $z = c_{uv}\, v(\tau, \tau_{uv})$ one can add a meromorphic form to $F(\tau, z)$ to cancel unwanted poles

The Integral Is a Mock Modular Form

For $\mathfrak{s}_{uv} = \mathfrak{s}(\mathcal{J})$ we find $Z_{\nu}^{CB,J_0} =$

$$\hat{g}_{\nu}(\tau_{uv}, \bar{\tau}_{uv})\ \Theta_{L_-}(\tau_{uv}) / \eta^{2\chi}(\tau_{uv})$$

$$g_{\nu} = 3 \sum_{n \geq 0} H(4n - 2\mu) q_{uv}^{n - \frac{\nu}{2}}$$

… but other $\mathfrak{s}$ generalize …

For $\mathbb{CP}^2$ & $c_{uv} = 1$ (acs $\Rightarrow c_{uv} = 3$)

$$\frac{\partial}{\partial \bar{\tau}_{uv}} Z_\nu = y_{uv}^{-\frac{3}{2}} \eta^{-2} \widehat{E_2} \, \Theta_\nu(-\bar{\tau}_{uv})$$

Including Observables

n	Hol. part of $\eta(\tau_{\mathrm{uv}})^6\,\Phi^{\mathbb{P}^2}_{1/2}[u^n_{\mathrm{D}}]$
0	$q_{\mathrm{uv}}^{3/4} + 3\,q_{\mathrm{uv}}^{7/4} + 3\,q_{\mathrm{uv}}^{11/4} + 6\,q_{\mathrm{uv}}^{15/4} + \dots$
1	$-m^2\left(\frac{3}{2}\,q_{\mathrm{uv}}^{7/4} + 12\,q_{\mathrm{uv}}^{11/4} + 35\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
2	$m^4\left(\frac{19}{16}\,q_{\mathrm{uv}}^{7/4} + \frac{31}{2}\,q_{\mathrm{uv}}^{11/4} + 89\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
3	$-m^6\left(\frac{15}{4}\,q_{\mathrm{uv}}^{11/4} + \frac{971}{16}\,q_{\mathrm{uv}}^{15/4} + \dots\right)$
4	$m^8\left(\frac{85}{32}\,q_{\mathrm{uv}}^{\frac{11}{4}} + \frac{15151}{256}\,q_{\mathrm{uv}}^{\frac{15}{4}} + \dots\right)$

1. Motivation
2. Preliminaries And Background
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Contributions Of The Cusps u_j

Near each cusp u_j , $j = 1,2,3$

the description of the vacuum changes:

We have a U(1) VM coupled to a charge 1 HM.

(In the appropriate duality frame) [Seiberg-Witten 94]

There is a separate contribution to the path integral coming from the path integral of these three LEET.

