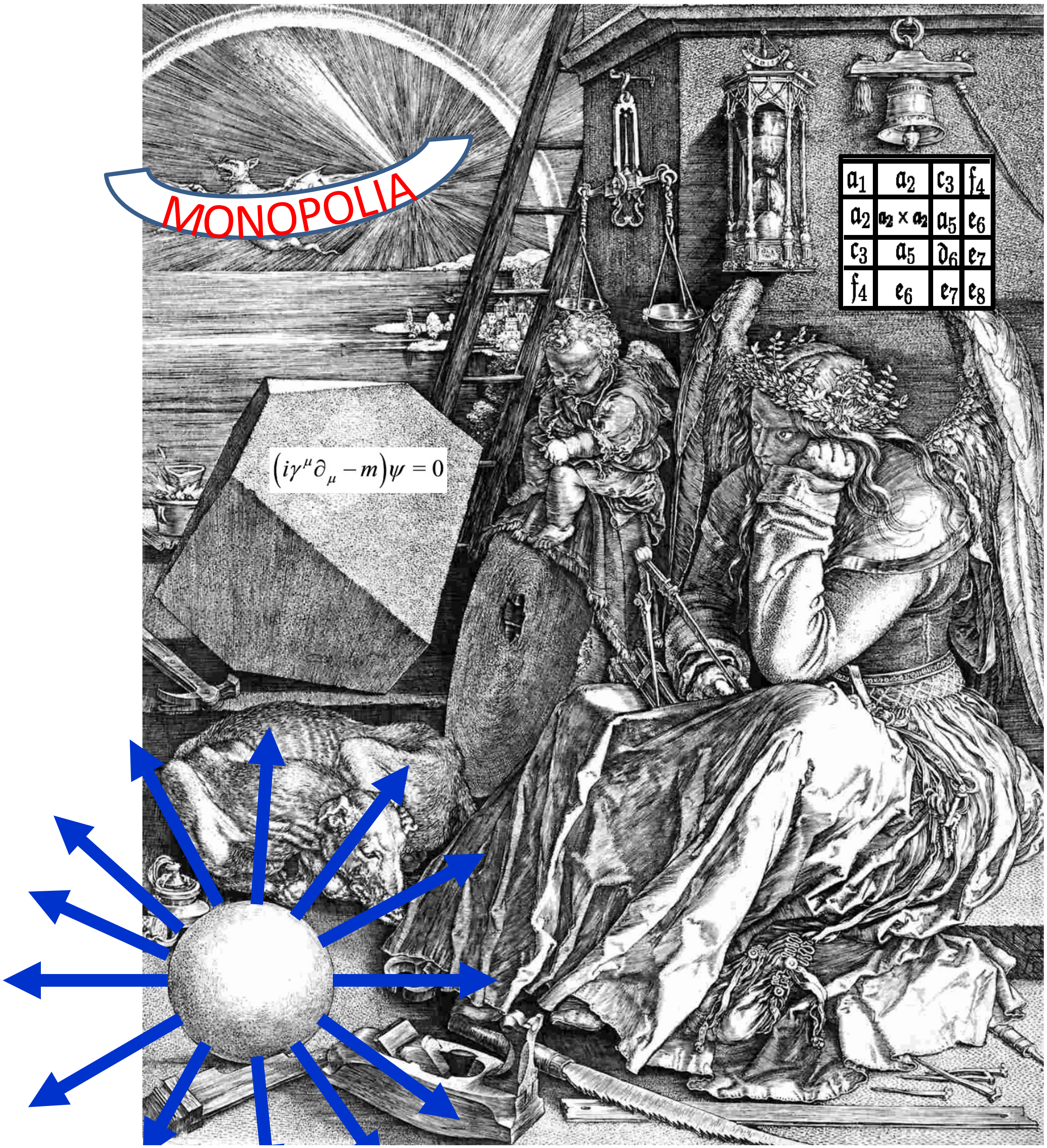

MONOPOLIA
$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0$
a_1 a_2 c_3 f_4
a_2 $a_2\times a_2$ a_5 e_6
c_3 a_5 d_6 e_7
f_4 e_6 e_7 e_8

Robbert Dijkgraaf's Thesis Frontispiece

Late Edition

``Making the World a Stabler Place''

The BPS Times

Today, BPS degeneracies, wall-crossing formulae. **Tonight**, Sleep. **Tomorrow**, K3 metrics, BPS algebras, p.B6

Est. 1975 www.bpstimes.com SEOUL, FRIDAY, JUNE 28, 2013 ₩ 2743.75

INVESTIGATORS SEE NO EXOTICS IN PURE SU(N) GAUGE THEORY

Use of Motives Cited

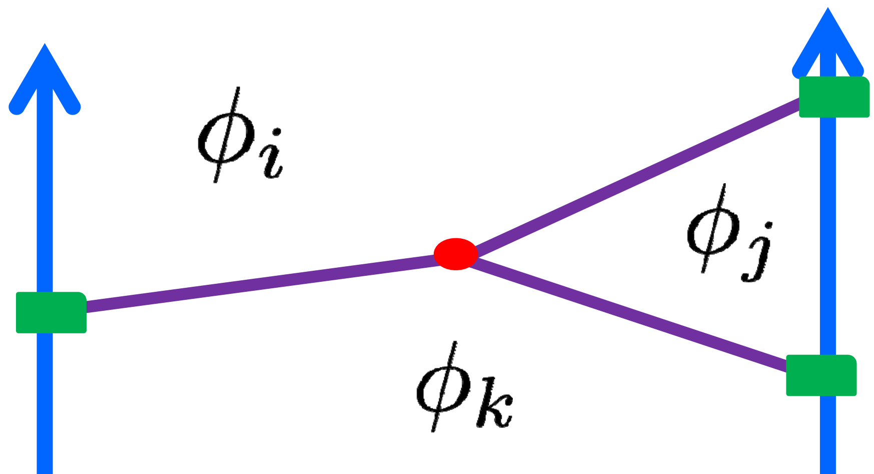
By E. Diaconescu, et. al.

RUTGERS – An application of results on the motivic structure of quiver moduli spaces has led to a proof of a conjecture of GMN. p.A12

Semiclassical, but Framed, BPS States

By G. Moore, A. Royston, and D. Van den Bleeken

RUTGERS – Semiclassical framed BPS states have been constructed as



An instanton correction to the differential

Operadic Structures Found in Infrared Limit of 2D LG Models

NOVEL CONSTRUCTION OF d ON INTERVAL

Hope Expressed for Categorical WCF

By D. Gaiotto, G. Moore, and E. Witten

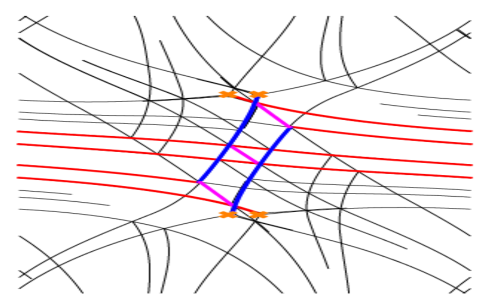
PRINCETON - A Morse-theoretic formulation of LG models has revealed ∞-structures familiar from String Field Theory. LG models are nearly trivial in

WILD WALL CROSSING IN SU(3)

EXPONENTIAL GROWTH OF Ω

By D. Galakhov, P. Longhi, T. Mainiero, G. Moore, and A. Neitzke

AUSTIN – Some strong coupling regions exhibit wild wall crossing. ``I didn't think this could happen,'' declared Prof. Nathan Seiberg of the Institute for Advanced Study in Princeton. *Continued on p.A4*



Goal Of Our Project

Recently there has been some nice progress in understanding BPS states in d=4, N=2 supersymmetric field theory:

No Exotics Theorem & Wall-Crossing Formulae

What can we learn about the differential geometry of monopole moduli spaces from these results?

1. **Brane bending and monopole moduli**
Gregory W. Moore (Rutgers U., Piscataway), Andrew B. Royston (Texas A-M), Dieter Van den Bleeken (Bogazici U.). Apr 28, 2014. 49 pp.
Published in **JHEP 1410 (2014) 157**
MIFPA-14-14
DOI: 10.1007/JHEP10(2014)157
e-Print: **arXiv:1404.7158 [hep-th]** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | E
ADS Abstract Service; Link to Article from SCOAP3
Detailed record - Cited by 7 records

2. **Parameter counting for singular monopoles on $\mathbb{R}^3$**
Gregory W. Moore (Rutgers U., Piscataway), Andrew B. Royston (Texas A-M), Dieter Van den Bleeken (Bogazici U.). Apr 22, 2014. 60 pp.
Published in **JHEP 1410 (2014) 142**
MIFPA-14-13
DOI: 10.1007/JHEP10(2014)142
e-Print: **arXiv:1404.5616 [hep-th]** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service; Link to Article from SCOAP3
Detailed record - Cited by 8 records

Papers 3 & 4 ``almost done''

1. Introduction
2. Monopoles & Monopole Moduli Space
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4. Singular Monopole Moduli: Dimension & Existence
5. Semiclassical N=2 d=4 SYM: Collective Coordinates
6. Semiclassical (Framed) BPS States
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Lie Algebra Review: 1/4

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$.

