On Flux Quantization in M-Theory

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work in progress

Introduction - I

This work grew out of thinking about the E_8 formalism for the C-field of M-theory. But I have written the talk so that you don't need to know that formalism.

The motivating question today is the following:

M-theory has a 3-form potential C, with 4-form fieldstrength G. There is a well-known quantization law (Witten, 96)

$$[G] = a - \frac{1}{2}\lambda \qquad\qquad a \in H^4(X,\mathbb{Z})$$

(Torsion will not play a role in this talk)

This rule follows from the existence of fundamental $M2$-branes:

$$\exp\left(2\pi i \int_\Sigma C\right)$$

However, there are electromagnetically dual $M5$-branes. Thus there must be some kind of dual fieldstrength G_7 of the potential which couples to the $M5$ brane.

What is the electromagnetic dual quantization law on $[G_7]$?

Introduction - II

Answer: The dual quantization condition must be understood in terms of the quantum mechanics of the C-field.

Using Gauss' law for C-field gauge transformations we will define a flux G_7 with $[G_7]$ conserved.

The cohomology class $[G_7]$ is often called the Page charge. We will find that, in the presence of $[G_4] \neq 0$ it is noncommutative.

Lattice of fluxes in $H^7(X, \mathbb{Z}) \longrightarrow$ a nonabelian group.

Most of the talk is development of formalism - but there are some applications perhaps of broader interest:

• Flux compactifications of M-theory on 7-folds. (Clarification of G_2 superpotentials.)

• 5-brane partition functions. (We can derive Witten's theta function directly from M-theory. This leads to some technical clarifications.)

A Useful Analogy - 3D abelian CS & Landau Problem

A closely related theory is 3d massive abelian CS:

$$S = \int_{\Sigma\times \mathbb{R}} -\frac{1}{2e^2} F * F + 2\pi k \int_{\Sigma\times \mathbb{R}} AdA$$

On $\Sigma \times \mathbb{R}$ the dynamics of the topological (flat) modes of A is that of an electron on a torus $H^1(\Sigma; U(1))$ in a constant magnetic field.

Recall:

$$\Omega = \int_\Sigma \delta\Pi \wedge \delta A$$

$$\mathcal{H} = \frac{1}{2}\int_\Sigma e^2 \tilde{\Pi} * \tilde{\Pi} + \frac{1}{e^2} F * F$$

$$\tilde{\Pi} = \Pi - 2\pi k A$$

In this case, the long distance limit is $e^2 \to \infty$.

Low energy physics is governed by the harmonic modes of A $\Rightarrow$ same mathematics as the Landau problem.

For simplicity, take $\Sigma = T^2$.

The quantum mechanics of these modes is identical to the Landau problem of an electron on the torus with $B = 2\pi k$.

Landau Problem - I

Begin with the Landau problem on the x, y-plane:

$$\mathcal{H} = (\tilde{\Pi}^x)^2 + (\tilde{\Pi}^y)^2 = \left(-i\frac{\partial}{\partial x} - \frac{B}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{B}{2}x\right)^2$$

$$x, y \leftrightarrow \text{harmonic modes of } C$$

$$B \leftrightarrow G$$

$$\text{Electron wavefunction} \leftrightarrow C\text{-field wavefunction.}$$

Note that neither Π^i nor $\tilde{\Pi}^i$ are conserved...

In the Landau problem on the plane we can define magnetic translation operators:

$$P^x = -i\frac{\partial}{\partial x} + \frac{B}{2}y \qquad\qquad P^y = -i\frac{\partial}{\partial y} - \frac{B}{2}x$$

These commute with $\mathcal{H}$, and hence are conserved.

However, they do *not* commute with themselves:

$$[P^x, P^y] = iB$$

$$P^x, P^y \leftrightarrow \text{Page charges}$$

Landau Problem - II

Now consider the Landau problem on the torus. So we identify $x \sim x+1, y \sim y+1$:

$$x \sim x+1 \leftrightarrow \text{Large } C\text{-field gauge transformations.}$$

In the Landau problem the wavefunction is a section of a line bundle with connection over T^2 with $c_1 = k$, where $B = 2\pi k$, $k \in \mathbb{Z}$, and $\mathcal{H}$ is the Laplacian.