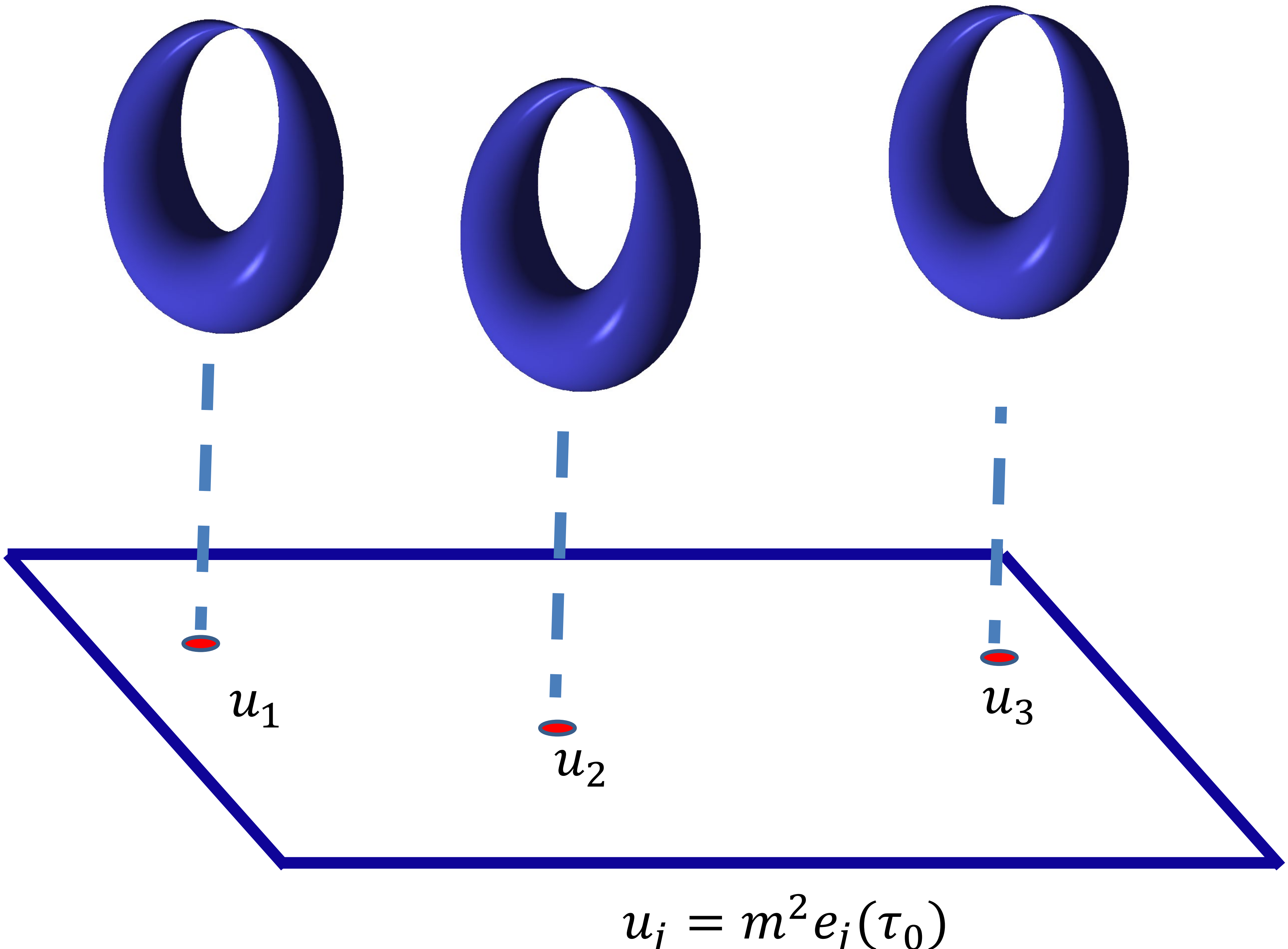
We add the contributions, because we sum over vacua:

$$Z_\nu = Z_\nu^{CB} + \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

When $b_2^+ > 1$ $\mathrm{Z}_\nu^{\mathrm{CB}}$ vanishes –
- we get true topological invariants:

$$Z_\nu = \sum_{j=1}^{3} Z_{\nu,j}^{SW}$$

So it is quite interesting to determine the three effective actions



u_1
u_2
u_3
$u_j = m^2 e_j(\tau_0)$

MW97: The behavior of the CB integral at u_j uniquely fixes $Z_{\nu,j}^{SW}$

Determination Of Effective Action

$$Z_{\nu,j}^{SW} = \sum_{c_{ir}} SW(c_{ir}) \prod_{n=1}^{12} F_{n,j}(\tau_{uv};t)^{\Delta_n}$$

$$\chi_h = \frac{1}{4}(\chi + \sigma) \qquad \lambda = 2\chi + 3\sigma \qquad \ell = \frac{c_{uv}^2 - \lambda}{8}$$

$$x^2 := \left(\frac{c_{uv} - c_{ir}}{2}\right)^2 \qquad S \cdot c_{ir} \qquad S^2 \qquad S \cdot c_{uv}$$

$$\boxed{\begin{aligned} Z_{SW,1,\boldsymbol{\mu}}(\tau_{\rm uv}) &= \left(-2\,\eta(2\tau_{\rm uv})^{12}\right)^{-\chi_{\rm h}} \left(4\,t^3\,\eta(\tau_{\rm uv})^4\,\vartheta_3(2\tau_{\rm uv})^4\right)^{-\ell} \left(\frac{\eta(\tau_{\rm uv})^2}{\vartheta_3(2\tau_{\rm uv})}\right)^{\lambda} \\ &\times \sum_{\boldsymbol{x}=2\boldsymbol{\mu} \mod 2L} {\rm SW}(c_{\rm ir}) \left(\frac{\vartheta_3(2\tau_{\rm uv})}{\vartheta_2(2\tau_{\rm uv})}\right)^{\boldsymbol{x}^2}. \end{aligned}}$$

