Lecture 1 Practice :

Put up **BEFORE** Lecture :

Notes @ www.physics.rutgers.edu/~gmoore/SCGP-FourManifoldsNotes-2017.pdf

(homepage talk # ....)

I

0. Historical / Topological Background.
1. Formal structure of CohTFT : Mathai-Quillen rep$^{\underline{n}}$ of Thom class. Fields-~~Eq~~ Equations-Symmetries.
2. How top. twisted $N=2$ SYM Fits in: Relation to Donaldson Polynomials

II

~~3. Seiberg-Witten LEETM~~ Q-Closed Observables.

3. Seiberg-Witten LEET for $N=2$ SYM.
   - Classical Vacua
   - Quantum Vacua

III

4. Gravitational Couplings in LEET
   - Coulomb branch
   - Higgs branch
5. Mapping Op's UV $\longrightarrow$ IR
6. General Form of Z_{Higgs}
7. ~~Coulomb~~ $Z_{Coulomb}$: The u-plane integral
8. Determination of unknown couplings via WC
9. Relation of D to SW invts. Simple type + Witten conj.

10. Physics Postdictions & Predictions

11. Open Problems + Future Directions

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X cpt, conn., orientable, 4-manifold.

$$H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\bar{H}^2(X) \subset H^2(X;\mathbb{R})$$

Q_X

Milnor: $X_1 \underset{h.e.}{\sim} X_2$ iff $Q_{X_1} \approx Q_{X_2}$

1982 Freedman: $\forall\, Q = Q_X$

- even $\quad !$
- odd $\quad 2 \quad$ at most one is smooth.

1983 - ... Donaldson

- X smooth Q_X definite $\longrightarrow$ diagonalizable
- polynomials on $H_0(X) \oplus H_2(X)$
  invts of smooth structure $b_2^+ > 1$

ASD eq. $\quad P \xrightarrow{G} X$, $g_{\mu\nu}$

$$F + *F = 0$$

$$\mathcal{M}(P, g)$$

1988 Witten – TFT

QFT : $N=2$ $G=SU(2)$ SYM.

Atiyah + Jeffrey

$g_{\mu\nu} \to t g_{\mu\nu}$, $t \to \infty$, $L \sim \frac{\hbar}{E}$

UV $\to$ IR : LEET

QCD: UV YM $G=SU(3)$
Coupled Dirac $N_f(3 \oplus \bar{3})$

IR LEET: NLσM $\pi: \mathbb{M}^{1,3} \longrightarrow SU(N_f)$

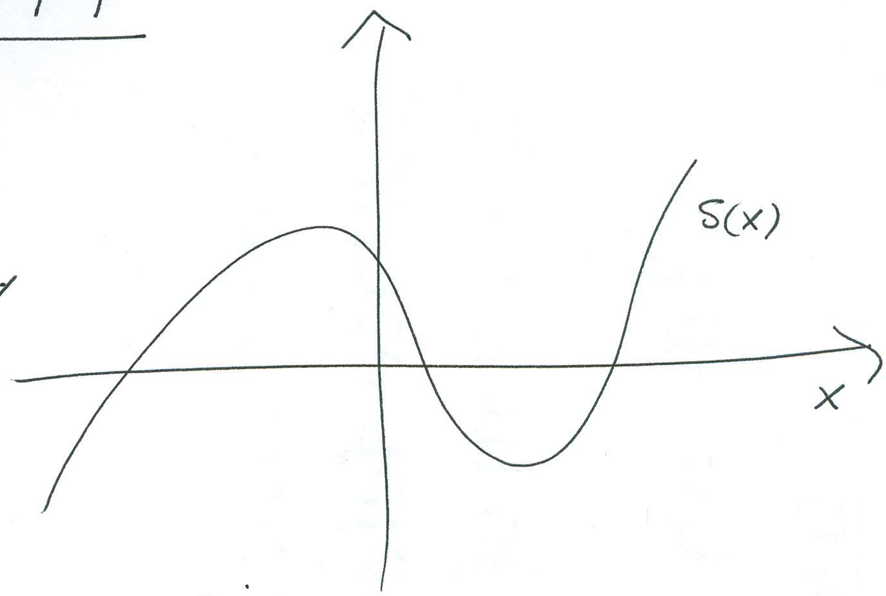
1994 Seiberg + Witten : LEET for $SU(2)$ SYM
Witten: SW eqs.