The magnetic translation operators P^x, P^y are not well defined on the torus: Lattice translations are generated by e^{iP^x}, e^{iP^y} and the quantum wavefunction satisfies:

$$e^{iP^x}\psi = \psi \qquad\qquad e^{iP^y}\psi = \psi$$

Translation by $\vec{n}$ is generated by $U(\vec{n}) = e^{in_x P^x + in_y P^y}$:

$$U(\vec{n}) P^x U(\vec{n})^{-1} = P^x + 2\pi k n_y$$

Landau problem - III

The operators

$$T(\vec{a}) := \exp\bigl(ia_x P^x + ia_y P^y\bigr)$$

are lattice-invariant iff (a_x, a_y) is a k-torsion point. Moreover, these operators form an Heisenberg group

$$T(\vec{a})T(\vec{b}) = e^{-2\pi ik|\vec{a}\times\vec{b}|}T(\vec{b})T(\vec{a})$$

This is the kind of nonabelian group which will replace the naive lattice of flux quantization.

The lowest Landau level is of dimension $= k$, and provides a representation of the Heisenberg group - this is analogous to the 5-brane partition function.

In the lowest Landau level $\delta\tilde{\Pi} = 0 \Rightarrow$ a topological theory with

$$\Omega = \int \delta\tilde{\Pi}\delta A + 2\pi \int_\Sigma k\delta A\delta A \to 2\pi \int_\Sigma k\delta A\delta A$$

$T(\vec{a}) \sim$ Wilson, 't Hooft, or Verlinde operators, depending on context

Wavefunction $\sim$ conformal blocks.

Why Page charge has not really been defined, yet

The usual discussion goes like this:

$$\ell^{-3} d * G = \frac{1}{2} G^2 - I_8$$

So, if $G = dC$ then

$$d(-\ell^{-3} * G + \frac{1}{2} CG - I_7) := dP = 0$$

Now consider $Y = X \times \mathbb{R}$ so

$$P = P_X + P_0 dt$$

$d_X P_X = 0 \Rightarrow$ define $[P_X] \in H^7_{DR}(X)$,

$\dot{P}_X = d_X P_0 \Rightarrow [P_X]$ is conserved.

This conserved cohomology class is the Page charge.

Criticism:

- $[G] \neq 0 \Rightarrow C$ does not exist.

- $[P_X]$ is not quantized because we can shift C by any real harmonic form θ. $C \to C + \theta \Rightarrow$

$$[P_X] \to [P_X] + \frac{1}{2}[\theta] \wedge [G]$$

e.g. $X = R^3 \times S^3 \times S^4$, $G = \omega_{S^4}$, $\theta = x\omega_{S^3}$, $0 < x < 1$.

Introducing a basepoint

The first difficulty is minor.

Topological sectors are labelled by $a \in H^4(Y, \mathbb{Z})$.

For each topological sector choose a basepoint $C_\bullet$

$$C = C_\bullet + c, \qquad G = G_\bullet + dc$$

$c \in \Omega^3(Y)$, BUT! $C_\bullet \notin \Omega^3(Y)$.

$$\begin{aligned} S = 2\pi \int_Y -\frac{1}{2\ell^3} G \wedge *G \\ +2\pi \int_Y \left\{ c\Big(\frac{1}{2} {G_\bullet}^2 - I_8(g)\Big) + \frac{1}{2} c dc G_\bullet + \frac{1}{6} c (dc)^2 \right\} + \Phi_\bullet \end{aligned}$$

In this way we treat the CS term as a function.

But when $\partial Y = X$ it is really a section of a line bundle with connection.

Ignoring this fact can lead to problems.

Digression: The E_8 formalism

You do not need to know the E_8 model for the C-field to follow this talk.

However, for those who know, recall the main elements:

1. The gauge invariant information in C is encoded in the membrane coupling, which defines an element of $\check{H}^4(Y, U(1))$.

