Topics In The Relation Of Four-Manifold Invariants And Supersymmetric Field Theory

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1 Motivation

2 Topologically Twisted d=4 N=2 Field Theory

3 Review Of The Donaldson-Witten Paradigm

4 Massive surprise from SQCD

5 d=5: ``K-theoretic Donaldson invariants''

6 d=6: ``Elliptic Donaldson invariants''

7 Family Donaldson invariants

Glorious History Of
4d Field Theory & Four-Manifold Topology

Instantons (BPST) (1975) →

Donaldson invariants (1982) →

TQFT (1988) →

Seiberg-Witten Invariants (1994)

→ Revolution of 1995

Main motivation for this talk is the question:

Can we learn more about 4-manifolds using QFT (or string theory) ?

Some Ways To Generalize The Donaldson-Witten/Seiberg-Witten Paradigm

Other 4d N=2 theories

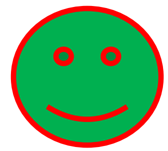
5d theories

6d theories

Coupling to background supergravity…

Why ?

1. Learn more about QFT, topology of moduli spaces, analytic number theory along the way



2. Learn new properties of Donaldson and Seiberg-Witten invariants.

3. Discover smooth invariants of four-manifolds that are independent of SW invariants.

?????

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``Standard'' Four-Manifolds

X: $\mathrm{d} = 4$, Smooth, compact, oriented, $\partial X = \emptyset$.

For **simplicity**: Connected & $\pi_1(X) = 0$

Essential in Donaldson & Seiberg-Witten theory:
X admits an almost complex structure

⟺ $b_2^+(X)$ *is* odd

Misses ``half'' the world of four-manifolds!

We relax that condition in part 7 of the talk.

Witten's Topological Field Theory

Witten constructed an $SU(2)$ gauge theory whose ``partition function'' is a function

$$Z_W : H_*(X) \to \mathbb{C}$$

Z_W only depends on:

1. oriented diffeomorphism type of X

2. A choice of $w_2(P_g)$ $('t\ Hooft\ flux)$ for the $SO(3)$ gauge bundle P_g

For other $d = 4, N = 2$ theories one

must

choose tangential structures

to define the (uv) twisted theory.

Example: $N = 2^*$ one must choose a spin-c structure

(Labastida-Marino; Manschot-Moore)

What is the general tangential structure we must choose to define a TFT?

Toplogical Twisting Of 4d $\mathcal{N} = 2$ Supersymmetric Theories 2411.14396

(with Vivek Saxena and Ranveer Singh)
Describe topological twisting of *arbitrary* d=4 N=2 theories,

even those with no Lagrangian.

Transfer Of Structure Group

$$\varphi : G_1 \to G_2$$

Homomorphism of groups

$$\varphi_* : B^\nabla G_1 \to B^\nabla G_2$$

Field Theory As Representation Of Geometric Bordism Category

$$Z: Bord_d^{\mathcal{F}} \rightarrow VECT$$

$\mathcal{F}$: Background fields (Sheaf on Man_d^{op})

$\mathcal{F}(X) = \{ Orientations\ on\ X \}$ (or other tangential structures)

$\mathcal{F}(X) = MET(X)$ (Aspirational.)

$\mathcal{F}(X) = \{Principal\ G - bundles\ with\ connection: (P, \nabla)\}$

$\mathcal{F}(X) = \{gerbes\ with\ \ connection\}$

$VECT$: Suitable category of vector spaces (Kontsevich-Segal; Wedeen)

Specialize to d=4, $\mathcal{N} = 2$ Field Theories

<u>Begin</u> with $\tilde{Z}: Bord_4^{\tilde{\mathcal{F}}} \to VECT$

$$\tilde{\mathcal{F}}(X) = \{(\tilde{P} \to X, \widetilde{\nabla})\}$$

$$\tilde{G} = Spin(4) \times SU(2)_R \times G^{flavor}$$

p_1 p_2 p_3

$Spin(4)$ $SU(2)_R$ G^{flavor}

Relation of $Spin(4)$ to Structure Group of TX

$$\pi: Spin(4) \rightarrow SO(4)$$

$$(\pi \circ p_1)_*(\tilde{P}, \widetilde{\nabla}) = (SOFr(TX), \nabla^{LC})$$

$(p_1)_*(\tilde{P}, \widetilde{\nabla})$: Spin bundle with spin connection

$\Rightarrow \tilde{Z}: Bord_4^{\tilde{\mathcal{F}}} \rightarrow VECT$ is only defined on spin manifolds

To define theories on a larger domain we consider a slightly different set of background fields $\mathcal{F}$

$$\mathcal{F}(X) = \{\,(P, \nabla)\}$$

$$G := \tilde{G}/C \quad \text{with} \quad C \subset \; Z(\tilde{G})$$

$$Bord_4^{\tilde{\mathcal{F}}} \xrightarrow{\ \tilde{Z}\ } VECT$$

$$\pi_* : Bord_4^{\tilde{\mathcal{F}}} \to Bord_4^{\mathcal{F}}$$

$$Z : Bord_4^{\mathcal{F}} \to VECT$$

This can be done for suitable C so that we can work with all standard 4-manifolds

Now we describe how to define topological twisting of such theories Z

Twisting= Special choice of background fields

Witten's Homomorphism

$$\varphi_W : SO(4) \rightarrow \frac{Spin(4) \times SU(2)_R}{\langle (-1, -1) \rangle}$$

$\langle(-1,-1)\rangle \cong \mu_2$ acts trivially on vectormultiplets

$$(u_1, u_2) \in SU(2) \times SU(2) \cong Spin(4)$$

$$\varphi_W([u_1, u_2]) := [(u_1, u_2), u_1]$$

Pulls back spinorial $\otimes$ spinorial representations of $Spin(4) \times SU(2)_R$ to vector representations.

Topological Twisting For General d=4 N=2 Theory

When the background fields $\mathcal{F}$ are determined by transfer from a bundle with connection for a suitable structure group G^{tw} the partition function Z will be

<u>locally constant</u>

as a function of ∇^{tw}

(in particular, metric-independent)

``Suitable'' means:

$$G^{tw} := (Spin(4) \times G^{flavor})/C^{tw}$$

$$\begin{array}{ccc} G^{tw} & \xrightarrow{\varphi^{tw}} & G = \tilde{G}/C \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ \dfrac{Spin(4)}{\langle -1 \rangle} & \xrightarrow{\varphi_W} & \dfrac{Spin(4) \times SU(2)_R}{\langle (-1,-1) \rangle} \end{array}$$

Background fields of the ``physical theory'' are determined by

$$\varphi_*^{tw}(P^{tw}, \nabla^{tw})$$

$$(p_1)_*(P^{tw}, \nabla^{tw}) = (SOFr(TX), \nabla^{LC})$$

Is a generalized Spin − c connection

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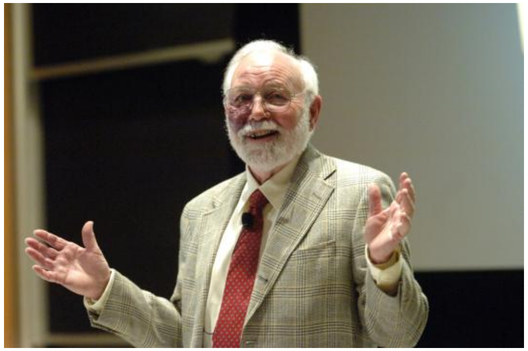
Q-Symmetry

Witten 1988;

Baulieu & Singer, 1988



$\boldsymbol{P_g, \nabla_g}$ **: Dynamical gauge bundle and connection.**

$$Q\,A_\mu = \psi_\mu \quad Q\,\psi_\mu = -D_\mu \phi \qquad Q\,\phi = 0$$

$$A \in \mathcal{A}(P_g) \qquad \phi \in \Omega^0(X, ad\, P_g \otimes \mathbb{C})$$

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P_g)$

$\mathcal{G} := Aut(P_g)$ Group of gauge transformations

Q −closed observables: $\mathcal{O}(pt) = Tr\left(\phi^2(pt)\right)$

$$\{K, Q\} = d \Rightarrow \quad \mathcal{O}_j := K^j \mathcal{O}$$

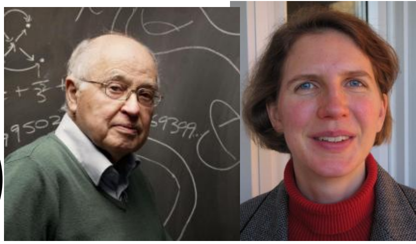
$\mathcal{O}\left(\Sigma_j\right) := \int_{\Sigma_j} \mathcal{O}_j$ only depends on $\left[\Sigma_j\right] \in H_j(X)$