$X \in \mathfrak{g}$ is ***regular*** if Z(X) has minimal dimension.

Then Z(X) = $\mathfrak{t}$ is a Cartan subalgebra.

$T = \exp[2\pi \mathfrak{t}]$ is a Cartan subgroup.

$\Lambda_G^{\vee} := \mathrm{Hom}(T, U(1))$ character lattice

$\Lambda_G := \mathrm{Hom}(U(1), T)$ $\exp(2\pi X) = 1$

$$\Lambda_{rt} \subset \Lambda_G^{\vee} \subset \Lambda_{wt} \subset \mathfrak{t}^{\vee}$$

$$\Lambda_{cr} \subset \Lambda_G \subset \Lambda_{mw} \subset \mathfrak{t}$$

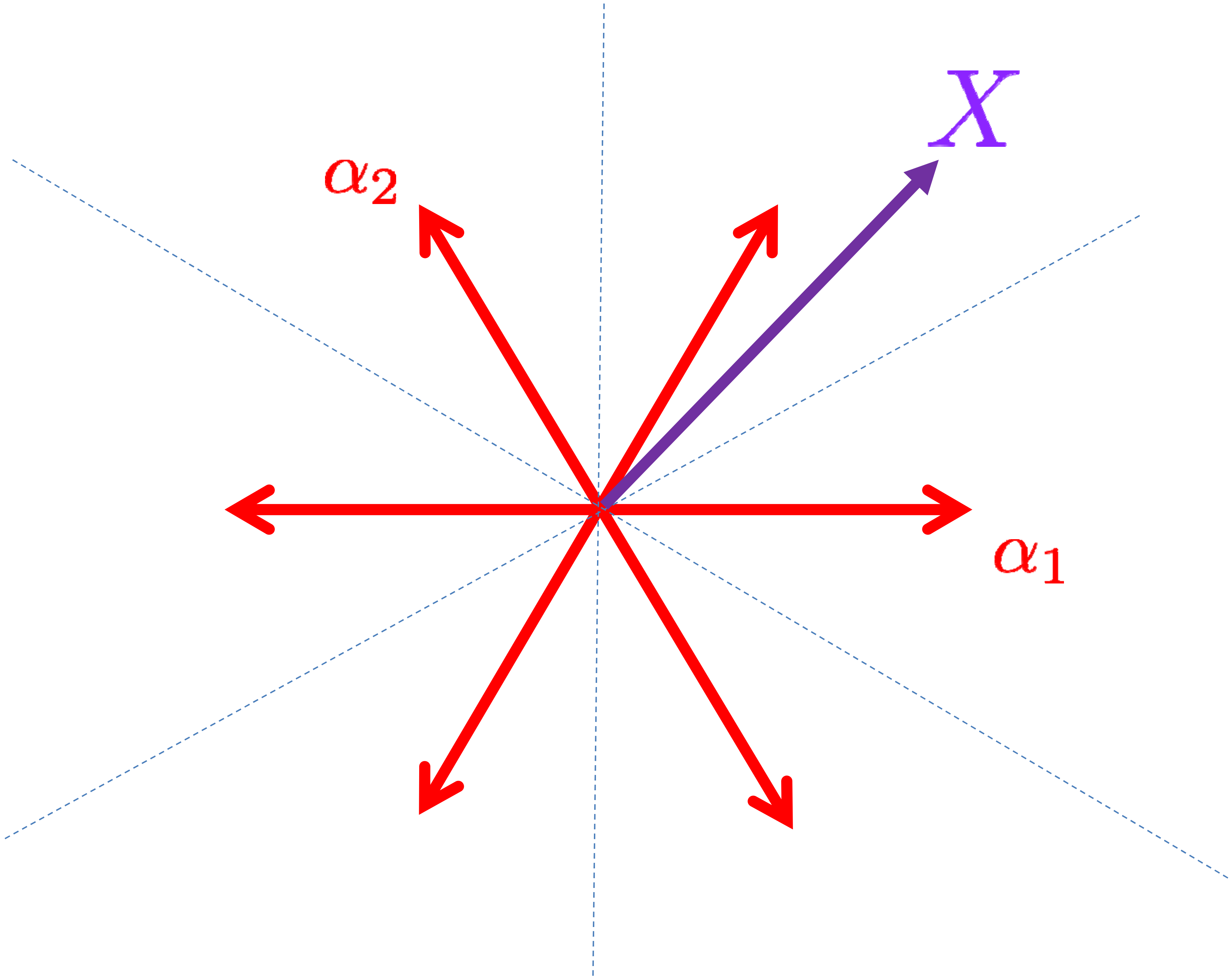
Lie Algebra Review: 2/4

Moreover, a regular element X determines a set of simple roots $\alpha_I \in \mathfrak{t}^\vee$

and simple coroots $H_I \in \mathfrak{t}$

$$\Lambda_{rt} = \oplus_I \mathbb{Z}\alpha_I \subset \mathfrak{t}^\vee$$

$$\Lambda_{cr} = \oplus_I \mathbb{Z}H_I \subset \mathfrak{t}$$



X
α_2
α_1

Examples:

$$H_1 = -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad h^1 = -\frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H_1 = -\mathrm{i}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad h^1 = -\frac{\mathrm{i}}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$H_2 = -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad h^2 = -\frac{\mathrm{i}}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Nonabelian Monopoles

Yang-Mills-Higgs system for compact simple G

(A, X) $\int_{\mathbb{R}^4} \mathrm{Tr}(F * F + DX * DX)$

$F = *DX$ on $\mathbb{R}^3$

$F = \gamma_m \mathrm{vol}(S^2) + \cdots$ $X \to X_\infty - \frac{\gamma_m}{2r} + \cdots$

$X_\infty \in \mathfrak{g}$ regular ➡ $\mathfrak{t}$ α_I H_I

$$\gamma_m \in \Lambda_{cr} \subset \mathfrak{t} \subset \mathfrak{g}$$

$$\gamma_m = \sum_{I=1}^r n_m^I H_I \qquad n_m^I \in \mathbb{Z}$$

Monopole Moduli Space

$\mathcal{M}(\gamma_m; X_\infty)$ SOLUTIONS/GAUGE TRANSFORMATIONS

Gauge transformations: $g(x) \longrightarrow 1$ for $r \longrightarrow \infty$

If $\mathcal{M}$ is nonempty then [Callias; E. Weinberg]:

$$\dim \mathcal{M}(\gamma_m; X_\infty) = 4 \sum_I n_m^I$$

Known: $\mathcal{M}$ is nonempty iff all magnetic charges nonnegative and *at least one* is positive (so $4 \leq \dim \mathcal{M}$)

$\mathcal{M}$ has a hyperkahler metric. Group of isometries with Lie algebra:

$$\mathbb{R}^3 \oplus \mathfrak{so}(3) \oplus \mathfrak{t}$$

Translations | Rotations | Global gauge transformations

Action Of Global Gauge Transformations

$H \in \mathfrak{t}$ ⟹ $G(H)$ Killing vector field on $\mathcal{M}$

$$\hat{A} = A_i dx^i + X dx^4 \qquad \hat{F} = *\hat{F}$$

Directional derivative along G(H) at $[\hat{A}] \in \mathcal{M}$ $\quad \frac{d\hat{A}}{ds} = -\hat{D}\epsilon$

$$\epsilon : \mathbb{R}^3 \to \mathfrak{g}$$

$$\lim_{x\to\infty} \epsilon(x) = H \qquad \hat{D}^2\epsilon = 0$$

Strongly Centered Moduli Space

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

Orbits of translations

Orbits of $G(X_\infty)$

$$\mathcal{M}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_0}{\mathbb{Z}}$$

Higher rank is different!

$$\mathcal{M}(\gamma_m; X_\infty) \neq \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_0}{\mathbb{Z}_r}$$

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Singular Monopoles

$$F = \gamma_m \mathrm{vol}(S^2) + \cdots \qquad X \to X_\infty - \frac{\gamma_m}{2r} + \cdots$$

$$\vec{x} \to \infty$$

<u>AND</u>

$$F = P\mathrm{vol}(S^2) + \cdots \qquad X \to -\frac{P}{2r} + \mathcal{O}(r^{-1/2})$$

$$\vec{x} \to 0$$

Use: construction of 't Hooft line defects (``line operators'')