GOAL

1. Explain Witten's 1988 interp. of D.P.'s
2. " Seiberg Witten LEET
3. Show how it implies the "Witen conjecture."

12:47

12:48 CohTFT



$$Z = \int_{-\infty}^{\infty} \prod_i \frac{dx_i}{\sqrt{2\pi\hbar}} \cancel{s'(x)} \det\frac{\partial s^i}{\partial x_j} \, e^{-\frac{1}{2\hbar}(s(x))^2}$$

$$= \sum_{x_\ell : s(x_\ell)=0} \frac{\cancel{s'(x_\ell)}}{|\cancel{s'(x_\ell)}|} \operatorname{sgn} \det \frac{\partial s^i}{\partial x^j}$$

$$s : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

- Z Counting soln's to eqs w/ signs
- $\cap \#$ $\Gamma(s) \cap \Gamma(0)$ def. invt.
- $\hbar \longrightarrow 0$ localize to $Z(s) = \{x \mid s(x) = 0\}$
  S.P. appxt. exact.

"Supersymmetry"

$$\hat{M} = \Pi T^* M$$

x^i loc. coords, ψ^i

$$Q \circlearrowright C^\infty(\hat{M}) \cong \Omega^*(M)$$

$$\mathcal{O}_\omega \longleftarrow \omega$$

$$\mathcal{O} \longrightarrow \omega_{\mathcal{O}}$$

$$\psi^i \longleftrightarrow dx^i$$

$$gh\#(\mathcal{O}_\omega) = \deg \omega$$

$$Q x^i = \psi^i \quad Q\psi^i = 0$$

choose orient.

$$\int_{\hat{M}} \mathrm{Ber}(x|\psi)\, \mathcal{O}_\omega = \int_M \omega$$

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$\chi_a \quad a = 1, \dots, n$

$$Z = \int_{\widehat{\mathbb{R}}^n} \mathrm{Ber}(x|\psi) \int_{\Pi(\mathbb{R}^n)} \prod_{a=1}^{n} d\chi_a dH_a$$

$$\exp\Big(\overbrace{\cancel{-\frac{1}{2\hbar} s^a(x) s^a(x)} + i\chi_a \frac{\partial s^a}{\partial x^j}\psi^j \underline{- \frac{\hbar}{2} H_a H_a - i H_a s^a}}^{S'} \Big)$$

Leave on board

$$\underset{\substack{\uparrow \\ gh\# = -1}}{Q\chi_a} = \underset{\substack{\uparrow \\ gh\# = 0.}}{H_a} \qquad Q H_a \; 0$$

$$S' = Q(\Psi)$$

Leave on board

$$\Big[\; \Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a + \frac{\hbar}{2}\chi_a \# \Theta^{ab}_{i} \psi^i \chi_b$$

- Invariance under pert's.

$$\psi \longrightarrow \psi + \Delta\psi$$

- $(\chi_a, \underset{\substack{\uparrow \\ \text{auxiliary}}}{H_a})$ antighost mult.

- Localizing to Q- fixed points

$$Q\chi = H \underset{\substack{\uparrow \\ G}}{=} -is_a/\hbar \;\underline{\underline{=0}}$$

$$\int d\theta = 0.$$

All t_7 geom. PDE's are Q-fixed pt eqs. of a susy field/string theory.