Function on $H_*(X)$: $Z_W(\Sigma) = \left\langle e^{\mathcal{O}(\Sigma)} \right\rangle_{tw}$

Witten (1988):

Z_W independent of metric $g_{\mu\nu}$ on X

Witten (1988) & Atiyah& Jeffrey(1990)



$Z_W(\Sigma)$ path integral localizes to an integral over

$$\mathcal{M}^{inst} \subset \mathcal{A}(P_g)/\mathcal{G}$$

$$\mathcal{M}^{inst} := \{A \in \mathcal{A}\left(P_g\right) : F(A)^+ = 0\}/\mathcal{G}$$

$$F^+ := \frac{1}{2}(F + * F)$$

Donaldson Invariants

Donaldson: $\mu: H_*(X) \to H^*(\mathcal{M}^{inst})$

$$Z_D: H_*(X) \to \mathbb{Q}$$

$$Z_D(\Sigma) = \int_{\mathcal{M}^{inst}} e^{\mu(\Sigma)}$$

$\mathcal{M}^{inst}$ depends on $g_{\mu\nu}$, but Z_D does not

Main Statement

$$Z_W = Z_D \quad =: Z_{DW}$$

Evaluation Of $Z_{DW}(\Sigma)$

Z_{DW} independent of $g_{\mu\nu}$ on X

Consider metric $L^2 g_{\mu\nu}$ in the limit $L \to \infty$

$\Rightarrow$ Use Seiberg-Witten LEET

$$Z_{DW} = Z_{Coul}^{J} + Z_{SW}^{J}$$

Witten 94
Moore-Witten 97

$b_2^+ = 1$: J: oriented line $H^{2,+}(X) \subset H^2(X)$

One can derive Z_{SW}^{J} from $\mathrm{Z}_{\mathrm{Coul}}^{J}$

Probably related to Mochizuki's work

$\Rightarrow$ start with $\mathrm{Z}_{\mathrm{Coul}}^{J}$

Z_{Coul}^{J} : Integral over complex plane parametrized by $u \in \mathbb{C}$

Compute $\mathrm{Z}_{\mathrm{Coul}}^{J}$ using LEET of Seiberg & Witten + couplings to background gravity and flavor gauge fields determined in other ways…

$$Z^J_{Coul}(\Sigma) = \int_{\mathcal{F}(\Gamma^0(4))} d\tau d\bar{\tau}\ \mathcal{H}(\tau)\frac{\partial}{\partial\bar{\tau}}\ \widehat{G}^J(\tau,\bar{\tau},\Sigma)$$

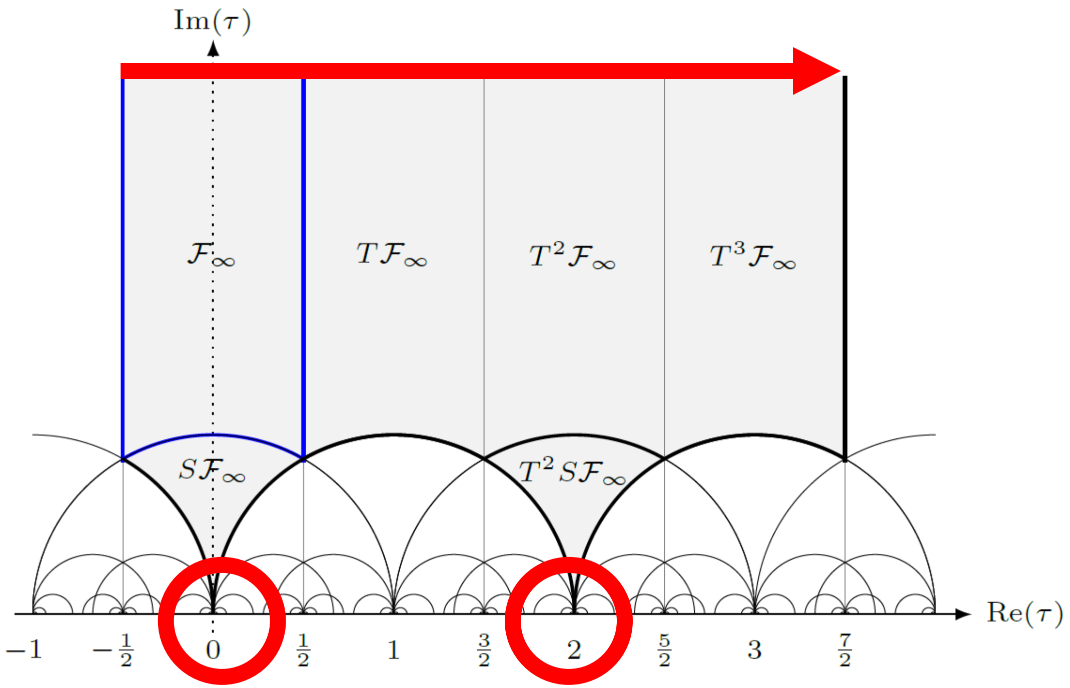
$\widehat{G}^J$ varies with metric on X

$\mathcal{H}$ & $\widehat{G}^J$ only depend on topology of X through cohomology ring $H^*(X)$

$\widehat{G}^J(\tau,\bar{\tau},\Sigma)$: Is a mock Jacobi form

G. Korpas, J. Manschot, G. Moore, I. Nidaiev (2019)

$$Z^J_{Coul}(\Sigma) \sim \sum_{cusps} [\mathcal{H}(\tau) G^J(\tau,\Sigma)]_{q^0}$$



Cusps = $\infty \cup \tau = 0$ $[Monopole]$ $\cup$ $\tau = 2$ $[Dyon]$

$u = \Lambda^2$ $\quad$ $u = -\Lambda^2$

Singularities at $u = \pm\Lambda^2$ spoil topological invariance.

Restore it with path integral for LEET valid in a germ of neighborhood around $\pm\Lambda^2$

$$Z^{J}_{SW,\pm\Lambda^2}(\Sigma) = \sum_{c\in Spin^c(X)} SW^J(c)\; f_{c,\pm\Lambda^2}(\Sigma)$$

$SW^J(c) \in \mathbb{Z}$ Seiberg-Witten invariants

$f_{c,\pm\Lambda^2}(\Sigma)$ ***is computed from*** $Z^J_{Coul}(\Sigma)$

⇒ Derivation of the Witten conjecture

That was for SU(2) SYM with no extra fields.

What about the infinite collection of other N=2 theories, all of which can be topologically twisted?

What Do The Other (Lagrangian) Theories Compute?

The path integral for topologically twisted Lagrangian theories localizes to ``integration of coho classes on $\mathcal{M}$''

$\mathcal{M}$: moduli stack of the Nonabelian monopole equations

[Labastida-Marino 1997; Losev-Shatashvili-Nekrasov1997]

(instanton moduli space is a special case)

Nonabelian Monopole Equations

aka Nonabelian Seiberg-Witten Equations

$$G^{tw} = \left(\tilde{G}_g \times Spin(4) \times G^{flavor}\right)/C^{tw}$$

$(p_2)_*(P^{tw}, \nabla^{tw}) = (SOF(TX), \nabla^{LC})$ is <u>fixed</u>

Matter (hypers): V : complex representation of $\widetilde{G_g}$

$V \otimes S^+$: Representation of G^{tw}

$\mathcal{V}$: Associated bundle: $M \in \Gamma(\mathcal{V})$

$$F(A_g)^+ = q(M, \bar{M}) \qquad D\, M = 0$$

N.B.! In contrast to the Seiberg-Witten equations

The spin-c connection, and flavor connection (if present) are fixed background fields.