Where Does The ‘t Hooft Charge P Live?

$$P \in \mathfrak{t} \qquad P \in \Lambda_G$$

$$\gamma_m \in \Lambda_{cr} + P$$

Example: Rank 1

$$H_1 = -\mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SU(2) Gauge Theory: Minimal P $P = \pm H_1$

SO(3) Gauge Theory: Minimal P $P = \pm \frac{1}{2} H_1 = \pm h^1$

Example: A Singular Nonabelian SU(2) Monopole

$$X = \frac{1}{2}h(r)H \qquad A = \frac{1}{2}(\pm 1 - \cos\theta)d\phi H$$

$$+\frac{1}{2}f(r)\left[e^{\pm i\phi}(-d\theta - i\sin\theta d\phi)E_+ + c.c.\right]$$

Bogomolnyi eqs:

$$f'(r) + f(r)h(r) = 0$$

$$r^2 h'(r) + f(r)^2 - 1 = 0$$

$$h(r) = m_W \coth(m_W r + c) - \frac{1}{r} \qquad f(r) = \frac{m_W r}{\sinh(m_W r + c)}$$

(‘t Hooft; Polyakov; Prasad & Sommerfield took c = 0)

c > 0 is the singular monopole: Physical interpretation?

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Singular Monopole Moduli Space

$$\overline{\underline{\mathcal{M}}}(P;\gamma_m;X_\infty)$$

SOLUTIONS/GAUGE TRANSFORMATIONS

Now g(x) must commute with P for x $\longrightarrow$ 0 .

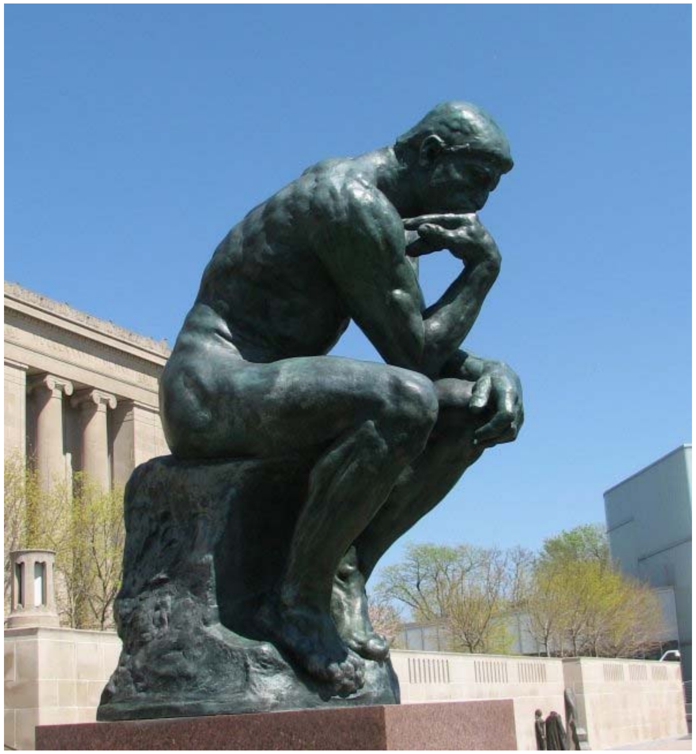
When is it nonempty?

What is the dimension?

If P = γ_m is P screened or not ?

Is the dimension zero?

or not?



Dimension Formula

Assuming the moduli space is nonempty repeat computation of Callias; E. Weinberg to find:

$$\dim \overline{\mathcal{M}} = 2\mathrm{ind}(L) = \lim_{\epsilon \to 0^+} \mathrm{Tr}\left(\frac{\epsilon}{L^\dagger L + \epsilon} - \frac{\epsilon}{L L^\dagger + \epsilon} \right)$$

For a general 3-manifold we find:

$$\dim \overline{\mathcal{M}} = \int_{M_3 - \mathcal{S}} dJ^{(\epsilon)} = 4 \sum_I \tilde{n}_m^I$$

Relative magnetic charges.

Dimension Formula

$$\dim \overline{\mathcal{M}} = 4 \sum_I \tilde{n}^I_m$$

$$\sum_I \tilde{n}^I_m H_I = \gamma_m - P^-$$

γ_m from $r \longrightarrow \infty$ and $-P^-$ from $r \longrightarrow 0$

P^- : Weyl group image such that $\langle \alpha_I, P^- \rangle \leq 0$

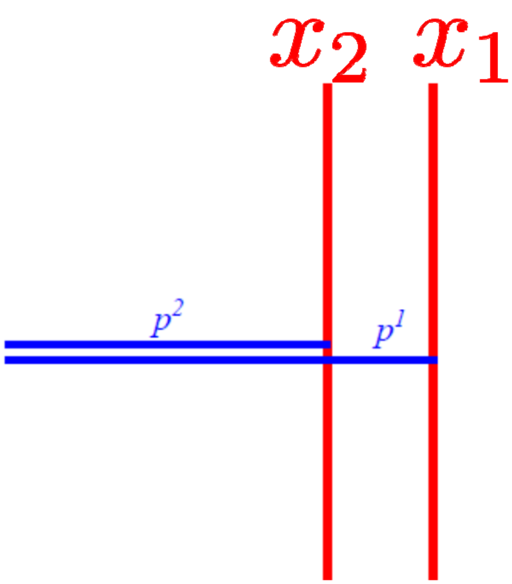
(Positive chamber determined by X_∞)

Existence

Conjecture:

$$\overline{\mathcal{M}}(P;\gamma_m;X_\infty) \neq \emptyset \quad \Longleftrightarrow \quad \forall I, \tilde{n}^I_m \geq 0$$

Intuition for relative charges comes from D-branes. Example:
Singular SU(2) monopoles from D1-D3 system

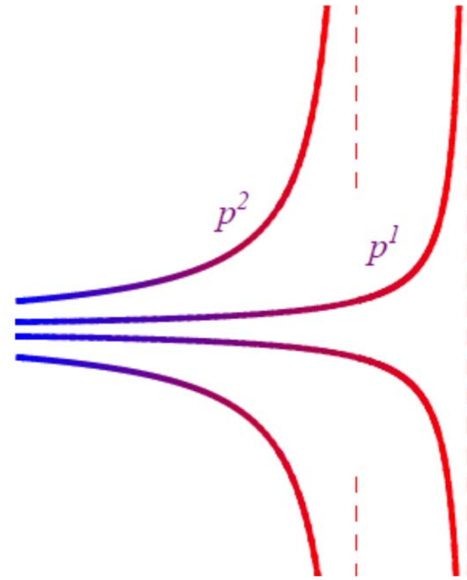


$$X = \begin{pmatrix} x_1 - \frac{p_1}{2r} & 0 \\ 0 & x_2 - \frac{p_2}{2r} \end{pmatrix}$$

$$\gamma_m = P = (p^1 - p^2)\tfrac{1}{2}H$$

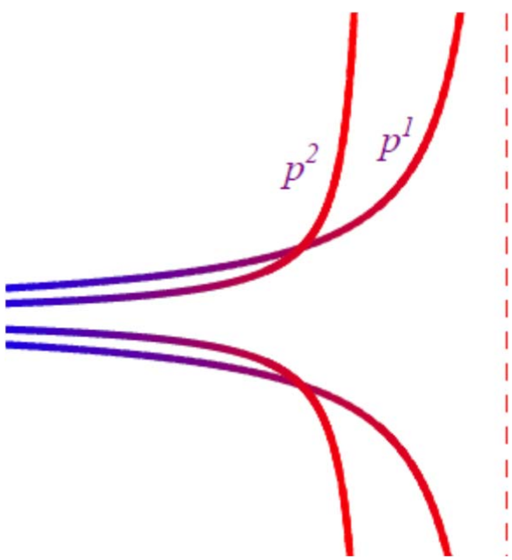
$$p^1 < p^2 \Rightarrow \gamma_m = P^-$$

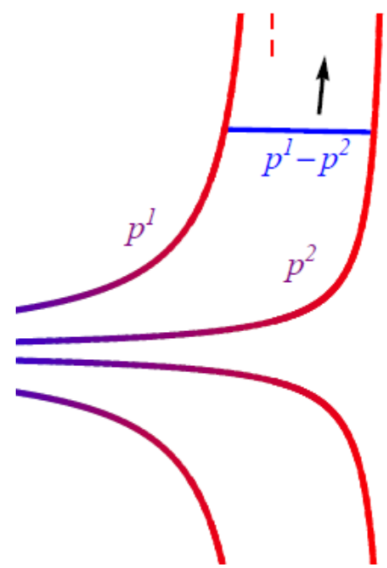
$$\Rightarrow \dim \underline{\overline{\mathcal{M}}} = 0$$



$$p^1 > p^2 \Rightarrow \gamma_m = -P^-$$

$$\Rightarrow \dim \underline{\overline{\mathcal{M}}} = 4(p^1 - p^2)$$





Application: Meaning Of The Singular 't Hooft-Polyakov Ansatz

$$X = (m_W \coth(m_W r + c) - \frac{1}{r})\frac{1}{2}H$$

$$\gamma_m = P = H \Rightarrow \tilde{n}_m = 2$$

$$\Rightarrow \dim \underline{\overline{\mathcal{M}}} = 8$$

Two smooth monopoles in the presence of minimal SU(2) singular monopole.