$$\mathbf{+}$$

$$\boxed{\begin{aligned} Z_{SW,2,\boldsymbol{\mu}}(\tau_{\rm uv}) &= 2\,\left(2\,\eta(\tau_{\rm uv}/2)^{12}\right)^{-\chi_{\rm h}} \left(-t^3\,\eta(\tau_{\rm uv})^4\,\vartheta_3(\tau_{\rm uv}/2)^4\right)^{-\ell} \left(\frac{2\eta(\tau_{\rm uv})^2}{\vartheta_3(\tau_{\rm uv}/2)}\right)^{\lambda} \\ &\times \sum_{\boldsymbol{x}\in L} {\rm SW}(c_{\rm ir})\,(-1)^{2B(\boldsymbol{x},\boldsymbol{\mu})} \left(\frac{\vartheta_3(\tau_{\rm uv}/2)}{\vartheta_4(\tau_{\rm uv}/2)}\right)^{\boldsymbol{x}^2}. \end{aligned}}$$

$$\mathbf{+}$$

$$\boxed{\begin{aligned} Z_{SW,3,\boldsymbol{\mu}}(\tau_{\rm uv}) &= 2\,e^{2\pi i\boldsymbol{\mu}^2} \left(-t^3\,\eta(\tau_{\rm uv})^4\,\vartheta_3((\tau_{\rm uv}+1)/2)^4\right)^{-\ell} \\ &\times \left(2\,\eta((\tau_{\rm uv}+1)/2)^{12}\right)^{-\chi_{\rm h}} \left(\frac{2\,\eta(\tau_{\rm uv})^2}{\vartheta_3((\tau_{\rm uv}+1)/2)}\right)^{\lambda} \\ &\times \sum_{\boldsymbol{x}\in L} {\rm SW}(c_{\rm ir})\,(-1)^{2B(\boldsymbol{x},\boldsymbol{\mu})} \left(\frac{\vartheta_3((\tau_{\rm uv}+1)/2)}{\vartheta_4((\tau_{\rm uv}+1)/2)}\right)^{\boldsymbol{x}^2}. \end{aligned}}$$

Including Observables

j	$B(S, c_{\rm ir})$: $\log(\mathcal{S}_j)$	S^2: $\log(\mathcal{T}_j)$	$B(S, c_{\rm uv})$: $\log(\mathcal{U}_j)$
1	$-\frac{it}{4}\vartheta_3^2\vartheta_4^2$	$\frac{t^2}{144}\left[-3\vartheta_3^4\vartheta_4^4 + E_2(\vartheta_3^4+\vartheta_4^4) + E_2^2\right]$	$\frac{t}{12i}[\vartheta_3^4 + \vartheta_4^4 + 2E_2]$
2	$-\frac{it}{4}\vartheta_2^2\vartheta_3^2$	$\frac{t^2}{144}\left[-3\vartheta_2^4\vartheta_3^4 - E_2(\vartheta_2^4+\vartheta_3^4) + E_2^2\right]$	$\frac{t}{12i}[-\vartheta_2^4 - \vartheta_3^4 + 2E_2]$
3	$-\frac{t}{4}\vartheta_2^2\vartheta_4^2$	$\frac{t^2}{144}\left[3\vartheta_2^4\vartheta_4^4 + E_2(\vartheta_2^4-\vartheta_4^4) + E_2^2\right]$	$\frac{t}{12i}[\vartheta_2^4 - \vartheta_4^4 + 2E_2]$