1:03 "Nonzero index"

$$s: \mathbb{R}^n \longrightarrow V \cong \mathbb{R}^m$$

(χ_a, H_a) $\quad a = 1, \dots, m$

$$\widehat{\mathrm{Eul}}_s = \int_{\tilde{V}} \underbrace{\mathrm{Ber}(\cancel{H}/\chi)}_{\prod_1^m dH_a d\chi_a} e^{Q(\Psi)} \qquad gh\# = m$$

$$\tilde{V} = \Pi T V$$

$$\int_{\mathbb{R}^n} \mathrm{Ber}(x/\psi)\, \underset{\substack{\uparrow \\ Q\text{-closed}}}{\mathcal{O}_\omega(x,\psi)}\, \widehat{\mathrm{Eul}}_s$$

- ~~[illegible]~~ Can only be nonzero if $gh\,\mathcal{O} = n - m$
- only depends $[\mathcal{O}] \in H^*_Q$

$$= \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s = \int_{Z(s)} \iota^*(\omega)$$

Thom $\quad \pi : E \underset{\curvearrowleft s}{\longrightarrow} M \qquad \operatorname{rk} E = m, \quad \dim M = n$

$$H^i(M) \cong H^{i+m}_{\cancel{V\text{-cpt}}\ \mathrm{R.D.}}(E)$$

$$\omega \longmapsto \pi^*(\omega)\, \Phi(E)$$

$s^* \Phi(E)$ Euler class.

$$\int_M \omega\, s^* \Phi(E) = \int_{Z(s)} \iota^*(\omega)$$

Physics $\quad E$: "bundle of eqs"

put metric on fibers of E

∇ on $E \qquad H^{ab}_{j}$

$$S = -\frac{1}{2\hbar} s^a s^a + i \chi_a (\nabla_j s^a) \psi^j + \frac{\hbar}{2} \chi_a \chi_b F^{ab}_{ij} \psi^i \psi^j$$

$$\widehat{\mathrm{Eul}}_s(E, \nabla) = \int \prod \frac{d\chi_a\, dH_a}{2\pi i}\, e^{S'}$$

$$S' = Q(\Psi)$$

$$\nabla s : TM \longrightarrow E$$

$$0 \to \operatorname{Im} \nabla s \longrightarrow E \longrightarrow \operatorname{Cok} \nabla s \to 0$$

$$\int_{\hat{E}} \mathrm{Ber}(x, H \mid \psi, \chi)\, e^{S}\, \mathcal{O}$$

$$= \int_{\hat{M}} \mathrm{Ber}(x \mid \psi)\, \widehat{\mathrm{Eul}}_s(E, \nabla)\, \mathcal{O}$$

$$= \int_{Z(s)} \iota^*(\omega_{\mathcal{O}})\, \mathrm{Eul}(\mathrm{Cok} \nabla s)$$

- $\widehat{\mathrm{Eul}}_s(E, \nabla)$ pullback of MQ
- Equiv. $\Psi \longrightarrow \Psi + \Psi_{\text{proj}}$

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Equivariance (Gauge Theories)

$$\begin{array}{c} E \\ s \uparrow\downarrow \\ M \xrightarrow{\mathcal{G}} \bar{M} \end{array} \qquad \text{principal } \mathcal{G} \text{ bundle}$$

$$\mathcal{M} = Z(s)/\mathcal{G} \hookrightarrow \bar{M}$$

Equivariant coho. Cartan model $\phi \in \mathrm{Lie}(\mathcal{G})$

$gh\#(\phi) = 2$

$\bar{\phi}, \eta$ $\mathcal{W}(\mathrm{Lie}\,\mathcal{G})$

$-2, -1$

$$\mathcal{Z} = \int_{\mathrm{Lie}(\mathcal{G})} [d\phi] \int_{\hat{E} \times W(\mathrm{Lie}\,\mathcal{G})^\vee} \mathrm{Ber} \quad \mathcal{O} \quad e^{Q(\Psi + \Psi_{proj})}$$

path integral

$$= \int_{\mathcal{M}} \omega_{\mathcal{O}} \quad \mathrm{Eul}\big(\mathrm{cok}(\mathbb{F})\big)$$