The nonabelian monopole equations should be regarded as equations for the dynamical nonabelian gauge field A_g and the monopole/quark fields $M \in \Gamma(\mathcal{V})$

4d, 5d, 6d path integrals require knowledge of $w_1(\mathcal{M})$, $w_2(\mathcal{M})$ and $p_1(\mathcal{M})$, respectively

Discussions with D. Freed and M. Hopkins settle the case of w_1, w_2 for instanton moduli space $\mathcal{M}^{inst}$

For instanton moduli space there is a KO class T on $\mathcal{A}(P_g)/\mathcal{G}$ that restricts to the tangent bundle on the instanton moduli stack,

$$-T = \Omega_X^0(adP_g) - \Omega_X^1(adP_g) + \Omega_X^{2,+}(adP_g)$$

FHM draft computes $w_i(T)$ for $i = 1, 2$

!!! turns out to be cohomological in $w_i(X), w_i(P_g), \mathrm{i} = 1, \dots, 4$

a surprise since $w_i(T)$ are mod-two indices in KO theory.

$$p\colon BG^X \times X \to BG^X \qquad ev\colon BG^X \times X \to BG$$

$$x_0 \in X, \qquad \mathfrak{g}_{x_0} \coloneqq p^* ev_{x_0}^*(ad\,P) \in KO(BG^X \times X)$$

$$w_i \coloneqq w_i(TX) \qquad w_i' \coloneqq w_i(-ev^*(ad\,P))$$

$$w_1(T) = \frac{\chi + \sigma}{2}\, w_1(\mathfrak{g}_{x_0})$$

$$w_2(T) = w_2(\mathfrak{g}_{x_0}) \int_X w_2 w_2' + w_1(\mathfrak{g}_{x_0}) \int_X w_2 w_3'$$

$$+ \int_X w_2 w_4' + w_2\left(-\frac{\chi + \sigma}{2}\, \mathfrak{g}_{x_0}\right)$$

What Do The Other NON-Lagrangian Theories Compute?

WE DON'T KNOW!!!

There is no known description in terms of intersection theory on some moduli space of solutions of some pde

It is unlikely there is such a description.

Still, we CAN write the answers.

AD3: X of $SWST$ and $b_2^+ > 1$:

$$Z^{AD3}(p,S) = \sum_{\mathrm{c} \in \mathrm{Spinc}} e^{2\pi i\, \mathrm{c}\cdot w_2} SW(\mathrm{c}) \cdot$$

$$\cdot \left\{ (\mathrm{c}\cdot S)^{\mathfrak{B}-2} \left(24 (\mathrm{c}\cdot S)^2 + \mathfrak{B}(\mathfrak{B}-1) S^2 \right) \right\}$$

$$\mathfrak{B} = \chi_h - c_1^2 = -\ (7\,\chi + \ 11\sigma)/4$$

Surprise! p drops out: U(p) is a ``null vector''

(but NOT if X is not of SWST)

Moore-Nidaiev 2018:

Are all the invariants for non-Lagrangian theories expressed in terms of SW ?

WE DON'T KNOW!!!

(Manschot-Moore 2104.06492 section 8.4 some relevant remarks hinting that the answer is "no" for class S theories.)

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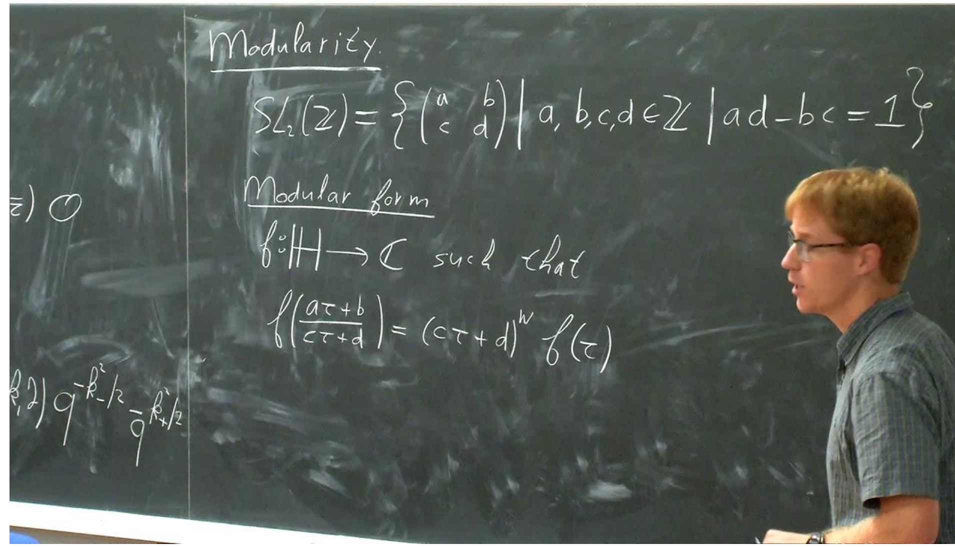
A Surprise From SQCD



Work with Elias Furrer
and Jan Manschot

SQCD

$$\frac{SU(2)_g \times Spin(4) \times U(1)^f}{C} \qquad f = 1,2,3,4$$

$\mathcal{V} = (E \otimes S^+)^{\oplus f}$ $\quad E$: Fundamental rep of SU(2)

$$\left(F^a_{\dot\alpha\dot\beta}\right)^+ = \frac{i}{2}\sum_j \bar{M}^j_{(\dot\alpha}\, T^a\, M^j_{\dot\beta)}$$

$$D\, M^j = 0$$

Defines moduli space $\mathcal{M}^Q$

$$vdim\mathcal{M}_k^{inst} = 8k - \frac{3}{2}(\chi + \sigma)$$

$$IndexD = -k + \frac{c(\mathfrak{s})^2 - \sigma}{4}$$

$$d_k^Q = vdim\mathcal{M}_k^Q = 2(4-f)k - \frac{3}{2}(\chi + \sigma) + \frac{f}{2}(c(\mathfrak{s})^2 - \sigma)$$

Nonabelian monopole equations have $U(1)^f$ symmetry

$$M_j \to \zeta_j M_j \quad \zeta_j \in U(1)$$

Work equivariantly with parameters m_j , $j = 1, \ldots, f$.

$$Z(m) = \int_{\mathcal{M}^Q} (Eul(Cok(D); m)^f$$

Localization wrt $U(1)^f$: $\mathcal{M}^Q = \mathcal{M}^{inst} \cup \mathcal{M}^{red}$

$$Z(m) = Z^{inst}(m) + Z^{red}(m)$$

Losev, Nekrasov, Shatashvili 97: $Z^{inst}(m) =$

$$\sum_k \Lambda_f^{d_k^Q} \prod_j m_j^{I_k} \int_{\mathcal{M}_k^{inst}} \prod_j \{ \sum_\ell \left(\frac{c_\ell(Ind(D))}{m_j^\ell} \right) \}$$

Physics Answer Derived From Seiberg-Witten Special Kahler geometry

SW curve is $y^2 = x^3 - g_2 x - g_3$

g_2, g_3: Polynomials in Λ_f, m_j, u , $f = 1,2,3$

for $N_f = 1$:

$$g_2 = \frac{4u^2}{3} - \Lambda_1^3 m \ ,$$

$$g_3 = \frac{8u^3}{27} - \frac{1}{3}\Lambda_1^3 mu + \frac{\Lambda_1^6}{16} \ ,$$

for $N_f = 2$:

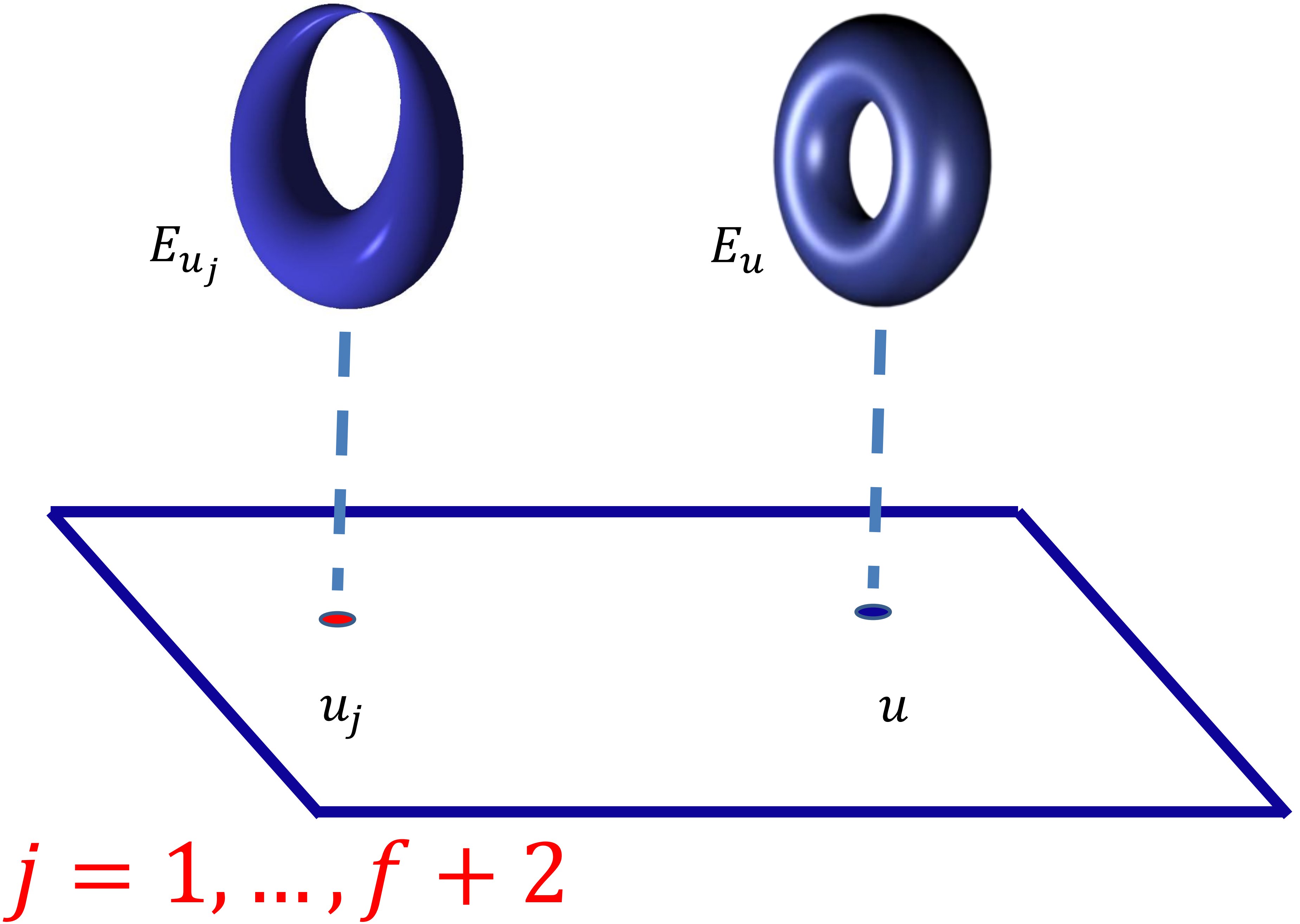
$$g_2 = \frac{4u^2}{3} + \frac{\Lambda_2^4}{16} - \Lambda_2^2 m_1 m_2 \ ,$$

$$g_3 = \frac{8u^3}{27} - \frac{1}{24}u\left(\Lambda_2^4 + 8\Lambda_2^2 m_1 m_2\right) + \frac{1}{16}\Lambda_2^4\left(m_1^2 + m_2^2\right) \ ,$$

Family over the u-plane degenerates with I_1 singularities at $f+2$ distinct points

$$u_j \quad j = 1, \ldots, f$$

$$g_2(u_j)^3 = 27\, g_3(u_j)^2$$



E_{u_j}
E_u
u_j
u
$j = 1, \ldots, f + 2$

Moore-Witten 97

$$b_2^+ > 1: \quad Z(m) = \left(\sum_j R_j\right) \mathcal{SW}$$

$$\mathcal{SW} = \sum_c (-1)^{\frac{K(K+c)}{2}} SW(c)$$

$$R_j = (-1)^{f\,\chi_h} 2^{c_1^2+1} 3^{c_1^2-4\,\chi_h} \Lambda_f^{(2f-3)\chi_h - c_1^2} \times$$

$$\times \left(\frac{g_2(u_j)}{g_3(u_j)}\right)^{\frac{\chi_h - c_1^2}{2}} \frac{g_2(u_j)^{6\chi_h} g_3(u_j)^{-3\chi_h}}{\Delta'(u_j)^{\chi_h}}$$

$$\Delta(u) := \prod_j (u - u_j)$$

Looking at hundreds of examples shows the sum is amazingly simple:

$$Z(m) = C \cdot \frac{Q}{V(m)^{2\left[\frac{1}{2}\chi_h\right]} P_{sc}^{Max\left[0,\left[\frac{\chi_h - c_1^2}{4}\right]\right]}}$$

$$C = (-1)^{f\chi_h} \; (pos.power\ of\ 2) \, \Lambda_f^{(2f-3)\chi_h - c_1^2}$$

V, P_{sc}, Q : Rel. prime polynomials in m_j & Λ_f

$$V(m) = \prod_{i<j} (m_i^2 - m_j^2)$$

$$\Delta(u) := \prod_j (u - u_j) \qquad Disc(\Delta) = P_{sc}^3\ V(m)$$

P_{sc} : Minimal polynomial of the locus

$$\{\, (m_j, \Lambda_f):\ \ \exists u\ \ g_2 = g_3 = 0 \,\}$$

Argyres-Douglas superconformal theories

[Argyres&Douglas; Argyres,Plesser,Seiberg,Witten]

$$P_1^{\mathrm{AD}} = 27\Lambda_1^3 - 64m^3,$$

$$P_2^{\mathrm{AD}} = \Lambda_2^6 - 12m_1m_2\Lambda_2^4 + 3(9m_1^4 + 9m_2^4 - 2m_1^2m_2^2)\Lambda_2^2 - 64m_1^3m_2^3,$$

Q : Polynomial in m_j, Λ_f

WITH INTEGRAL COEFFICIENTS!

The mathematical significance of these integers is unclear.

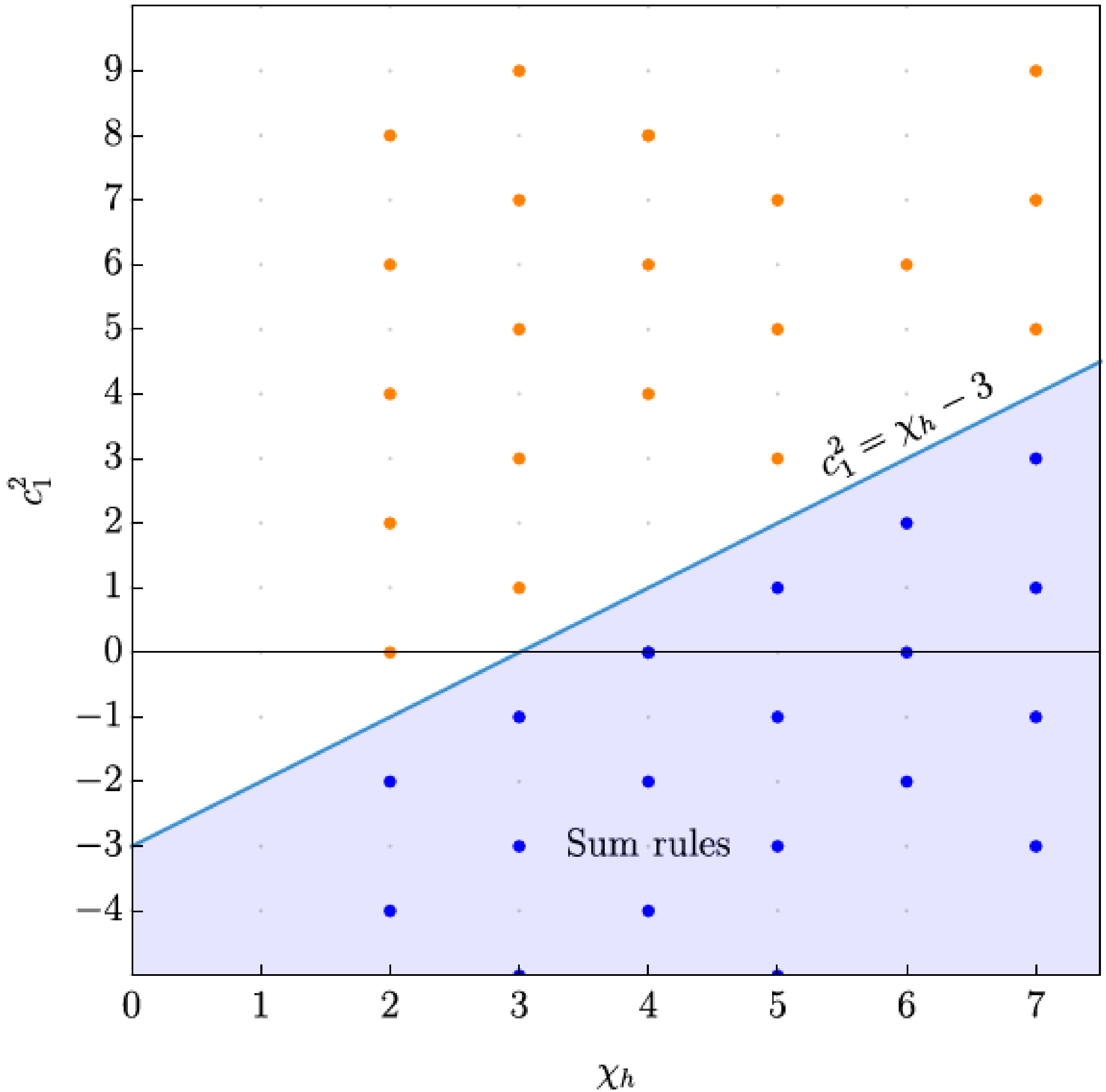
c_1^2	Q^1_{2,c_1^2}
-8	$36\Lambda_1^2$
-6	$576\Lambda_1^2 m^2$
-4	$64\Lambda_1 m$
-2	$8\Lambda_1(64m^3 - 9\Lambda_1^3)$
0	$4m^2$
2	$8m(4m^3 - \Lambda_1^3)$
4	$9\Lambda_1^6 + 256m^6 - 128\Lambda_1^3 m^3$
6	$32m^2(8m^3 - 3\Lambda_1^3)^2$
8	$16m(-27\Lambda_1^9 + 1024m^9 - 1024\Lambda_1^3 m^6 + 316\Lambda_1^6 m^3)$
10	$2(243\Lambda_1^{12} + 65536m^{12} - 81920\Lambda_1^3 m^9 + 35328\Lambda_1^6 m^6 - 5760\Lambda_1^9 m^3)$