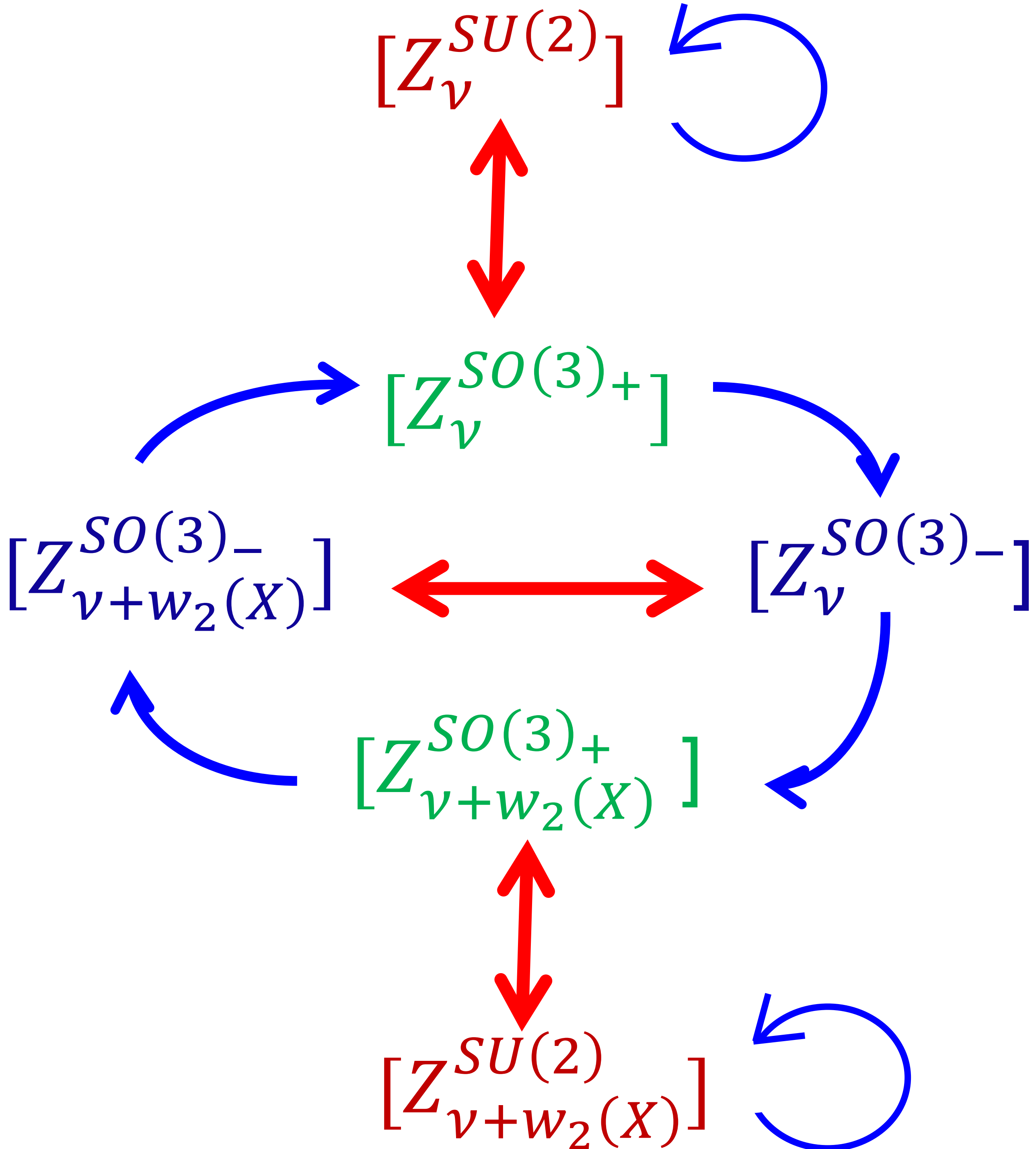
Full Modular Transformation Law

$$Z_\nu\left(\tilde{p}, \tilde{S}, \frac{a\tau_{uv} + b}{c\tau_{uv} + d}\right) = (c\tau_{uv} + d)^w \sum_\mu B_{\mu,\nu}(\gamma) Z_\mu(p, S, \tau_{uv})$$

$$w = -\frac{\chi}{2} - 4\ell$$

$$\tilde{S} = \frac{S}{(c\tau + d)^2}$$

$$\tilde{p} = \frac{1}{(c\tau + d)^2}\left(p - 2\pi i\, c\,(c\tau + d) S^2\right)$$



$[Z_{\nu}^{SU(2)}]$
$[Z_{\nu}^{SO(3)_+}]$
$[Z_{\nu+w_2(X)}^{SO(3)_-}]$
$[Z_{\nu}^{SO(3)_-}]$
$[Z_{\nu+w_2(X)}^{SO(3)_+}]$
$[Z_{\nu+w_2(X)}^{SU(2)}]$

CONCLUDING REMARKS

Relation To Previous Results

For $\mathfrak{s}(\mathcal{I})$ and $m \to 0$ we recover and generalize formulae of [VW;DPS] for VW invariants.

For $c_{uv} = 0$ we recover formulae of Labastida-Lozano

For $m \to \infty$, $\mathrm{q_{uv}} \to 0$ <u>*after suitable renormalization*</u> we recover the ``Witten conjecture'' for the Donaldson invariants in terms of the Seiberg-Witten invariants.