They sit on top of the singular monopole but have a relative phase: $e^{-c} = \sin(\psi/2)$

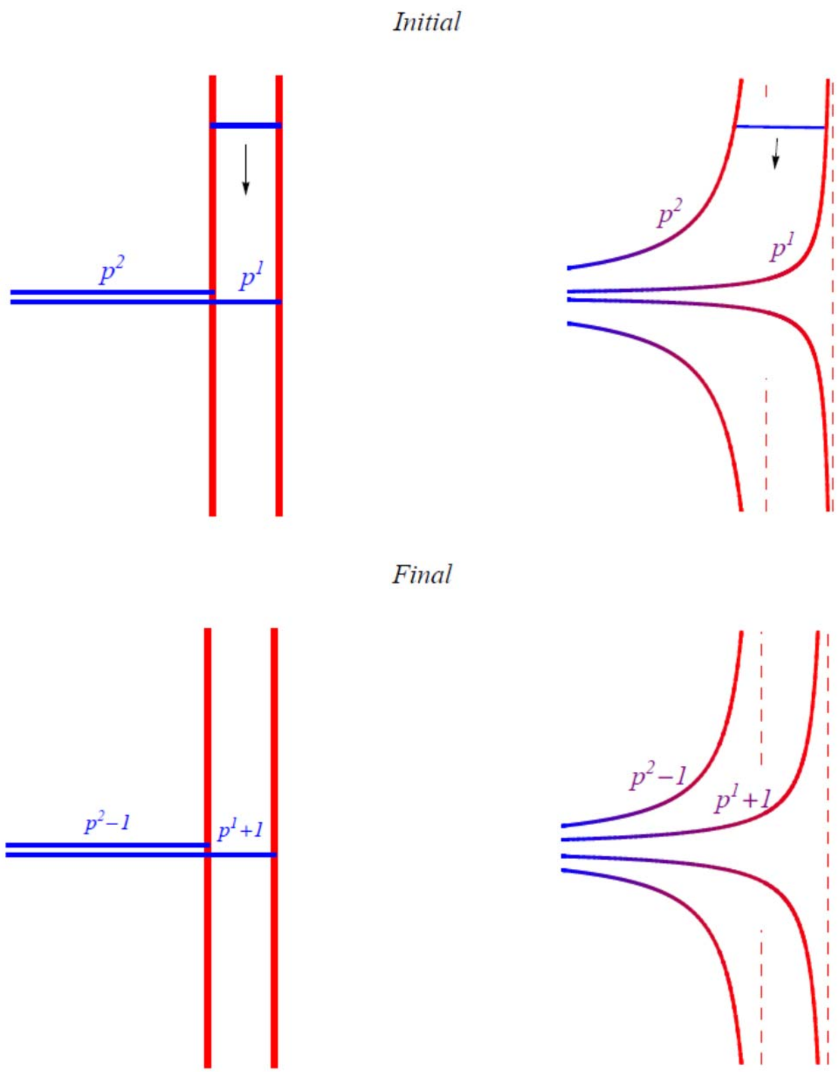
Two D6-branes on an $O6^-$ plane;

Moduli space of d=3 N=4 SYM with two massless HM

Properties of $\overline{\mathcal{M}}$

$\overline{\mathcal{M}}$ Hyperkähler (with singular loci - monopole bubbling)

[Kapustin-Witten]



Isometries of $\overline{\mathcal{M}}$

$\overline{\mathcal{M}}$ has an action of $\mathfrak{so}(3) \oplus \mathfrak{t}$

$\mathfrak{so}(3)$: spatial rotations

$\mathfrak{t}$-action: global gauge transformations commuting with X_∞

$$H \in \mathfrak{t} \Longrightarrow G(H) \in \mathrm{VECT}(\overline{\mathcal{M}})$$

$\mathcal{N}$=2 Super-Yang-Mills

Second real adjoint scalar Y

Vacuum requires $[X_\infty, Y_\infty]=0$.

$$\zeta^{-1}\varphi = Y + \mathrm{i}X$$

Meaning of ζ: BPS equations on $\mathbb{R}^3$ for preserving

$$Q + \zeta^{-1}\bar{Q}$$

$$F = B = *DX \qquad E = DY$$

ζ And BPS States

Framed case: Phase ζ is part of the data describing 't Hooft line defect L

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(L,\gamma;u) \qquad u \in \mathcal{M}_{\mathrm{Coulomb}}$$

Smooth case: Phase ζ will be related to central charge of BPS state

$$\mathcal{H}^{\mathrm{BPS}}(\gamma;u) \qquad \zeta = -Z_{\gamma}(u)/|Z_{\gamma}(u)|$$

Semiclassical Regime

Definition: Series expansions for $a_D(a;\Lambda)$ converges: $|\langle \alpha, a \rangle| > c|\Lambda|$

Local system of charges has natural duality frame:

$$\Gamma = \Lambda_{rt} \oplus \Lambda_{mw}$$

$$\gamma = \gamma^e \oplus \gamma_m$$

(Trivialized after choices of cuts in logs for a_D.)

$\Lambda(t) = e^{-\pi t/h^\vee} \Lambda_0$ $\lim_{t \to +\infty} \mathcal{H}^{\rm BPS}(\gamma; u_t)$

In this regime there is a well-known semiclassical approach to describing BPS states.

Collective Coordinate Quantization

At weak coupling BPS monopoles with magnetic charge γ_m are heavy: Study quantum fluctuations using quantum mechanics on monopole moduli space

The semiclassical states at (u,ζ) with electromagnetic charge $\gamma^e \oplus \gamma_m$ are described in terms of supersymmetric quantum mechanics on

$$\underline{\overline{\mathcal{M}}}(P, \gamma_m; X_\infty) \quad \text{OR} \quad \mathcal{M}(\gamma_m; X_\infty)$$

What sort of SQM? How is (u,ζ) related to X_∞?

How does γ^e have anything to do with it?

What Sort Of SQM?

(Sethi, Stern, Zaslow; Gauntlett & Harvey ; Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi; Bak, Lee, Lee, Yi; Stern & Yi)

N=4 SQM on $\mathcal{M}(\gamma_m, X_\infty)$ with a potential:

$$S = \int \left(\| \dot{z} \|^2 - \| G(\mathcal{Y}^{\mathrm{cl}}_\infty) \|^2 + \cdots \right)$$

$$\mathcal{Y}^{\mathrm{cl}}_\infty := \frac{4\pi}{g_0^2} Y_\infty + \frac{\theta_0}{2\pi} X_\infty$$

$$\{Q, z^\mu\} \sim \chi^\mu$$

$\Rightarrow$ States are spinors on $\mathcal{M}$

$$Q_4 = \chi^\mu (D + G(\mathcal{Y}^{\mathrm{cl}}_\infty))_\mu := \mathbf{D}$$

How is (u,ζ) related to X_∞?

Need to write X_∞, $\mathcal{Y}_\infty$ as functions on the Coulomb branch

$$X_\infty := \mathrm{Im}(\zeta^{-1} a(u)) := X$$

$$\mathcal{Y}_\infty := \mathrm{Im}(\zeta^{-1} a_D(u;\Lambda)) := \mathcal{Y}$$

Framed case: Phase ζ: data describing 't Hooft line defect L

Smooth: Phase ζ will be related to central charge of BPS state

What's New Here?

Include singular monopoles: Extra boundary terms in the original action to regularize divergences: Requires a long and careful treatment.

Include effect of theta-term: Leads to nontrivial terms in the collective coordinate action

Consistency requires we properly include one-loop effects:

Essential if one is going to see semiclassical wall-crossing. (failure to do so lead to past mistakes…)

We incorporate one-loop effects, (up to some reasonable conjectures): Use the above map to X, $\mathcal{Y}$.