$$Z(m) = C \cdot \frac{Q}{V(m)^{2\left[\frac{1}{2}\chi_h\right]} \, P_{sc}^{Max\left[0,\left[\frac{\chi_h - c_1^2}{4}\right]\right]}}$$

Denominator P_{sc} turns on when $\chi_h - c_1^2 > 3$

But physically there should not be any singularity at the zero locus of P_{sc}

Therefore we must have $\mathcal{SW} = 0$ for $\chi_h - c_1^2 > 3$



$\mathcal{SW} = 0$

Reproduces special case of MMP 98

$$Z(m) = \begin{cases} C \dfrac{Q}{V^{2\left[\frac{\chi_h}{2}\right]}} \mathcal{SW} & c_1^2 \geq \chi_h - 3 \\ 0 & c_1^2 < \chi_h - 3 \end{cases}$$

Large Mass Limit

$$|m_j| \gg |\Lambda_f| \quad j = 1, \ldots, f$$

$f + 2$ singularities on the u-plane split into two groups

$$u_+ = \Lambda_0^2 + \cdots$$

$$u_- = -\Lambda_0^2 + \cdots$$

$$u_j = m_j^2 + \cdots$$

$$j = 1, \ldots, f$$

$$Z(m) = \left(Z_{u_+} + Z_{u_-} \right) + \sum_{j=1}^{f} Z_{u_j}$$

Conjecture:

$U(1)^f$-localization: $Z(m) = Z^{inst}(m) + Z^{red}(m)$

$$Z(m) = \left(Z_{u_+} + Z_{u_-} \right) + \sum_{j=1}^{f} Z_{u_j}$$

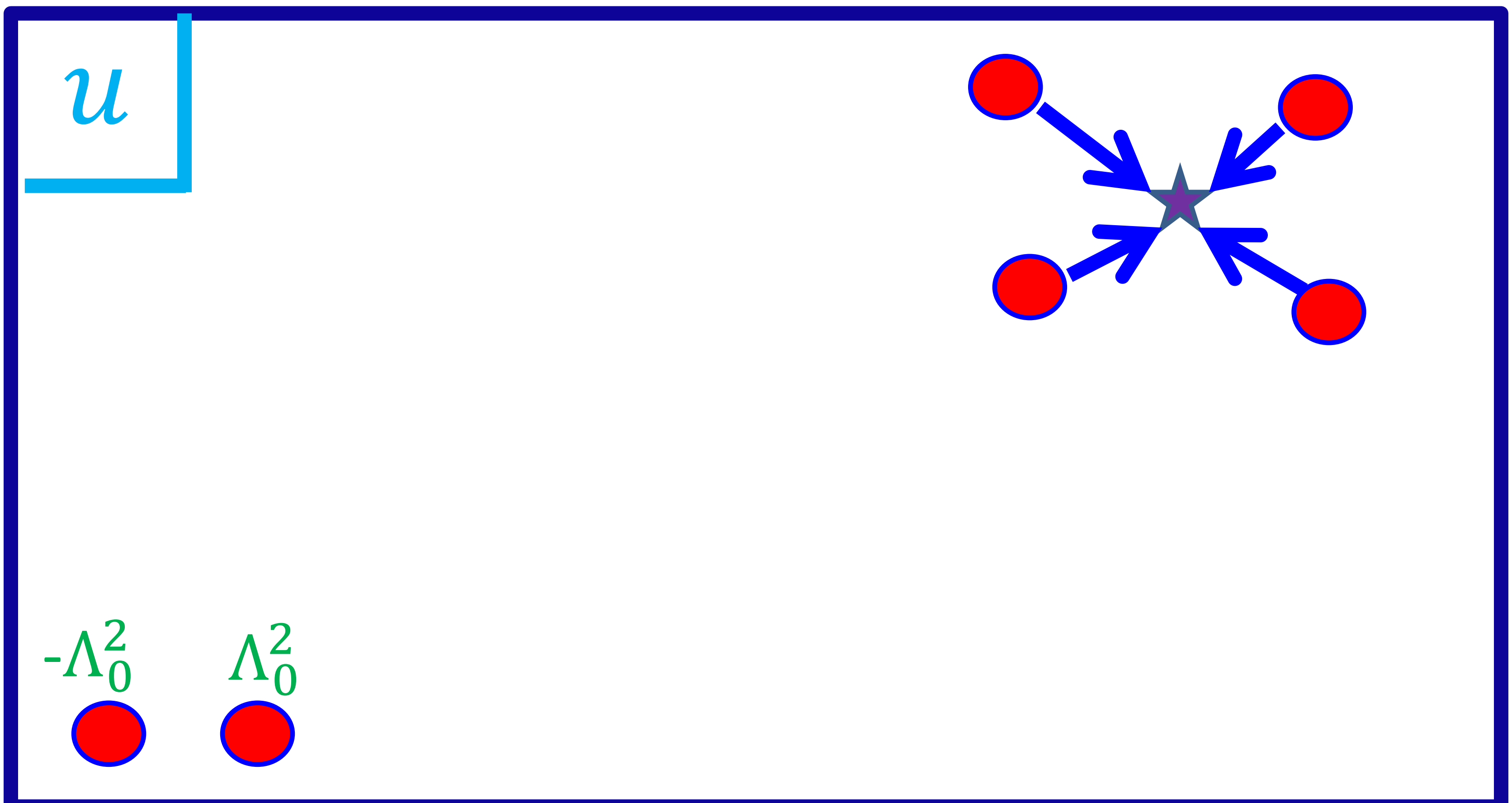
$$Z^{inst}(m) = Z_{u_+} + Z_{u_-}$$

Manschot-Moore: S-duality of the N=2*

In a separate work Furrer & Manschot have verified this conjecture in a highly nontrivial way by reproducing intricate formulae of Gottsche & Kool for Segre' numbers of moduli spaces of sheaves of algebraic surfaces.

Singularities when V(m)=0: Collision Of Mutually Local Singularities

Interesting things when mass parameters m_j take values so that mutually local u_j collide.