Recover and generalize explicit evaluation of u-plane integral for $\mathbb{CP}^2, S^2 \times S^2$ of Moore-Witten, Malmendier-Ono

A generalization and unification of the 1990's formulae:

Recent Discussions Of Holomorphic Anomaly

Duality and Mock Modularity

Atish Dabholkar,[1] **Pavel Putrov,**[1] **Edward Witten**[2]

Gauge theories on compact toric manifolds

Giulio Bonelli[‡1], Francesco Fucito[† 2], Jose Francisco Morales [† 3], Massimiliano Ronzani [4],
Ekaterina Sysoeva[‡ 5], Alessandro Tanzini[‡ 6]

VIRTUAL REFINEMENTS OF THE VAFA-WITTEN FORMULA

LOTHAR GÖTTSCHE AND MARTIN KOOL

with an appendix by Lothar Göttsche and Hiraku Nakajima

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

REFINED $\mathrm{SU}(3)$ VAFA-WITTEN INVARIANTS AND MODULARITY

LOTHAR GÖTTSCHE AND MARTIJN KOOL

VIRTUAL SEGRE AND VERLINDE NUMBERS OF PROJECTIVE SURFACES

L. GÖTTSCHE AND M. KOOL

SHEAVES ON SURFACES AND VIRTUAL INVARIANTS

L. GÖTTSCHE AND M. KOOL

$$\boxed{\begin{aligned} Z_{SW,1,\boldsymbol{\mu}}(\tau_{\rm uv}) &= \left(-2\,\eta(2\tau_{\rm uv})^{12}\right)^{-\chi_{\rm h}} \left(4\,t^3\,\eta(\tau_{\rm uv})^4\,\vartheta_3(2\tau_{\rm uv})^4\right)^{-\ell} \left(\frac{\eta(\tau_{\rm uv})^2}{\vartheta_3(2\tau_{\rm uv})}\right)^{\lambda} \\ &\times \sum_{\boldsymbol{x}=2\boldsymbol{\mu} \mod 2L} \mathrm{SW}(c_{\rm ir}) \left(\frac{\vartheta_3(2\tau_{\rm uv})}{\vartheta_2(2\tau_{\rm uv})}\right)^{\boldsymbol{x}^2}. \end{aligned}}$$

$$\boldsymbol{+}$$

$$\boxed{\begin{aligned} Z_{SW,2,\boldsymbol{\mu}}(\tau_{\rm uv}) &= 2\,\left(2\,\eta(\tau_{\rm uv}/2)^{12}\right)^{-\chi_{\rm h}} \left(-t^3\,\eta(\tau_{\rm uv})^4\,\vartheta_3(\tau_{\rm uv}/2)^4\right)^{-\ell} \left(\frac{2\eta(\tau_{\rm uv})^2}{\vartheta_3(\tau_{\rm uv}/2)}\right)^{\lambda} \\ &\times \sum_{\boldsymbol{x}\in L} \mathrm{SW}(c_{\rm ir})\,(-1)^{2B(\boldsymbol{x},\boldsymbol{\mu})} \left(\frac{\vartheta_3(\tau_{\rm uv}/2)}{\vartheta_4(\tau_{\rm uv}/2)}\right)^{\boldsymbol{x}^2}. \end{aligned}}$$

$$\boldsymbol{+}$$

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Comparing With GKNW

X: General four-manifold admitting acs.

X: Projective algebraic surface

Arbitrary spin-c structure $\mathfrak{s}_{uv}$

Canonical spin-c structure determined by the complex structure

?!?!?!?!?

Further refinement of invariants computing χ_y -genus

VERLINDE FORMULAE ON COMPLEX SURFACES I: K-THEORETIC INVARIANTS

L. GÖTTSCHE, M. KOOL, AND R. A. WILLIAMS

Conjecture 1.2. *Let S be a smooth projective surface with $p_g(S) > 0$, $b_1(S) = 0$, and $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Let* vd *be defined by (1). Then $y^{-\frac{\mathrm{vd}}{2}}\chi^{\mathrm{vir}}_{-y}(M_S^H(2,c_1,c_2),\mu(L))$ equals the coefficient of x^{vd} of*