Moreover, we propose that all the quantum effects relevant to BPS wall-crossing (in particular going beyond the small $\mathcal{Y}_\infty$ approximation) are captured by the ansatz:

$$X_\infty := \mathrm{Im}(\zeta^{-1} a(u))$$

$$\mathcal{Y}_\infty := \mathrm{Im}(\zeta^{-1} a_D(u; \Lambda))$$

NEW!

$$
\begin{aligned}
H_{\rm c.c.} = M^{\rm cl}_{\gamma_{\rm m}} + \frac{g_0^2}{8\pi}\Big\{ \pi_m g^{mn} \pi_n + g_{mn} {\rm G}(\mathcal{Y}^{\rm cl}_{\infty})^m {\rm G}(\mathcal{Y}^{\rm cl}_{\infty})^n + \frac{4\pi i}{g_0^2} \chi^m \chi^n \nabla_m {\rm G}(\mathcal{Y}^{\rm cl}_{\infty})_n \Big\} + \\
+ i\tilde{\theta}_0 \left(i{\rm G}(X_\infty)^m \pi_m + \frac{2\pi}{g_0^2} \chi^m \chi^n \nabla_m {\rm G}(X_\infty)_n \right) + O(g_0^2) \ .
\end{aligned}
$$

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Semiclassical BPS States: Overview

$$Q_4 = \chi^{\mu}(D + G(\mathcal{Y}))_{\mu} := \mathbf{D}$$

Semiclassical framed or smooth BPS states with magnetic charge γ_{m} will be:

a Dirac spinor Ψ on $\mathcal{M}(\gamma_{\mathrm{m}})$ or $\overline{\mathcal{M}}(\gamma_{\mathrm{m}})$ $\quad \mathbf{D}\Psi = 0$

Must be suitably normalizable: $\quad \ker_{L^2} \mathbf{D}$

Must be suitably equivariant…

Many devils in the details….

States Of Definite Electric Charge

$\overline{\mathcal{M}}$ has a t-action: G(H) commutes with **D**

$$\exp[2\pi G(H)] \cdot \Psi = \exp[2\pi i \langle \gamma^e, H \rangle] \Psi$$

$$\gamma^e \in \mathfrak{t}^\vee$$

Cartan torus T of adjoint group acts on $\overline{\mathcal{M}}$

$$T = \mathfrak{t}/\Lambda_{mw} \Longrightarrow \gamma^e \in \Lambda_{rt} \subset \mathfrak{t}^\vee$$

Organize L^2-harmonic spinors by T-representation:

$$\ker_{L^2} \mathbf{D} = \oplus_{\gamma^e} \ker_{L^2}^{\gamma^e} \mathbf{D}$$

Geometric Framed BPS States

$$\ker_{L^2}\mathbf{D} = \oplus_{\gamma^e \in \Lambda_{rt}} \ker_{L^2}^{\gamma^e}\mathbf{D}$$

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(P;\gamma;X,\mathcal{Y}) := \ker_{L^2}^{\gamma^e}\mathbf{D}$$

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(L,\gamma;u) = \underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(P;\gamma;X,\mathcal{Y})$$

$$X = \mathrm{Im}(\zeta^{-1}a(u))$$

$$\mathcal{Y} = \mathrm{Im}(\zeta^{-1}a_D(u;\Lambda))$$

BPS States From Smooth Monopoles
- The Electric Charge -

Spinors and **D** live on universal cover: $\tilde{\mathcal{M}}$

T acts on $\mathcal{M}$, so $\mathfrak{t}$ acts on $\tilde{\mathcal{M}}$

$$T = \mathfrak{t}/\Lambda_{mw}$$

States Ψ of definite electric charge transform in a definite character of $\mathfrak{t}$: (``momentum'')

In order to have a T-action the character must act trivially on Λ_{mw} $\quad \gamma^e \in \Lambda_{mw}^{\vee} \cong \Lambda_{rt}$

Smooth Monopoles – Separating The COM

$$\widetilde{\mathcal{M}}(\gamma_m; X_\infty) = \mathbb{R}^3 \times \mathbb{R} \times \mathcal{M}_0$$

No L^2 harmonic spinors on $\mathbb{R}^4$.

Only ``plane-wave-normalizable'' in $\mathbb{R}^4$

$$\mathbf{D} = \mathbf{D}_{\mathrm{com}} + \mathbf{D}_0$$

$$\Psi = \Psi_{\mathrm{com}} \otimes \Psi_0$$

$$\mathbf{D}_{\mathrm{com}} \Psi_{\mathrm{com}} = 0 \qquad \mathbf{D}_0 \Psi_0 = 0$$

Smooth Monopoles – Separating The COM

$$\mathbf{D} = \chi^{\mu}(D + G(\mathcal{Y}))_{\mu} = \mathbf{D}_{\mathrm{com}} + \mathbf{D}_0$$

Need orthogonal projection of $G(\mathcal{Y})$ along $G(X_{\infty})$.

$$(G(X_{\infty}), G(H))_{\mathrm{metric}} = (\gamma_m, H)_{\mathrm{Killing}}$$

X_{∞} : generic, irrational direction in $\mathfrak{t}$

A remarkable formula!

γ_m is a rational direction in $\mathfrak{t}$

Flow along γ_m in $T = \mathfrak{t}/\Lambda_{mw}$ will close.

Not so for flow along X_{∞}

Smooth Monopoles – Separating The COM

$$\mathbf{D}_{\rm com} = \sum_{i=1}^{3} \chi^i \frac{\partial}{\partial x^i} + \chi^4 \left(\frac{\partial}{\partial x^4} - \frac{(\mathcal{Y}, \gamma_m)}{(X, \gamma_m)} \right)$$

$$\Psi_{\rm com} = e^{{\rm i} q x^4} s_{\rm com} \qquad q = (\mathcal{Y}, \gamma_m)/(X, \gamma_m)$$

But for states of definite electric charge γ^e:

$$q = -\langle \gamma^e, X \rangle / (X, \gamma_m)$$

$$\Rightarrow \quad \langle \gamma^e, X \rangle + (\gamma_m, \mathcal{Y}) = 0$$

Dirac Zeromode Ψ_0

Ψ_0 with magnetic charge $\gamma_m \in \ker_{L^2} \mathbf{D}_0$

Note: The L^2 condition is crucial!

We do not want ``extra'' internal d.o.f.

Contrast this with the hypothetical ``instanton particle'' of 5D SYM.

Organize L^2-harmonic spinors by $\mathfrak{t}^{\perp}$-representation:

$$\ker_{L^2} \mathbf{D}_0 = \oplus_{\gamma_e} \ker_{L^2}^{\gamma_e^{\perp}} \mathbf{D}_0$$

$$\gamma_e^{\perp} \in \left(\Lambda_{mw} \cap \gamma_m^{\perp}\right)^{\vee} \subset \mathfrak{t}^{\vee}$$

Semiclassical Smooth BPS States

????

$$\mathcal{H}^{\mathrm{BPS}}(\gamma;u) = \ker^{q}(\mathbf{D}_{\mathrm{com}}) \otimes \ker_{L^2}^{\gamma_e^{\perp}} \mathbf{D}_0$$

$$X = \mathrm{Im}(\zeta^{-1} a(u))$$

$$\mathcal{Y} = \mathrm{Im}(\zeta^{-1} a_D(u;\Lambda))$$

$$\zeta = -Z_{\gamma}(u)/|Z_{\gamma}(u)|$$

$$\langle \gamma^{e}, X \rangle + (\gamma_m, \mathcal{Y}) = 0$$

Tricky Subtlety: 1/2

Spinors must descend to $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{D}$

$\mathbb{D} \cong \mathbb{Z}$ Generated by isometry ϕ

Subtlety: Imposing electric charge quantization only imposes invariance under a *proper* subgroup of the Deck group:

$$\exp[2\pi G(\lambda)]\Psi = \Psi \qquad \lambda \in \Lambda_{mw}$$

$$\exp[2\pi G(\lambda)] = \phi^{\mu(\lambda)}$$

Tricky Subtlety: 2/2

Conjecture: $$\boxed{\mu(\lambda) = (\lambda, \gamma_m)}$$

$\Rightarrow$ $$\exp[2\pi G(\lambda)] = \phi^{\mu(\lambda)}$$

only generate a subgroup ***r*** $\mathbb{Z}$, where ***r*** is, roughly speaking, the gcd(magnetic charges)

$\Rightarrow$ Extra restriction to $\mathbb{Z}/\mathrm{r}\mathbb{Z}$ invariant subspace:

$$\left(\ker^{q}(\mathbf{D}_{\mathrm{com}}) \otimes \ker_{L^2}^{\gamma_e^{\perp}} \mathbf{D}_0\right)^{\mathbb{Z}/\mathrm{r}\mathbb{Z}}$$

Combine above picture with results on N=2,d=4:

No Exotics Theorem

Wall-Crossing

(higher rank is different)

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Exotic (Framed) BPS States

$\overline{\underline{\mathcal{H}}}_{\gamma}^{\rm BPS}$ $\mathcal{H}_{\gamma}^{\rm BPS}$ $\mathfrak{so}(3)_{\rm rot} \oplus \mathfrak{su}(2)_R$ -reps

Smooth monopoles: $\mathcal{H}_{\gamma}^{\rm BPS} = \rho_{hh} \otimes \mathfrak{h}(\gamma)$

Half-Hyper from COM: $\rho_{hh} = (\frac{1}{2}; 0) \oplus (0; \frac{1}{2})$

Singular monopoles: No HH factor:

$$\overline{\underline{\mathcal{H}}}_{\gamma}^{\rm BPS} = \mathfrak{h}(\gamma)$$

Definition: ***Exotic BPS states***: States in $\mathfrak{h}(\gamma)$ transforming nontrivially under $\mathfrak{su}(2)_R$

No Exotics Conjecture/Theorem

Conjecture [GMN]: $\mathfrak{su}(2)_R$ acts trivially on $\mathfrak{h}(\gamma)$: exotics don't exist.