LEET near collision point has several light monopole fields

Q-fixed point equations of the LEET are the ***Abelian*** multimonopole equations:

$$F^{+}_{\dot{\alpha}\dot{\beta}} = \sum_{j=1}^{f} \bar{M}^{j}_{\dot{\alpha}} M^{j}_{\dot{\beta}}$$

Equations for spin-c connection

$f > 1$: Equivariant integral over this moduli space is expected to diverge like $V(m)^{-\chi_h}$ for $m_i^2 \sim m_j^2$

Dedushenko, Gukov, and Putrov 2017

Three-Slide Summary Of Remaining Material

Partially Twisted 5d SYM on $X \times S^1$

$$Z(\mathcal{R}, n) = \sum_k \mathcal{R}^{\frac{d_k}{2}} I_k$$

I_k: L^2-index of a spin-c Dirac operator on $\mathcal{M}_k^{inst}$ for spin-c structure depending on $n \in H^2(X, \mathbb{Z})$

Can evaluate the series explicitly.
There is a singularity when $\mathcal{R}^4 = 1$

The singularities suggest there is a generalization of SW invariants based on Abelian multi-monopole moduli space

Partially Twisted 6d SYM on $X \times \mathbb{E}$

$$Z \sim \sum_k \mathcal{R}^{\frac{d_k}{2}} Ell(\mathcal{M}_k; m, \tau_{\mathbb{E}})$$

$\mathcal{M}_k$: nonabelian monopole moduli space

Family Invariants

Couple d=4 N=2 twisted SYM to twisted truncated superconformal gravity

$\mathcal{G}_d$ −equivariant cohomology of $MET(X)$

$$Q\ g_{\mu\nu} = \Psi_{\mu\nu} \qquad Q\ \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu \qquad Q\Phi^\mu = 0$$

$Z\left[g_{\mu\nu}; \Psi_{\mu\nu}; \Phi^\mu\right]$: Represents cohomology class in $H^*\left(BDiff^+(X)\right)$

1 Motivation

2 Topologically Twisted d=4 N=2 Field Theory

3 Review Of The Donaldson-Witten Paradigm

4 Massive surprise from SQCD

5 d=5: ``K-theoretic Donaldson invariants''

6 d=6: ``Elliptic Donaldson invariants''

7 Family Donaldson invariants

5D SYM & K-Theoretic Invariants

For 5d SYM on $X \times S^1$

$\exists$ (partial) topological twisting making it topological on X

Not topological on S^1:

$$Z(\mathcal{R}) = \sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} I_k$$

$$Z(\mathcal{R}) = \sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} I_k$$

$$d_k = \dim_{\mathbb{R}} \mathcal{M}_k = 4h^{\vee}k - \dim G \frac{\chi+\sigma}{2}$$

$$\mathcal{H}_k = L^2(S \to \mathcal{M}_k)$$

$$I_k = Tr_{\mathcal{H}_k}\{(-1)^F \; e^{-\mathcal{R}\,D^2}\}$$

L^2 –index of the Dirac operator D

$\Rightarrow$ ``K-theoretic Donaldson invariants''

[Nekrasov, 1996; Losev, Nekrasov, Shatashvili, 1997] arXiv: 2509.23042

Kim, Manschot, Moore, Tao, Zhang

In general, the moduli space of instantons is not spin.

But if X admits an ACS, then spin-c structures exist on $\mathcal{M}_k$

They depend on a choice $n^{(I)} \in H^2(X, \mathbb{Z})$

$$\Rightarrow Z\left(\mathcal{R}, n^{(I)}\right)$$

$$Z(\mathcal{R}, n^{(I)}) =$$

$$\sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} \, Tr_{\mathcal{H}_k}\{(-1)^F \exp(-\mathcal{R}\, D^2_{\mathcal{L}^{(I)}} \,)\}$$

In good cases, the Witten index is the L^2 index of the Dirac operator $D_{\mathcal{L}^{(I)}}$

$\mathcal{L}^{(I)}$ is determined from $n^{(I)}$

arXiv: 2509.23042

Kim, Manschot, Moore, Tao, Zhang

We study $Z(\mathcal{R}, n^{(I)})$ using both the Coulomb branch integral

and,

for X_4 a toric Kahler manifold, toric localization.

We reproduce and generalize results of
Gottsche, Nakajima, Yoshioka
and
Gottsche, Kool, Williams
using totally different methods

Requires resolving some several subtle
issues in working with the 5d QFT

$\mathcal{N} = 1$ 5D SYM

$$G^{phys} = (SU(2)_g \times Spin(4) \times SU(2)_R \times U(1)^{(I)})/\mathbb{Z}_2$$

Vectormultiplet: $V = (A, \ldots)$

$$J^{(I)} = tr\, F^2 \;\Rightarrow\; V^{(I)} = \left(A^{(I)}, \ldots\right)$$

Seiberg 1995

$P^{(I)}$: Principal U(1) bundle over X_5.

Connection $A^{(I)} \in \mathcal{A}(P^{(I)})$.

$$n^{(I)} = c_1\left(P^{(I)}\right)$$

Electric Coupling Of Instanton Particle

$$\int_{X_5} A^{(I)} \wedge J^{(I)} = \int_{X_5} A^{(I)} \wedge tr\, F^2$$

Supersymmetrization gives entire 5d SYM action coupled to background $V^{(I)}$

Is $\int_{X_5} A^{(I)} \wedge trF^2$ globally well-defined?

Yes!*

$\left(P^{(I)}, A^{(I)}\right)$ corresponds to an element of $\check{H}^2(X_5)$

trF^2 is the fieldstrength of the Chern-Simons differential class in $\check{H}^4(X_5)$

$$\check{H}^2(X_5) \times \check{H}^4(X_5) \to \check{H}^1(pt) \cong \mathbb{R}/\mathbb{Z}$$

Is $\int_{X_5} A^{(I)} \wedge trF^2$ globally well-defined?

Well…. almost …

When $w_2(P_g) \neq 0$ instanton particles have fractional charge so the exponentiated electric coupling can have an anomaly

Expect: Cancelled by anomaly in the fermion determinant.

$Pfaff\left(D_{X_5}\right)$ not well-defined on $\mathcal{A}/\mathcal{G}$

Now take $X_5 = X_4 \times S^1$

$\theta \sim \oint_{S^1} A^{(I)}$ const. on X_4

$P^{(I)}$ & $F^{(I)}$ pulled back from X_4

We have a *partial topological twist* based on transfer of structure group

$$\varphi\colon \ \mathbb{Z}_2 \times SO(4) \to (Spin(5) \times SU(2)^R)/\mathbb{Z}_2$$

Background fields: $\varphi_*(\nabla^{LC})$

Topological on X_4 but a nontopological, spin, theory on S^1

$$Q^2 = \partial_t$$

SQM With Target $\mathcal{M}$

Topological on $X_4 \Rightarrow$
Can shrink $X_4 \Rightarrow$
Describe the twisted theory in terms of SQM with target space the moduli stack of instantons [Nekrasov, 1996]

In General The SQM Is Anomalous

1D: $Pfaff(\gamma \cdot D_{S^1})$ not well-defined on $L\mathcal{M}$

Anomaly if $\mathcal{M}(X_4)$ is not spin

[Witten ’85; Atiyah ‘85]

X admits an ACS and $\mathrm{G_g} = PSU(2N)$:

$\mathcal{M}$ is spin iff $w_2\left(P_g\right) \cdot w_2(TX_4) = 0$

$Consistent\ with\ formula\ quoted\ above$

Line Bundle On $\mathcal{M}$

$$\int_{X_4 \times S^1} A^{(I)} \wedge J^{(I)} = \int_{X_4 \times S^1} A^{(I)} \wedge tr\, F^2$$

$\Rightarrow$ SQM($\mathcal{M}$) couples to a
``line bundle with connection'' $\mathcal{L}^{(I)} \to \mathcal{M}$

$$\breve{H}^2(X_4) \times \breve{H}^4(X_4 \times \mathcal{A}(P_g)/\mathcal{G}) \to \breve{H}^2(\mathcal{A}(P_g)/\mathcal{G})$$

w/ Kim, Manschot, Tao, Zhang:

$\mathbb{Z}_2$: Global anomaly : $w_2(P_g) \cdot n^{(I)} \neq 0$

$$n^{(I)} := c_1(P^{(I)}) \in H^2(X_4, \mathbb{Z})$$

(Conjectural) Anomaly Cancellation

$$w_2(P_g) \cdot \left(w_2(X_4) + n^{(I)} \right) = 0$$

$$Pfaff(\gamma \cdot D_{S^1}) \cdot e^{i \oint \mathcal{A}^{(I)}}$$

is well-defined on $L\mathcal{M}$

Conjecture $\mathcal{A}^{(I)}$:

``U(1) gauge field'' of a Spin-c structure on $\mathcal{M}$

When global anomalies cancel, the partition function is a function of

1. Oriented diffeomorphism type of X_4

2. ‘t Hooft flux

3. $n^{(I)} \coloneqq c_1\left(P^{(I)}\right)$

4. $\mathcal{R}$

Total partition function is a sum of two terms

$$Z^J(\mathcal{R}, n^{(I)}) = \mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n^{(I)}) + Z^J_{SW}(\mathcal{R}, n^{(I)})$$

$\mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n^{(I)})$: Coulomb branch integral of the 4d theory from reduction

$$X_5 = X_4 \times S^1 \to X_4$$

One can deduce Z^J_{SW} from $\mathrm{Z}^J_{\mathrm{Coul}}$

The Coulomb branch integral is complicated

It is a branched double cover of modular curve for $\Gamma^0(4)$

One can deduce Z_{SW}^J from Z_{Coul}^J

Answer For $b_2^+ > 1$

Introduce the function:

$$S(\mathcal{R}, n) = \frac{2^{2\chi+3\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\frac{1}{2}n^2+\chi_h}} \sum_c SW(c) \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c\cdot\frac{n}{2}}$$

$Z(\mathcal{R}, n)$ = Terms in the power series with $\mathcal{R}^d$ with $d = \chi_h \ mod\ 4$

Agrees with, and generalizes, GKW Conjecture 1.1

E_1 Theory

Note singularity at $\mathcal{R}^4 = 1$

$$\mathcal{R}^4 = \exp\left[-8\,\pi^2 \frac{R}{g_{5d,YM}^2} + i\,\theta \right]$$

$g_{5d,sym}^2 = \infty$ corresponds to the E_1 5d superconformal theory [Seiberg 1995]

Most notably, the walls for wall-crossing at one of the strong coupling singularities is NOT at walls determined by spin-c structures!

$$\left(c - n^{(I)}\right) \cdot J = 0$$

$$c = w_2(TX) \ \ mod\ 2$$

$\Rightarrow$ Cannot be canceled by standard Seiberg-Witten invariants!

At this cusp there are multiple massless monopoles and the ``Higgs branch'' consists of solutions to the multi-monopole equations:

Spin-c connection ∇.

Monopole fields $M_j \in \Gamma\left(W^+ \otimes L_j\right)$

$$D_{W^+ \otimes L_j} M_j = 0$$

$$\sum_j \left(F(\nabla)^+ + 2\, F\left(\nabla_{L_j}\right)^+ \right) = \sum_j \bar{M}_j\, M_j$$

Walls: $\sum_j \left(c(\mathfrak{s}) + 2\, c_1\left(L_j\right)\right) \cdot J = 0$

Can we cancel the wall-crossing of the Coulomb branch at the walls

Walls at: $$\left(c + \left(c - 2n^{(I)}\right)\right) \cdot J = 0$$

using a generalization of SW invariants associated with these equations?

Toric Localization

When X_4 is a toric manifold we can construct an ``Ω-background''

$X_5 =$ Mapping cylinder for $U(1) \times U(1)$ action on X_4 with parameters (ϵ_1, ϵ_2)

$$\sum_{k=0}^{\infty} \mathcal{R}^{\frac{d_k}{2}} \; Tr_{\mathcal{H}_k}(-1)^F \, e^{-R\,D^2 + \epsilon_1 J_1 + \epsilon_2 J_2}$$

$Z(\mathcal{R}, n^{(I)}, \epsilon_1, \epsilon_2)$ can be expressed in terms of countour integrals of products of Nekrasov partition functions

But there are a lot of unresolved technical issues .

$\mathfrak{p}^{(I)}$	$Z^{\mathbb{CP}^2\times S^1}_{1/2,\mathfrak{p}^I}(\epsilon_1,\epsilon_2,\mathcal{R})$
{1,0,0}	$t_1^{1/2}t_2^{-1/4}+O(\mathcal{R}^{16})$
{0,1,0}	$t_1^{-1/4}t_2^{1/2}+O(\mathcal{R}^{16})$
{0,0,1}	$t_1^{-1/4}t_2^{-1/4}+O(\mathcal{R}^{16})$
{2,−1,0}	$t_1^{5/4}t_2^{-1}+O(\mathcal{R}^{16})$
{3,0,0}	$t_1^{3/2}t_2^{-3/4}+t_1^{7/2}t_2^{-7/4}\mathcal{R}^4+t_1^{11/2}t_2^{-11/4}\mathcal{R}^8+t_1^{15/2}t_2^{-15/4}\mathcal{R}^{12}+O(\mathcal{R}^{16})$
{1,1,1}	$1+\mathcal{R}^4+\mathcal{R}^8+\mathcal{R}^{12}+O(\mathcal{R}^{16})$
{5,0,0}	$t_1^{5/2}t_2^{-5/4}+t_1^{9/2}t_2^{-17/4}(t_1^2+t_2^2+t_1t_2+t_1^2t_2^2+t_1^2t_2+t_1t_2^2)\mathcal{R}^4+\cdots O(\mathcal{R}^{16})$
{-1, 0, 0}	$t_1^{-1/2}t_2^{1/4}+O(\mathcal{R}^{16})$
	[illegible]

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Moving Up To Six Dimensions

All this should generalize to (anomaly-free) 6d SYM theories on $X_4 \times \mathbb{E}$

Anomaly free $\Rightarrow$ Must work with moduli space of nonabelian monopole equations

$$Index\left(D_{\mathcal{L}^{(I)}}\right) \rightarrow Ell(\sigma(\mathcal{M}))$$

Once again there is a subtle electric coupling to instanton strings:

$$\int_{M_6} B \wedge tr\,(F^2) \in \mathbb{R}/\mathbb{Z}$$

Now we have a background gerbe connection (background tensormultiplet)

$\exists$ holomorphic-topological twisting on $\mathbb{E} \times X$ with $Q^2 \sim \partial_{\bar{z}}$

Reduction to $\mathbb{E}$ is expected to give a σ-model with target space $\mathcal{M}$

When anomalies cancel we expect

$$Z \sim \sum_k \mathcal{R}^{\frac{d_k}{2}} Ell(\mathcal{M}_k; m, \tau_{\mathbb{E}})$$

$$\mathcal{R} \sim \exp[-\frac{A(\mathbb{E})}{g_{6d}^2} + i \int_{\mathbb{E}} B]$$

What is the Coulomb branch parameter?

5d: $U = \langle \, Tr_{fund} \, Pexp \oint_{S^1} (A + i\,\sigma\,) \, \rangle$

6d: U = vev of 2d defect in the 6d theory wrapped on $\mathbb{E}$

Is there a suitable Seiberg-Witten curve that allows for computation using the Coulomb branch integral?

Is there a relation to tmf[2d (0,1) theory] Gukov, Pei, Putrov, and Vafa?

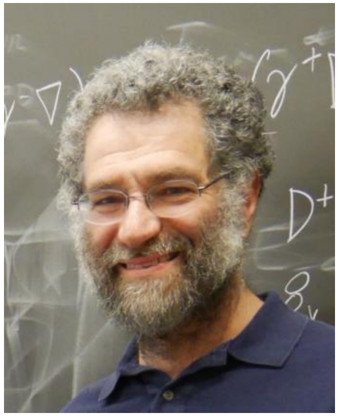
Family Donaldson Invariants

There is an interesting generalization to invariants for families of four-manifolds.

Mentioned by Donaldson long ago.

Studied in the math literature .

But not in the physics literature.



Families Of Metrics

Couple twisted theory to a family of metrics: $g_{\mu\nu}(x;s)$

$s \in \mathcal{P}$: Parameters of the family.

$Z_{DW}(g_{\mu\nu}(x;s))$ is independent of s.

A suitable coupling to background supergravity gives a partition function which is a closed differential form $\mathrm{Z}_{\mathrm{i}_1 \dots i_p} \mathrm{ds}^{i_1} \wedge \cdots ds^{i_p}$.

Periods of $\mathrm{Z}_{\mathrm{i}_1 \ldots i_p} \mathrm{ds}^{i_1} \wedge \cdots ds^{i_p}$ are the family Donaldson invariants.

No restriction on b_2^+. No assumption of ACS.

Does it see the other half of the world of four-manifolds?