2. A "C field" is defined to be

$$\check{C} = (A, c) \qquad\qquad A \in \mathcal{A}(P(a))$$

such that the membrane coupling is

$$\sim \exp 2\pi i \int_\Sigma [CS(A) - \frac{1}{2} CS(\omega) + c]$$

The gauge-invariant fieldstrength is:

$$G = \mathrm{tr} F^2 - \frac{1}{2}\mathrm{tr} R^2 + dc = G_\bullet + dc$$

In this framework, a choice of basepoint can be simply a choice of connection $A_\bullet$ together with a choice of metric.

N.B. The space of C-fields is fibered over the space of metrics! G depends on a metric.

Hamiltonian Formalism for the C-field

We have now solved the first difficulty with the standard discussion of Page charge. Let us turn to the second: The quantization of $[P_X]$.

We will discuss it in the Hamiltonian formalism.

$$Y = X \times \mathbb{R}$$

The Hamiltonian formalism is straightforward:

Phase space: $(c, \Pi) \in \Omega^3(X) \times \Omega^7(X)$

$$\Omega = \int_X \delta\Pi \wedge \delta c$$

$$H = \frac{1}{2}\int_X \left[\frac{\ell^3}{2\pi}\tilde{\Pi} * \tilde{\Pi} + \frac{2\pi}{\ell^3} G_X \wedge * G_X\right]$$

$$\tilde{\Pi} := \Pi - 2\pi\left(\frac{1}{2} c G_{\bullet,X} + \frac{1}{3} c dc\right)$$

Neither Π nor $\tilde{\Pi}$ is conserved....

Page charge in Hamiltonian formalism

What is conserved is:

$$P_X = \frac{1}{2\pi}\tilde{\Pi} + \left(G_{\bullet,X}\, c + \frac{1}{2} cdc\right) + \Xi_\bullet$$

where $\Xi_\bullet$ is *any* “trivialization”

$$d\Xi_\bullet = \frac{1}{2} {G_{\bullet,X}}^2 - I_8$$

Spatial components of EOM:

$$\mathcal{G} = d_X P_X = d\tilde{\Pi} + 2\pi(\frac{1}{2} G_X^2 - I_8) = 0$$

This is the *classical Gauss law* for C-field Gauge invariance:

$$L = \Pi \wedge \dot{c} - H + c_0 \mathcal{G}$$

$[P_X]$ is the conserved Page charge - expressed in Hamiltonian formalism.

We still need to understand why it is quantized.

Gauss Law, and a Paradox

The quantization of $[P_X]$ should be understood by thinking about the quantum wavefunction of the C-field.

$$\Pi \sim -i \frac{\partial}{\partial c}$$

Page charge quantization follows from the Quantum Gauss law for *large* C-field gauge transformations.

"Gauge invariance"

$$c \to c + d\Lambda \qquad \Lambda \in \Omega^2(X) \qquad \text{small}$$
$$c \to c + \omega \qquad \omega \in \Omega^3_{\mathbb{Z}}(X) \qquad \text{all}$$

(Actually, the gauge group is slightly bigger...)

Gauge invariant wavefunctions satisfy:

$$\psi(c + \omega) = (\text{phase})\psi(c)$$

How the quantum Gauss law quantizes $[P_X]$

As in ordinary gauge theory, the Gauss law is the generator of gauge transformations, so

$$e^{-2\pi i \int \Lambda d_X P_X}\psi = \psi \qquad\qquad \forall \Lambda \in \Omega^2(X)$$

So:

$$e^{2\pi i \int d_X \Lambda \wedge P_X}\psi = \psi \qquad\qquad \forall \Lambda \in \Omega^2(X)$$

Generalize to *large gauge transformations*:

$$e^{2\pi i \int \omega \wedge P_X}\psi \overset{?}{=} \psi \qquad\qquad \forall \omega \in \Omega^3_{\mathbb{Z}}(X)$$

Note

$$P_X = \frac{1}{2\pi}\Pi + \cdots$$

So

$$e^{i \int \omega \Pi}\psi(c) = \psi(c + \omega)$$

Morally, the *operator equation*

$$e^{2\pi i \int \omega P_X} = 1 \qquad\qquad \forall \omega \in \Omega^3_{\mathbb{Z}}(X)$$

means the eigenvalues of P_X satisfy $[P_X] \in H^7(X, \mathbb{Z})$, as in the naive discussion.

However, there is a problem...