Theorem: It's true!

Diaconescu et. al. : Pure SU(N) smooth and framed (for pure 't Hooft line defects)

Sen & del Zotto: Simply laced G (smooth)

Cordova & Dumitrescu: Any theory with ``Sohnius'' energy-momentum supermultiplet (smooth, so far…)

Geometry Of The R-Symmetry

Geometrically, $SU(2)_R$ is the commutant of the USp(2N) holonomy in SO(4N). It acts on sections of $T\mathcal{M}$ rotating the 3 complex structures;

Collective coordinate expression for generators of $\mathfrak{su}(2)_R$

$$I^r \sim \omega^r_{\mu\nu} \chi^\mu \chi^\nu$$

This defines a lift to the spin bundle.

Generators do not commute with Dirac, but do preserve kernel.

$\overline{\mathcal{M}}$ $\mathcal{M}$ have $\mathfrak{so}(3)$ action of rotations. Suitably defined, it commutes with $\mathfrak{su}(2)_R$.

Again, the generators do not commute with $\mathbf{D}_0$, $\mathbf{D}$, but do preserve the kernel.

$$\underline{\overline{\mathcal{H}}}^{\mathrm{BPS}}(P;\gamma;X,\mathcal{Y}) := \underbrace{\ker_{L^2}^{\gamma_e}\mathbf{D}}_{\mathfrak{so}(3)_{\mathrm{rot}} \oplus \mathfrak{su}(2)_R}$$

$$\mathfrak{h}^{\mathrm{BPS}}(\gamma;X,\mathcal{Y}) := \underbrace{\ker_{L^2}^{\gamma_e^{\perp}}\mathbf{D}_0}_{\mathfrak{so}(3)_{\mathrm{rot}} \oplus \mathfrak{su}(2)_R}$$

Geometrical Interpretation Of The No-Exotics Theorem -1

$$\rho : SU(2)_R \times USp(2N) \to \mathrm{Spin}(4N)$$

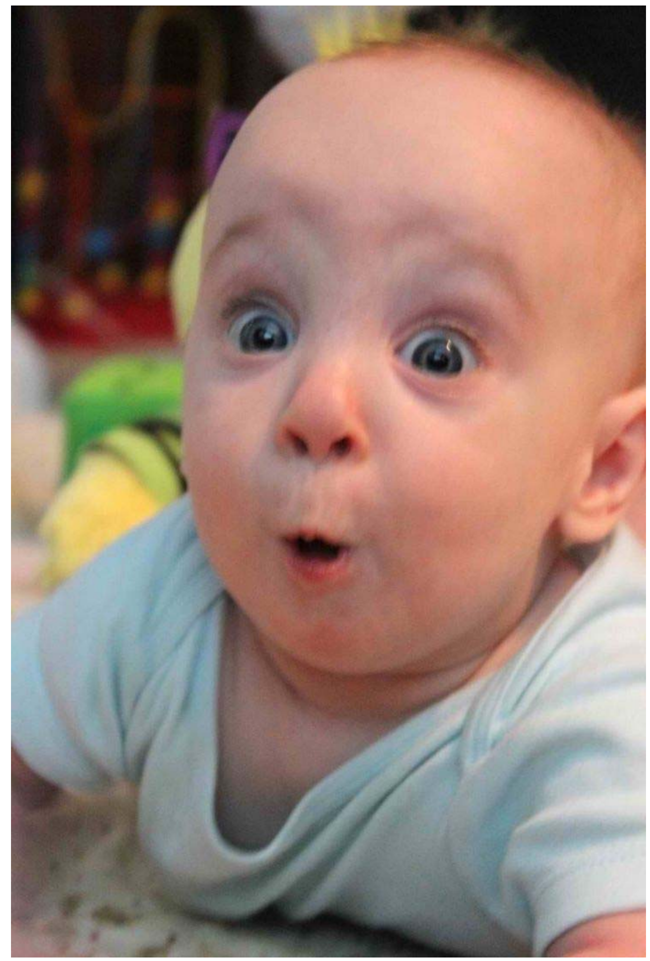
$$\rho : (-1, 1) \to \mathrm{vol}$$

⇒ All spinors in the kernel have chirality +1

⇒ $\mathrm{Ind}\mathbf{D}_0^+ = \dim \ker \mathbf{D}_0$

So, the ***absolute*** number of BPS states is the same as the BPS ***index***!

This kind of question arises frequently in BPS theory…



Geometrical Interpretation Of The No-Exotics Theorem - 2

Choose any complex structure on $\mathcal{M}$.

$$\mathcal{S} \cong \Lambda^{0,*}(T\mathcal{M}) \otimes K^{-1/2}$$

$$Q_3 + \mathrm{i}Q_4 \sim \bar{\partial} + G^{0,1}(\mathcal{Y})\wedge$$

$\mathfrak{su}(2)_R$ becomes ``Lefshetz $\mathfrak{sl}(2)$''

$$I^3|_{\Lambda^{0,q}} = \frac{1}{2}(q - N)\mathbf{1}$$

$$I^+ \sim \omega^{0,2}\wedge \qquad\qquad I^- \sim \iota(\omega^{2,0})$$

Geometrical Interpretation Of The No-Exotics Theorem - 3

$$H^{0,q}_{L^2}(\bar{\partial} + G^{0,1}(Y_\infty))$$

vanishes except in the middle degree q =N, and is primitive wrt ``Lefshetz $\mathfrak{sl}(2)$''.

Adding Matter

(work with Daniel Brennan)

Add matter hypermultiplets in a quaternionic representation R of G.

Bundle of hypermultiplet fermion zeromodes defines a real rank d vector bundle over $\mathcal{M}$: Structure group SO(d)

Associated bundle of spinors, $\mathcal{E}$, has hyperholomorphic connection.

(Manton & Schroers; Gauntlett & Harvey ; Tong; Gauntlett, Kim, Park, Yi; Gauntlett, Kim, Lee, Yi; Bak, Lee, Yi)

$$d = \sum_{\mu} \Big[\mathrm{sign}(\langle \mu, X \rangle + 2 m_I) \langle \mu, \gamma_m \rangle + |\langle \mu, P \rangle| \Big]$$

Sum over weights μ of R. $\zeta^{-1} m = m_R + i m_I$

Geometrical Interpretation Of The No-Exotics Theorem - 4

States are now L^2-sections of

$$S \otimes \mathcal{E} \to \mathcal{M}_0 \quad , \overline{\underline{\mathcal{M}}}$$

$$H^{0,q}_{L^2}(\bar{\partial}_{\mathcal{E}} + G^{0,1}(Y_\infty); \mathcal{E})$$

vanishes except in the middle degree q =N, and is primitive wrt ``Lefshetz $\mathfrak{sl}(2)$''.

SU(2) N=2* m $\longrightarrow$ 0 recovers the famous Sen conjecture

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Semiclassical Wall-Crossing: Overview

Easy fact: There are no L^2 harmonic spinors for ordinary Dirac operator on a noncompact hyperkähler manifold.

⇒ $\exists$ Semiclassical chamber ($\mathcal{Y}_\infty=0$) where all populated magnetic charges are just simple roots ($\mathcal{M}_0$ = pt)

Other semiclassical chambers have nonsimple magnetic charges filled.

⇒ Nontrivial semi-classical wall-crossing

(Higher rank is different.)

⇒ Interesting math predictions

Jumping Index

The L^2-kernel of D jumps.

No exotics theorem ⇒

Harmonic spinors have definite chirality

⇒ L^2 *index* jumps!   How?!