Finite dim'l integral.

$$\mathbb{F} = \nabla s \oplus \nu^{\dagger}$$

Fredholm

$$\mathrm{Lie}(\mathcal{G}) \xrightarrow{\nu} TM \xrightarrow{\nabla s} E$$

$$\cancel{\nu^{\dagger} : TM \longrightarrow \mathrm{Lie}(\mathcal{G})}$$

$\neq 0$ only if $gh(\mathcal{O}) = \mathrm{Index}(\mathbb{F})$

1:22

CohTFT

Fields: x^i

Equations: $s(x) = 0$

Symmetries: $\mathcal{G}$

1:24:30 Twisted $N=2$ SYM.

- cpt. Liegp G, $\mathfrak{g}$
- X, $g_{\mu\nu}$
- $P \xrightarrow{G} X$

M coord's x^i $\longleftrightarrow$ $\mathcal{A} = \mathrm{Conn}(P)$

$E \to M$, s $\longleftrightarrow$ $\mathcal{E} = \mathcal{A} \times \Omega^{2,+}(\mathrm{ad}P)$

$s(A) = F + *F = F^+$

$\mathcal{G}$ $\longleftrightarrow$ $\mathrm{Aut}(P)$

Coh TFT $\Rightarrow$ $A, H, \phi, \bar{\phi}, \psi, \chi, \eta$

Formally: Localizes to $\mathcal{M}(P, g)$

1:28:

Twisted $N=2$ SYM

Physics: $N=2$ field theory QFT with
$\mathcal{H}$ = unitary rep$^{\underline{n}}$ of $N=2$ S.P. algebra

Wick

$$SP^0 = \mathbb{R}^4 \rtimes \left(su(2)_- \oplus su(2)_+\right) \oplus su(2)_R \oplus U(1)_R$$

$$SP^1 = \overset{\bar{Q}_{\dot\alpha}^A}{(1;2;2)^1} \oplus \overset{Q_\alpha^A}{(2;1;2)^{-1}}$$

$$\mathrm{Sym}^2 SP^1 \longrightarrow \mathbb{R}^4$$

$$\{Q_\alpha^A, \bar{Q}_{\dot\alpha}^B\} = 2\epsilon^{AB}\sigma^\mu_{\alpha\dot\beta} P_\mu$$

$$\{Q_\alpha^A, Q_\beta^B\} = 0$$

Unitary $(Q_\alpha^A)^\dagger = \epsilon_{AB}\bar{Q}^B_{\dot\alpha}$

Two field repn's <u>VM's</u> + HM's.

A_μ	$(2,2,1)^0$	$\mathfrak{g}$	
$\bar{\psi}_{\dot\alpha}^A$	$(1,2,2)^{-1}$	"	
ψ_α^A	$(2,1,2)^{+1}$	"	
ϕ	$(1,1,1)^2$	$\mathfrak{g}\otimes\mathbb{C}$	
$\bar{\phi}$	$(1,1,1)^{-2}$	$\mathfrak{g}\otimes\mathbb{C}$	$\bar{\phi} = \phi^*$
D	$(1,1,3)^0$		

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On general 4-fold X

Add data $P_R \xrightarrow{SU(2)_R} X$; <u>ω_R connection</u>

- $A \in \mathrm{Conn}(P)$ — assoc'd $\underline{3}$ of $SU(2)_R$

$\phi, \bar{\phi} \in \Gamma(\mathrm{ad}P \otimes \mathbb{C})$

$D \in \Gamma(\mathrm{ad}P \otimes W)$ — $W \to X$

$$\psi, \bar{\psi} \in \Gamma\big(\underbrace{S^{\pm} \otimes S_R}\otimes ad(P)\big)$$