Page charges don't commute

It is convenient to define

$$P(\phi) = \int_X \phi \wedge P_X$$

for $\phi \in H^3(X)$.

Easy computation:

$$\boxed{[P(\phi_1), P(\phi_2)] = \frac{i}{2\pi} \int \phi_1 \wedge \phi_2 \wedge G}$$

$\Rightarrow$ Page charges don't commute!

This leads to an anomaly in the Gauss law:

$$U(\omega) = e^{2\pi i P(\omega)} = e^{2\pi i \int \omega P_X}$$

Now

$$U(\omega_1)U(\omega_2) = e^{-i\pi \int \omega_1 \omega_2 G} U(\omega_1 + \omega_2)$$

If the prefactor is a nontrivial phase we cannot consistently impose $U(\omega)\psi = \psi$ for all ω!

Anomaly in the Gauss law

(Nontrivial) fact: If X is a spin 10-manifold then

$$\int_X \omega_1 \omega_2 G \in \mathbb{Z} \qquad\qquad \forall \omega_1, \omega_2 \in \Omega^3_{\mathbb{Z}}$$

(Proof: Wu classes and Steenrod algebra....)

$\Rightarrow$ we only have a $\mathbb{Z}_2$ anomaly.

Still - if present it would mean we would have "1/2-membranes" ...

Also - when we come to the 5-brane - failure to account for this factor of 2 leads to the wrong (fourth power) of the 5-brane partition function.

Gauss law - Resolution

The anomaly arose because we failed to take into account the fact that the action of M-theory is really a section of a line bundle with connection.

To illustrate the point consider again the Landau problem

$$\psi(\vec{x}+\vec{n}) \stackrel{?}{=} e^{i\int_{\vec{x}}^{\vec{x}+\vec{n}} A}\psi(\vec{x})$$

We *can* say:

$$e^{in^x P_x}\psi = \psi \qquad \Leftrightarrow \qquad \psi(x+n^x, y) = e^{-i\int A}\psi(x,y)$$

$$e^{in^y P_y}\psi = \psi \qquad \Leftrightarrow \qquad \psi(x, y+n^y) = e^{-i\int A}\psi(x,y)$$

But then

$$e^{i(n^x P_x + n^y P_y)}\psi = e^{-i\int_{\triangle} F}\psi = e^{-i\pi k n^x n^y}\psi$$

Gauss law - Resolution II

Similarly, in M-theory

$$\exp 2\pi i \int \frac{1}{6} CGG - CI_8$$

is a section of a line bundle over the space of C-fields.

This bundle has connection and curvature

$$\mathcal{A} \sim 2\pi \int \delta C (\frac{1}{2} G^2 - I_8)$$

$$\mathcal{F} \sim \pi \int G \delta C \delta C$$

A (long) technical analysis leads to a consistent form of the Gauss law:

$$\exp\left(2\pi i \int \omega P_X\right) \psi = f_\bullet(\omega) \psi$$

Upshot:

1. $f_\bullet(\omega) = \pm 1$, is a function explicitly constructed from C. *It satisfies the $\mathbb{Z}_2$ cocycle law, and is not linear in ω!*

2. The Gauss law makes a specific choice of $\Xi_\bullet$

$$P_X = \frac{1}{2\pi} \tilde{\Pi} + \left(G_{\bullet,X} c + \frac{1}{2} c dc\right) + \Xi_\bullet$$

A word about the proof

Need to evaluate holonomies around cycles defined by $c \sim c + \omega$.

Using the connection find

$$\psi(c+\omega) = e_\omega(c)\psi$$

$$e_\omega(c) := \varphi(\check{C}_\bullet + c, \omega)^* e^{-2\pi i \int_X \left(\frac{1}{2}G - \frac{1}{6}dc\right)c\omega}$$

The subtle piece is $\varphi(\check{C}_\bullet + c, \omega)$ defined as follows (a construction first appearing in Witten's paper on 5-branes)

Given $\check{C}$ and ω on X we construct a *ω-twisted C-field on* $X \times S^1$.

(The E_8 model helps a lot here: We glue (A, c) to (A^g, c) where g is a nontrivial E_8 gauge transformation constructed from ω.

$$\varphi(\check{C}_\bullet + c, \omega) \sim \exp i\pi\eta(\not{D}_A) + \cdots)$$

The Page Charge Group

Now that we have a consistent Gauss law we can discuss the quantization of the Page charges.