Along hyperplanes in $\mathcal{Y}$-space zeromodes mix with continuum and D^+ fails to be Fredholm.

(Similar picture proposed by M. Stern & P. Yi in a special case.)

When Is D_0 Not Fredholm?

$\mathbf{D}_0^{\mathcal{Y}}$ is a function of $\mathcal{Y}$:

Translating physical criteria for wall-crossing implies :
ker $D^{\mathcal{Y}}_0$ on $\mathcal{M}(\gamma_m)$ only changes when

$$\exists \gamma_1, \gamma_2 \ \ \langle \gamma_1, \gamma_2 \rangle \neq 0 \quad \mathcal{H}(\gamma_i; X, \mathcal{Y}) \neq 0$$

$$\gamma_{1,m} + \gamma_{2,m} = \gamma_m$$

$$(\gamma_{i,m}, \mathcal{Y}) + \langle \gamma_{i,e}, X \rangle = 0, \quad i = 1, 2$$

($D^{\mathcal{Y}}_0$ only depends on $\mathcal{Y}$ orthogonal to γ_m
so this is real codimension one wall.)

When Is **D** on $\overline{\mathcal{M}}$ Not Fredholm?

$\mathbf{D}^{\mathcal{Y}}$ as a function of $\mathcal{Y}$ is not Fredholm if:

$$\exists \gamma_h \quad \mathcal{H}(\gamma_h; X, \mathcal{Y}) \neq 0$$

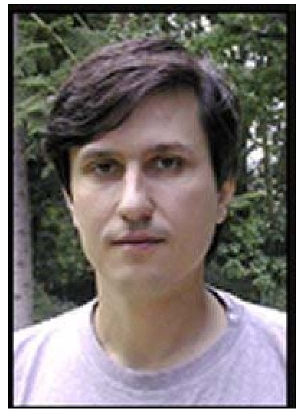
$$(\gamma_{h,m}, \mathcal{Y}) + \langle \gamma_{h,e}, X \rangle = 0$$

$\Rightarrow$ $\overline{\underline{\mathcal{H}}}(\gamma; X, \mathcal{Y})$ jumps across:

$$W(\gamma_h) := \{\mathcal{Y} | \text{above conditions}\}$$

How Does The BPS Space Jump?

Unframed/
smooth/
vanilla:



&

Framed:

&

Framed Wall-Crossing: 1/2

$$\underline{\overline{\Omega}}(L,\gamma;X,\mathcal{Y}) = \mathrm{Tr}_{\underline{\mathcal{H}}} y^{2J_3}$$

``Protected spin characters''

$$F(L) = \sum_{\gamma\in\Gamma} \underline{\overline{\Omega}}(L,\gamma;X,\mathcal{Y}) V_\gamma$$

$$V_{\gamma_1} V_{\gamma_2} = y^{\langle\gamma_1,\gamma_2\rangle} V_{\gamma_1+\gamma_2}$$

$F \to SFS^{-1}$ S: A product of quantum dilogs

Framed Wall-Crossing: 2/2

$$F(L) = \sum_{\gamma\in\Gamma} \overline{\underline{\Omega}}(L,\gamma;X,\mathcal{Y})V_\gamma$$

$$W(\gamma_h) := \{(X,\mathcal{Y}) : (\gamma_{h,m},\mathcal{Y}) + \langle\gamma_{h,e},X\rangle = 0\}$$

$$F(L) \to SF(L)S^{-1}$$

$$\Phi(z) = \prod_{k=1}^{\infty}(1+y^{2k-1}z)^{-1}$$

$$S = \prod_m \Phi((-y)^m V_{\gamma_h})^{a_{m,\gamma_h}}$$

$$\Omega(\gamma_h;u) = \sum_m a_{m,\gamma_h}(-y)^m$$

Example: Smooth SU(3) Wall-Crossing

[Gauntlett, Kim, Lee, Yi (2000)]

$$\mathfrak{g} = \mathfrak{su}(3) \qquad \mathfrak{t} \cong \mathbb{R}^2$$

$$\gamma_m = H_1 + H_2 = \gamma_{1,m} + \gamma_{2,m}$$

$$\mathcal{Y} = y_1 h^1 + y_2 h^2 \Rightarrow \mathcal{Y}^{\|} = y_1 + y_2$$

$$\gamma^e = n_1 \alpha_1 + n_2 \alpha_2 = \gamma_1^e + \gamma_2^e$$

$$\mathcal{M}_0(X; \gamma_{i,m}) = pt \qquad \mathcal{H}(\gamma_i; X, \mathcal{Y}) \neq 0$$

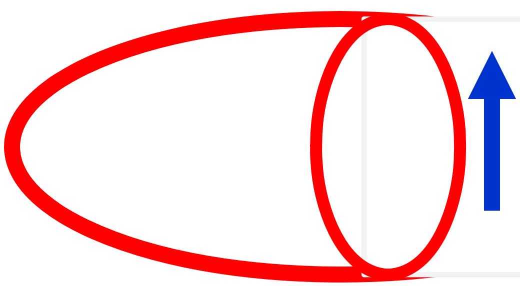
``Constituent BPS states exist''

$$\gamma_m = H_1 + H_2 \longrightarrow$$

$\mathcal{M}_0(X;\gamma_m)$ = Taub-NUT:

Zeromodes of **D**$_0$ can be ***explicitly*** computed [C. Pope, 1978]

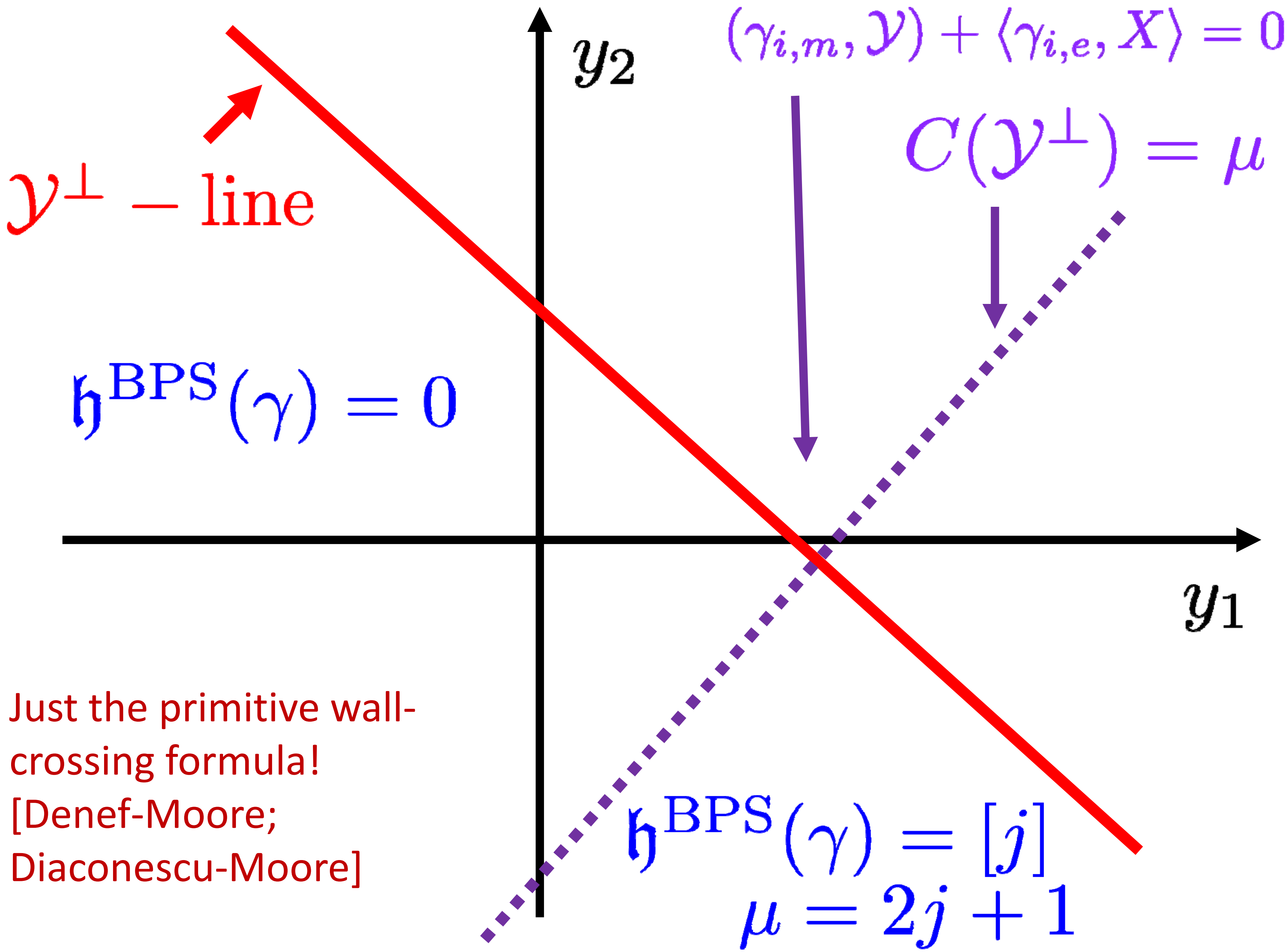
$\mathrm{t}^{\perp} - \mathrm{orbits}$ = orbits of standard HH U(1) isometry