$$
\begin{aligned}
&4\left(\frac{1}{2}\prod_{n=1}^{\infty}\frac{1}{(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{\chi(\mathcal{O}_S)}\left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{\frac{1}{2}})}\right)^{K_S^2}\\
&\cdot\left(\prod_{n=1}^{\infty}\left(\frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{n^2}\right)^{\frac{L^2}{2}}\left(\prod_{n=1}^{\infty}\left(\frac{1-x^{2n}y^{-1}}{1-x^{2n}y}\right)^{n}\right)^{LK_S}\\
&\cdot\sum_{a\in H^2(S,\mathbb{Z})}(-1)^{c_1a}\,\mathrm{SW}(a)\left(\frac{\theta_3(x,y^{\frac{1}{2}})}{\theta_3(-x,y^{\frac{1}{2}})}\right)^{aK_S}\\
&\cdot\left(\prod_{n=1}^{\infty}\left(\frac{(1-x^{2n-1}y^{\frac{1}{2}})(1+x^{2n-1}y^{-\frac{1}{2}})}{(1-x^{2n-1}y^{-\frac{1}{2}})(1+x^{2n-1}y^{\frac{1}{2}})}\right)^{2n-1}\right)^{\frac{L(K_S-2a)}{2}}.
\end{aligned}
$$

Comparison (t large) shows:

$$\int_{\mathcal{M}_{ab}} \cdots \quad \sim \quad Z^{SW}_{\nu,1} \qquad u_1 \sim \frac{m^2}{6}$$

$$\int_{\mathcal{M}_{inst}} \cdots \quad \sim \quad Z^{SW}_{\nu,2} + Z^{SW}_{\nu,3} \qquad u_2 \sim -\frac{m^2}{12} + \Lambda_0^2$$

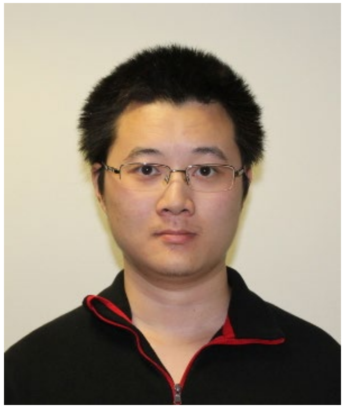
$$u_3 \sim -\frac{m^2}{12} - \Lambda_0^2$$

Showing that the instanton contribution alone cannot be S-duality covariant.

Future Directions

X complex: Compute Refined Versions From Physics

With X. Zhang:
Interesting generalization to 5d SYM



Derivation from 6d (2,0) theory?

Generalization of these techniques to class S

1 Motivation

2 Preliminaries And Background

3 Summary Of Main Results

4 Mathematical Formulation Of Partition Function

5 Coulomb Branch Integral: Measure & Evaluation

6 LEET Near Cusps & Explicit Results

FAQ 1

SW94 showed the SW curve for N=2*
is invariant under S-duality.
<u>What about the partition functions?</u>

*Partition functions are suitably
S-duality covariant with some
interesting nontrivial details.*

See previous talks for details.

FAQ 2:

SW94: Set $q_{uv} m^4 = \Lambda_0^4$ fixed and take $m \to \infty$. This gives the $N_f = 0$ SW curve.

<u>Does limit of Z_ν exist and give the DW partition function?</u>

Yes, sort of.

The limit does not exist.

But Z_ν is naturally a sum of three terms.

Throwing one away, and renormalizing the others, there is a well-defined limit.

It reproduces the DW function.

(With an interesting orientation issue.)

FAQ 3:

$$S = \int_X \tau_{uv}\, Tr\,(F \wedge F\,) \;+\; Q(*)$$

Is the partition function metric independent and holomorphic in τ_{uv} ?

Yes, *when* $b_2^+ > 1$.

Absolutely not when $b_2^+ = 1$.
In fact, most correlators vary continuously with metric.

$$\langle \mathcal{O}(S)^r \mathcal{O}(p)^n \rangle$$

Varies continuously with metric when

$$\begin{cases} \ell \leq n + r/2, & r \text{ even}, \\ \ell \leq n + (r+1)/2, & r \text{ odd}. \end{cases}$$

$$\ell = \frac{c(\mathfrak{s})^2 - 2\chi - 3\,\sigma}{8} \in \mathbb{Z}$$

(i.e. all but finitely many correlators)

We derive very explicit formulae for the holomorphic and metric anomalies.

FAQ 4:

Again using Q –symmetry the coupling to the background spin-c connection is expected to be holomorphic in $u = \langle Tr(\phi^2)\rangle$

$$S_{LEET} = \int_X \kappa_1(u)\, f \wedge F_{\mathfrak{s}} + \kappa_2(u) F_{\mathfrak{s}}^2$$

Shapere & Tachikawa

$F_{\mathfrak{s}}$: Fieldstrength of background spin-c connection

f: dynamical U(1) fieldstrength of the Coulomb branch LEET

Yes, for $\kappa_2(u)$. No for $\kappa_1(u)$.

This has important implications for the class S generalization

We also give explicit formulae for these couplings.