Recall $P(\phi) = \int \phi P_X$ and $[P(\phi_1), P(\phi_2)] = \frac{i}{2\pi} \int \phi_1 \phi_2 G$.

$$U(\omega) P(\phi) U^{-1}(\omega) = P(\phi) - \int \omega \phi G$$

Thus, $P(\phi)$ isn't even gauge invariant! (In the presence of G-flux.)

Since we wish to work with gauge invariant objects we define:

$$T(\phi) := e^{2\pi i P(\phi)}$$

so

$$U(\omega) P(\phi) U^{-1}(\omega) = e^{2\pi i \int \phi \omega G} T(\phi)$$

so $T(\phi)$ is gauge invariant if

$$\int \phi \omega G \in \mathbb{Z} \qquad \qquad \forall \omega \in H^3(X, \mathbb{Z})$$

$\Rightarrow$ we can have a nontrivial Heisenberg group

$$T(\phi_1) T(\phi_2) = e^{-i\pi \int \phi_1 \phi_2 G} T(\phi_1 + \phi_2) = e^{-2\pi i \int \phi_1 \phi_2 G} T(\phi_2) T(\phi_1)$$

When is the Heisenberg extension nontrivial?

We study the bilinear form on $H^3(X,\mathbb{Z})$:

$$\mathcal{B}(\omega_1,\omega_2) := \int \omega_1\omega_2 G$$

- $\mathbb{Z}$-valued
- antisymmetric

$\Rightarrow$ we can choose a basis of integral forms

$$\mathcal{B} = \begin{pmatrix} 0 & k_1 \\ -k_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & k_2 \\ -k_2 & 0 \end{pmatrix} \cdots \oplus \begin{pmatrix} 0 & k_r \\ -k_r & 0 \end{pmatrix} \oplus 0_{b_3-2r}$$

Case I: $k_i = 0$, so $\mathcal{B} = 0 \Rightarrow P(\phi)$ are all gauge invariant and simultaneously diagonalizable.

$$[P_X] \in H^7(X,\mathbb{Z})$$

Case II: $k_i \neq 0$.
For simplicity assume $\mathcal{B}$ is nondegenerate and $k_i = k$.

Then we can split:

$$H^3(X,\mathbb{Z}) = \Lambda_1 \oplus \Lambda_2$$

So that

$$\mathcal{B} = k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Nontrivial Page charge group

Choose a symplectic basis α^I, β_I so

The page charge group is generated by

$$U_I = T(\frac{1}{k}\alpha^I) \qquad\qquad V_I = T(\frac{1}{k}\beta_I)$$

Relations:

$$U_I^k = V_I^k = 1 \qquad U_I V_J U_I^{-1} V_J^{-1} = e^{\frac{2\pi i}{k}\delta_{IJ}}$$

The gauge invariant operators form a representation of a finite Heisenberg group

$$0 \to \mathbb{Z}/k\mathbb{Z} \to \mathcal{H} \to H^3(X, \mathbb{Z}/k\mathbb{Z}) \to 0$$

The long distance limit in M-theory

In the M-theory case the long distance limit $\ell \to 0$ is governed by the harmonic modes of C.

These modes are again goverend by an effective Landau problem and behave analogously to the topological field theory above with an effective symplectic form:

$$\Omega = \int \delta\tilde{\Pi}\delta c + \pi \int_\Sigma G\delta c\delta c \to \pi \int_\Sigma G\delta c\delta c$$

Thus, in the analogy to 3d abelian Chern-Simons

$$\boxed{k \to \frac{1}{2}[G]}$$

Application: The 5-brane partition function

The 5-brane partition function is interesting because

1. 6D (2,0) superconformal field theory

2. 5-brane instantons are important for moduli stabilization, and decay of deSitter vacua in string theory.

We will show how our Gauss law together with the quantum mechanics of the harmonic modes of the c-field can be used to derive the 5-brane partition function discussed by Witten.