$\frac{\partial}{\partial \psi}$

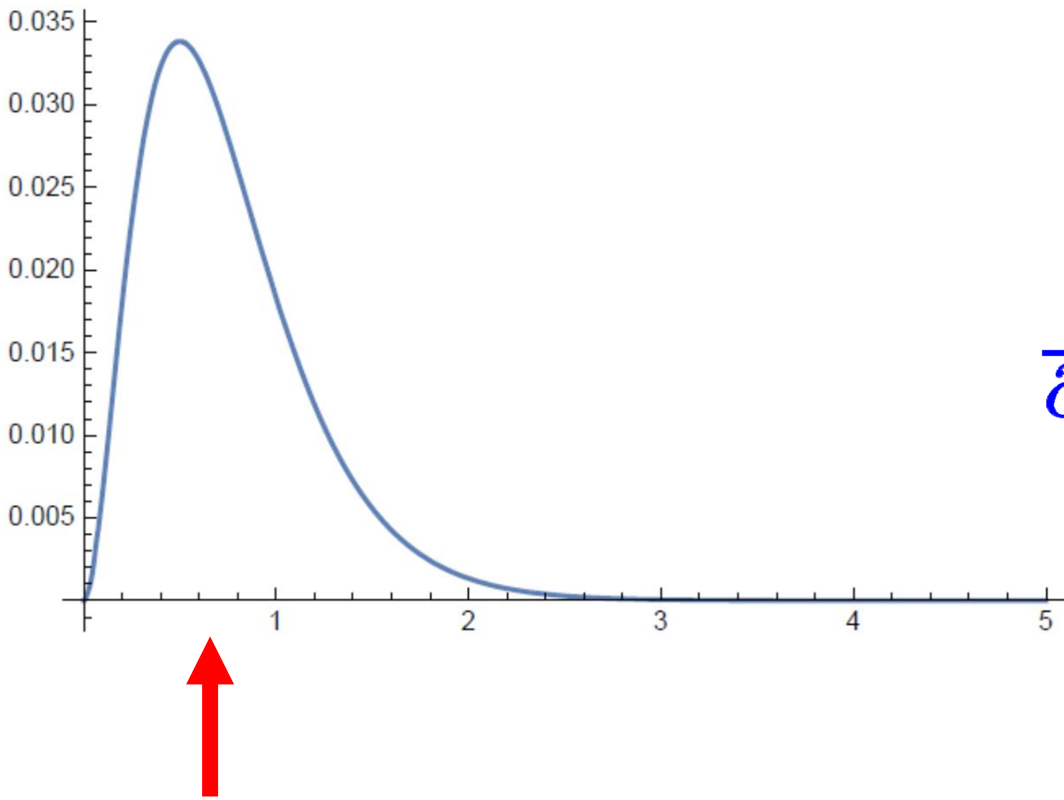
$$G(\mathcal{Y}) = C(\mathcal{Y}^{\perp})\frac{\partial}{\partial \psi}$$

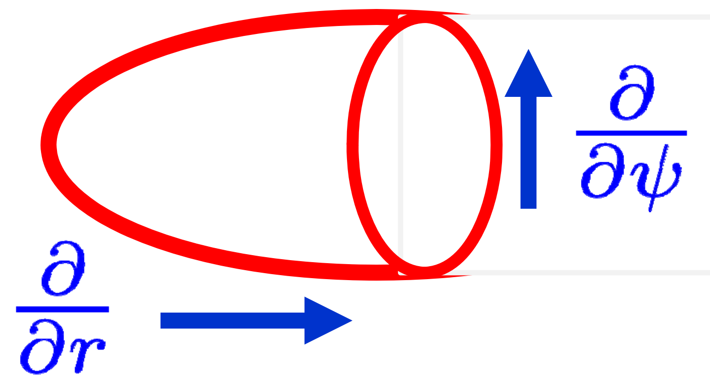
$$L_{\frac{\partial}{\partial \psi}} \Psi_0 = \mathrm{i}(n_1 - n_2)\Psi_0$$
$$= \mathrm{i}\mu\Psi_0$$



$(\gamma_{i,m}, \mathcal{Y}) + \langle \gamma_{i,e}, X \rangle = 0$
$C(\mathcal{Y}^{\perp}) = \mu$
y_2
y_1
$\mathcal{Y}^{\perp} - \text{line}$
$\mathfrak{h}^{\text{BPS}}(\gamma) = 0$
$\mathfrak{h}^{\text{BPS}}(\gamma) = [j]$
$\mu = 2j + 1$
Just the primitive wall-crossing formula!
[Denef-Moore; Diaconescu-Moore]

$$\Psi_0 \sim r^{(\mu-1)/2} e^{-|C-\mu| r/2}$$

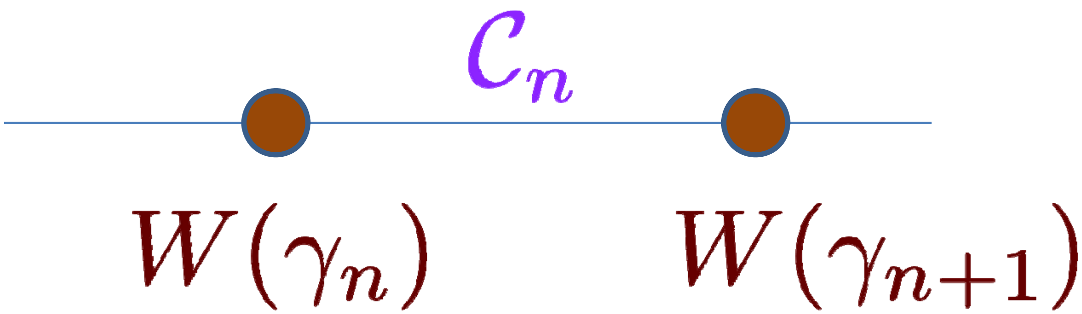




$$r_{\text{max}} = \frac{\mu}{|C-\mu|} = r_{\text{Denef}}$$

Example: Singular SU(2) Wall-Crossing

$$\mathfrak{g} = \mathfrak{su}(2) \qquad \mathfrak{t} \cong \mathbb{R}$$



Well-known spectrum of smooth BPS states [Seiberg & Witten]:

$$\gamma_n = n\alpha \oplus H$$

$$W(\gamma_h) := \{\mathcal{Y} | (\gamma_{h,m}, \mathcal{Y}) + \langle \gamma_{h,e}, X \rangle = 0\}$$

Line defect L: $P = \frac{p}{2} H$

$$F(L) = \sum_{\gamma \in \Gamma} \underline{\overline{\Omega}}(L, \gamma; X, \mathcal{Y}) V_\gamma$$

Explicit Generator Of PSC's

$$V_1 V_2 = y V_2 V_1$$

$$V_\gamma = V_{n^e \alpha + n_m H} = y^{-\frac{1}{2} n^e n_m} V_2^{n^e} V_1^{n_m}$$

$$F(\mathcal{C}_\ell) = \left[y^{2\ell} V_1^{-1} V_2^{-\ell} \left(\mathcal{U}_\ell(f_\ell) - y^2 V_2^{-1} \mathcal{U}_{\ell-1}(f_\ell) \right) \right]^p$$

$$\mathcal{U}_\ell(\cos\theta) := \frac{\sin((\ell+1)\theta)}{\sin\theta}$$

$$f_\ell = \frac{1}{2} \left[y^{-2} V_2 + y^2 V_2^{-1} \left(1 + y^{-1} V_1^2 V_2^{2\ell+2} \right) \right]$$

Predictions for ker **D** for infinitely many moduli spaces of arbitrarily high magnetic charge.

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So, What Did He Say?

Recent new old

Recent results on N=2 d=4 imply new results about the differential geometry of old monopole moduli spaces.

Future Directions

Add matter and arbitrary Wilson-'t Hooft lines. (In progress with Daniel Brennan)

Understand better how Fredholm property fails by using asymptotic form of the monopole metric.

Combine with result of Okuda et. al. and Bullimore-Dimofte-Gaiotto to get an interesting L^2-index theorem on (noncompact!) monopole moduli spaces ?