<u>X not spin:</u> $\quad P_R \quad SO(3)_R$ bundle.

$$w_2(P_R) = w_2(X)$$

<u>Action:</u>

$$S_{phys} = \int \frac{1}{g_0^2} \operatorname{tr}\left(F*F + D\phi * D\phi^* - \frac{1}{4}[\phi, \phi^*]^2 \mathrm{vol}\right)$$
$$+ \int \frac{\theta_0}{8\pi^2} \operatorname{tr} F\wedge F + \ldots$$

$$Z(\omega^-, \omega^+, \omega_R) = \int [dA\, d\psi \ldots] e^{S_{phys}}$$

Amazing Fact: $\quad \omega_R = \omega^+ \quad \Rightarrow$ Dependence on

top. twisting $\quad\quad \omega^{\pm}$ drops out!

$$SU(2)_- \oplus \left(SU(2)_+' := \mathrm{Diag.} \subset SU(2)_+ \oplus SU(2)_R\right)$$

$$Q = \delta^{\dot\alpha}_A \bar{Q}^A_{\dot\alpha} \quad \text{Scalar susy } Q^2 = 0.$$

$$\boxed{Q\, A_\mu = \psi_\mu}$$

$$\psi_\alpha{}^A \longrightarrow \psi_\mu$$

$\bar{\Psi}_{\dot\alpha}{}^{A} \qquad 2 \otimes 2 \quad \xrightarrow{\text{symm}} \chi_{\mu\nu} \in \Lambda^{+}$

$\qquad\qquad\qquad\qquad \xrightarrow{\text{anti}} \eta \in \Lambda^{0}$

$\phi, \bar{\phi}$

$D \longrightarrow \quad D \in \Gamma(\Lambda^{+} \otimes \mathrm{ad}P) \quad (D \approx H$ gen. dir.

$Q\chi_{\mu\nu} = D_{\mu\nu}$

$$S_{phys} = Q(\Psi) + \text{const.} \int \mathrm{tr}\, F \wedge F$$

$S(A) = F^{+}$

$$\frac{\delta}{\delta g^{\mu\nu}} S_{phys.} = \sqrt{\det g_{\mu\nu}}\, T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$$

Formal independence. $b_2^{+}(X) = 1$.

$\boxed{1:48}$

Q – Fixed: $\quad Q\,\chi_{\mu\nu} = i\left(F_{\mu\nu}^{+} - D_{\mu\nu}^{+}\right)$

$\mathcal{M}(P, g)$:

$$0 \to \Omega^{0}(\mathrm{ad}\,P) \xrightarrow{\nabla_A} \Omega^{1}(\mathrm{ad}\,P) \xrightarrow{\nabla_A^{+}} \Omega^{2,+}(\mathrm{ad}\,P)^{\oplus}$$

$$D^{+} : \Gamma(S^{-} \otimes E) \longrightarrow \Gamma(S^{+} \otimes E)$$

$$E = \mathrm{ad}\,P \otimes S^{+}$$

- $\mathrm{vdim}\,\mathcal{M}(P,g) = 4h^\vee k - \dim G\left(\frac{\chi+\sigma}{2}\right)$

$$k = p_1(\mathrm{ad}P)/4h^\vee$$

$$\stackrel{su(2)}{=} 8k - 3(b_2^+ - b_1 + 1)$$

- $\mathcal{M}(P,g)$ singular, not cpt.

- Q - fixed pt. $\quad D_A\phi = 0$

$$[\bar\phi, \phi] = 0$$

- Fermions $\bar\psi, \psi$ $\quad U(1)_R$ charge ± 1

top: $\quad S_R = S^+$

index for $U(1)_R$ precisely $\not{D}_{AHS}$.

$$\mathrm{index}(\mathbb{F}) = \text{anomaly in } U(1)_R$$
$$= \quad '' \quad '' \quad \text{ghost\#}$$

1:55