Use AdS/CFT

$$ds^2 = (k^{2/3}\ell^2)\left[dr^2 + e^{2r}ds^2_{D_6} + \frac{1}{4}ds^2_{S^4}\right]$$

$X = D_6 \times S^4$

$$G \to G_\infty = k\omega_{S^4} + \bar{G}$$

So we expect $(k \gg 1)$

$$Z(M/Y) = Z^{M5}_{U(k)}$$

Singleton sector

We can say something about the "singleton sector"

$$U(k) = \frac{SU(k) \times U(1)}{\mathbb{Z}_k}$$

$U(1) \sim$ C.O.M. for k 5-branes - couples to the harmonic modes $c_{\infty,h}$.

To write the general form of the partition function we use the symplectic form

$$\langle \omega_1, \omega_2 \rangle = \int_{D_6} \omega_1 \wedge \omega_2$$

$$H^3(D_6, \mathbb{Z}) = \Lambda_1 \oplus \Lambda_2$$

Then

$$Z^{M5}_{U(k)} = \sum_{\beta \in \Lambda_1 / k\Lambda_1} \zeta^\beta \Psi_\beta(c_{\infty,h})$$

• ζ^β is the contribution of the $SU(k)/\mathbb{Z}_k$ $(0,2)$ theory. β labels "'t Hooft sectors."

Note that dimensional reduction on S^1, $D_6 = D_5 \times S^1$, yields $SU(k)/\mathbb{Z}_k$ gauge theory and we have a natural symplectic splitting with

$$\Lambda_1 = H^2(D_5, \mathbb{Z}/k\mathbb{Z})$$

so β is literally an 't Hooft sector label. (Witten).

Page charge is dual to 't Hooft sector

• We'll derive a formula for Ψ_β below in terms of theta functions. It shows that the Heisenberg group of Page charges acts via

$$T(\phi_1)\Psi_\beta = \Psi_{\beta+\phi_1} \qquad \phi_1 \in \Lambda_1/k\Lambda_1$$
$$T(\phi_2)\Psi_\beta = e^{2\pi i k\langle\phi_2,\beta\rangle}\Psi_\beta \qquad \phi_2 \in \Lambda_2/k\Lambda_2$$

So the AdS dual interpretation of the 't Hooft sector label is the "Page charge."

• Note the sense in which the AdS partition function is a "vector in a Hilbert space" is that - as a function of $c_{\infty,h}$ it is in the linear span of Ψ_β.

Sketch of proof

1. As in the holographic RG, evolution in r = Euclidean time evolution.

2.

$$\Psi(r) = e^{-\int_{r_0}^{r} dr' e^{2r'} H} \Psi(r_0)$$

so $r \to +\infty$ projects to the groundstate of

$$H = \frac{1}{2} \int_X \left[\frac{\ell^3}{2\pi} \tilde{\Pi} * \tilde{\Pi} + \frac{2\pi}{\ell^3} G_X \wedge * G_X \right]$$

3.

$$c \to c_\infty = c_h + c' + c'' \in \mathcal{H}(X) \oplus \mathrm{Im}\delta_X \oplus \mathrm{Im} d_X$$

$\ell \to 0 \Rightarrow$ use a BO approximation ("integrated out massive modes") to isolate c_h dependence:

$$\Psi = \Psi_0(c', c'')\Psi_h(c_h)$$

Sketch proof - II

4. Choose an integral basis: $c_h = \sum_1^{b_3} c_a \omega^a$,

$$h^{ab} = \int \omega^a * \omega^b \qquad \mathcal{B}^{ab} = \int_X G\omega^a\omega^b = k\int_{D_6}\omega^a\omega^b$$

$$H_h = h_{ab}\big(-i\frac{\partial}{\partial c_a} - \pi\mathcal{B}^{aa'}c_{a'}\big)\big(-i\frac{\partial}{\partial c_b} - \pi\mathcal{B}^{bb'}c_{b'}\big)$$

5. This is literally the Landau problem $\Rightarrow$ overcomplete basis of states:

$$\psi \sim e^{-\frac{\pi}{2}Bz\bar{z}+vz}$$

Translating to the geometrical language:

$$\Psi_v(c) = e^{-\frac{\pi k}{2}\int_{D_6} c*c+\int v(1+i*)c}$$

6. These states are not gauge invariant. The average over large gauge transformations gives a projection to gauge invariant states:

$$\overline{\Psi_v} = \sum_{\omega\in\mathcal{H}^3_{\mathbb{Z}}(X)} (e_\omega(c))^*\Psi_v(c+\omega)$$

Sketch proof - III

$$\overline{\Psi_v} = \sum_{\omega \in \mathcal{H}^3_{\mathbb{Z}}(X)} (e_\omega(c))^* \Psi_v(c+\omega)$$

7. Written out explicitly this is

$$\bar{\Theta}_v = \sum_{\omega \in \mathcal{H}^3_{\mathbb{Z}}(Z_6)} \exp\left\{ -\frac{\pi k}{2} \int_{D_6} (c+\omega) * (c+\omega) - i\pi k \int_{D_6} c \wedge \omega \right\}$$
$$\varphi(\check{C}_\bullet, \omega) \exp\left\{ \int_{D_6} v \wedge (1+i*)(c+\omega) \right\}$$

where:

- $\varphi(\check{C}, \omega)$ is computed for $X = D_6 \times S^4$, *and depends on the the metric.*

- $H^3(D_6, \mathbb{R})$ has complex structure $J\omega = - * \omega$. Thus ω couples to $c^{(1,0)}$ and $v^{(0,1)}$.

- The span of the functions $\bar{\Theta}_v$ is *finite dimensional*, of dimension k: To see that we need the chiral splitting.

The chiral splitting

We want to interpret this as a sum over flux sectors in the $(2,0)$ theory.

But fluxes in the $(2,0)$ theory should be self-dual!

8. Choose a Lagrangian decomposition

$$H^3 = \Lambda_1 \oplus \Lambda_2$$

Then Poission summation: $\Rightarrow$

$$\overline{\Psi_v} = \sum_{\beta \in \Lambda_1/k\Lambda_1} \zeta^\beta(v)\Psi_\beta(c)$$

$$\Psi_\beta(c) = e^{-\frac{\pi k}{2}\int_{D_6} c * c + c^{(1,0)}\mathrm{Im}\tau c^{(1,0)}}\Theta_{\beta,k/2}(c^{1,0} + \theta + \tau\phi, \tau)$$

The characteristics are defined by the cocycle:

$$\varphi(C_\bullet, \omega) = \exp\left[2\pi i(\theta^I n_I + \phi_I m^I) + i\pi k \sum n_I m^I\right]$$

Comments on the result

1. Only derived for large k but we expect it is true for all k, including $k=1$.

2. Recall $G = \mathrm{tr}F^2 - \frac{1}{2}\mathrm{tr}R^2 + dc$, the space of C-fields is fibered over metrics. If we change the metric, $g_1 \to g_2$:

$$\frac{\varphi(\check{C}_1,\omega)}{\varphi(\check{C}_2,\omega)} = \exp[2\pi ik \int_{Z_6} \omega CS(g_1,g_2)]$$

Slogan: Characteristic $= CS(g)$.

$\Rightarrow$ characteristics are smooth functions of the metric $\Rightarrow$ potential *suppression* of 5-brane amplitudes:

$$\Theta \sim q^{\theta^2/2} + \cdots$$

3. If we did *not* twist the Θ function by φ then the chiral splitting leads to the wrong theta functions. $\Theta_{\beta,2k} \sim (\Theta_{\beta',k/2})^4$.

4. There can be a further shift of $v \to v + \kappa$ due to quantum effects.

Conclusions

1. We identified the quantization of $[G_7]$ with the quantum mechanical Gauss law for large C-field gauge transformations.

2. When $[G] \neq 0$ the CS term is a section of a line bundle with curvature

$$\mathcal{F} = \pi \int G \delta C \delta C$$

3. There is a strong analogy with the Landau problem:

Page charge $\leftrightarrow$ magnetic translation operator

4. The Page charge group in general is *not* the naive $H^7(X, \mathbb{Z})+$ shift, but rather a Heisenberg group determined by

$$\mathcal{B}(\omega_1, \omega_2) = \int G \omega_1 \omega_2$$

5. These remarks have applications to 5-brane partition functions.

6. We expect analogous phenomena in the typeII string. This should have important implications for the relation to K theory for nontorsion H-flux.