Universal Family

$$\mathcal{P} = Met(X)/Diff^+(X)$$

$$\pi_j\left(\frac{Met(X)}{Diff^+(X)}\right) \cong \pi_{j-1}\left(Diff^+(X)\right)$$

$$\pi_1\left(\frac{Met(X)}{Diff^+(X)}\right) \cong \pi_0\left(Diff^+(X)\right)$$

$\pi_0\left(Diff^+(X)\right)$: 4d mapping class group

Donaldson-Witten a la Baulieu-Singer

$P_g \to X \qquad\qquad \mathcal{G} := Aut(P_g)$

$\mathcal{G}$ −equivariant cohomology of $\mathcal{A}(P_g)$

$$\left(\Omega^* \left(\mathcal{A}(P_g)\right) \otimes S^*(Lie\mathcal{G}) \right)^{\mathcal{G}}$$

$Q\, A_\mu = \psi_\mu \qquad Q\, \psi_\mu = -D_\mu \phi \qquad Q\, \phi = 0$

Atiyah & Jeffrey + Baulieu & Singer

Z_{DW} : Pushforward in $\mathcal{G}$ −equivariant cohomology.

$$\mathcal{G}_d \coloneqq Diff^+(X)$$

$\mathcal{G}_d$ −equivariant cohomology of $MET(X)$

$$Q\ g_{\mu\nu} = \Psi_{\mu\nu} \quad Q\ \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu \quad Q\Phi^\mu = 0$$

These arise from truncated twisted N=2 superconformal gravity

Φ^μ : Ghost field

Action e^{-S} is a closed equivariant class in the $\mathcal{G} \rtimes \mathcal{G}_d$ $-$ equivariant cohomology of $MET(\mathbb{X}) \times \mathcal{A}(P)$

Push-forward in $\mathcal{G}$ $-$equivariant cohomology is a $\mathcal{G}_d$ $-$equivariant class on $MET(\mathbb{X})$

Thanks to heroic computations by JC and VS we have explicit actions e^{-S} obtained by coupling to truncated & twisted $N = 2$ conformal supergravity

Coupling To Twisted Truncated Background Supergravity

$$S[g,\Psi,\Phi] = S_{DW} + \int \sqrt{g} \, (\Psi^{\mu\nu}\Lambda_{\mu\nu} + \Phi^{\mu} Z_{\mu} + \Psi^{\mu\sigma}\Psi^{\nu}{}_{\sigma} \ \Upsilon_{\mu\nu} \ )$$

$$T_{\mu\nu}^{SYM} = \{Q, \Lambda_{\mu\nu}\} \qquad \Lambda_{\mu\nu} = Im\, \tau_{IJ} \left(F_{\rho\mu}^{-,I} \chi_{\nu}^{\rho,J} \right) + \cdots$$

$$Z_{\mu} = \mathcal{F}_{IJK}\, \psi_{\mu}^{I} \ F_{\rho\sigma}^{+,J} \ \chi^{\rho\sigma\, K} + \cdots$$

$$\Upsilon_{\mu\nu} = Im\, \tau_{IJ} \ \chi_{\mu\rho}^{I} \, \chi_{\nu}^{\rho,J} + \cdots$$

$\gamma \subset \frac{Met(X)}{Diff^+(X)}$ nontrivial 1-cycle from some nontrivial element of $\pi_0\left(Diff^+(X)\right)$

$$\oint_\gamma ds \int_X \mathrm{vol}(g) \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{ds}} \langle \Lambda^{\mu\nu} \rangle$$

$$Q\left(\Lambda_{\mu\nu}\right) = T_{\mu\nu}^{SYM} + \cdots$$

This raises several questions:

$\Lambda^{\mu\nu}$ is NOT Q-closed!!!

Does our path integral localize to moduli spaces of instantons?

Does tree-level exactness (of LEET) persist?

Assuming the tree-level exactness theorem persists we give a formulae for wall-crossing of k $-$dimensional families when $b_2^+ = k$.

Summary: What We Did

Topological twisting the arbitrary d=4 N=2 theories

Complete treatment of orientation and spinnability of ASD moduli space.

Unusual properties of the answer for SQCD.

Electric coupling to instanton current defines a $Spin^c$ structure on ASD moduli space

Evaluation of K-theoretic invariants using U-plane integral and localization.

Formulation of family invariants using coupling to background twisted and truncated conformal supergravity.

What We Would Like To Do

Invertible theory governing orientation & spin of nonabelian SW moduli space

Explain the unusual properties of the answer for SQCD.

Global anomalies of 5D SYM …

New invariants at $\mathcal{R}^4 = 1$???

Generalization to elliptic invariants from 6d theories on $X \times E$

Puzzles concerning the family generalization of Donaldson invariants

Other puzzles and directions I did not have time to mention ….


NOT
That's all Folks!

Supplemental Slides On Coulomb Branch

Coulomb Branch For $X_5 = X_4 \times S^1$

For 5d SYM gauge group of rank 1:

Coulomb branch = $\mathbb{C}$

parametrized by :

$$U = \langle\, Tr_F P exp \oint_{S^1} (\sigma dx^5 + i\, A)\, \rangle$$

``1-form symmetry'' $\wp: U \to -U$

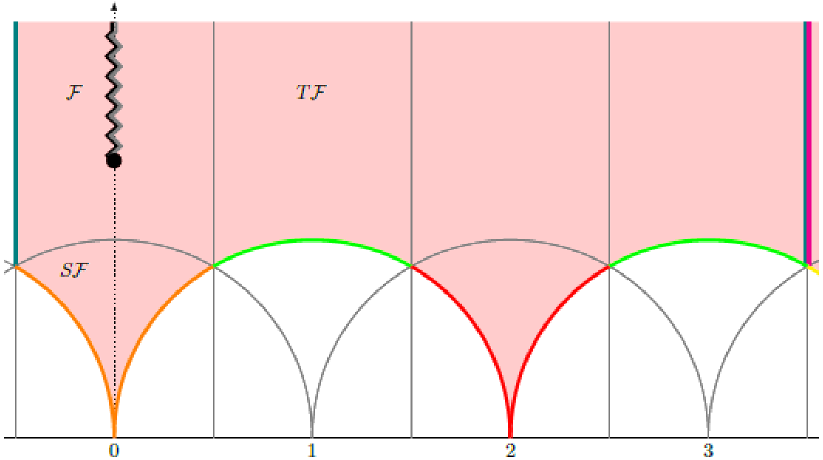
Modular Parametrization Of U −plane

The Coulomb branch is a branched double cover of the modular curve for $\Gamma^0(4)$

$$U^2 = 4\,(\mathcal{R}^4 - 2\,\mathcal{R}^2\,u(\tau) + 1)$$

$$u(\tau) = \frac{\vartheta_2(\tau)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_3(\tau)^2}{\vartheta_2(\tau)^2}$$

Hauptmodul for $\Gamma^0(4)$



Similar to Aspman, Furrer & Manschot

$$U^2 = 4\,(\mathcal{R}^4 - 2\,\mathcal{R}^2\,u(\tau) + 1)$$

$\Rightarrow$ $\wp$ is also the deck transformation of the double cover

$$\mathrm{Z}^J_{\mathrm{Coul}}(\mathcal{R}, n) = \int_{\mathbb{C}_U} \Omega_{Coul}$$

$$\wp^* \Omega_{Coul} = (-1)^{w_2(P^g)\cdot\left(w_2(X)+n^{(I)}\right)}\,\Omega_{Coul}$$

$\Rightarrow$ $Z^J_{Coul} = 0$ when there is a global anomaly

$$Z^{J}_{\mathrm{Coul}}(\mathcal{R}, n^{(I)}) = 2\int_{\mathcal{F}(\Gamma^0(4))} d\tau d\bar{\tau}\, \frac{\nu}{U}\, C^{n^2}\ \Psi^{J}\left(\tau, \frac{n^{(I)}}{2}\,\zeta\right)$$

$$\nu(\tau) = \frac{\vartheta_4^{13-b_2}}{\eta^9}$$

$C(\tau, \mathcal{R})$ Suitably modular invariant and holomorphic ``contact term''

$\Psi^{J}\left(\tau, \frac{n^{(I)}}{2}\,\zeta\right)$ Photon partition function

$$\Psi^J(\tau,z) = \sum_{k\in H^2(X,\mathbb{Z})} \left(\frac{\partial}{\partial\bar{\tau}} E_k^J\right) q^{-\frac{k^2}{2}} e^{-2\pi i\, k\cdot z} (-1)^{k\cdot K}$$

$$E_k^J = Erf\left(\sqrt{Im\tau}\,\left(k + \frac{Im\, z}{Im\, \tau}\right)\cdot J\right)$$

Not holomorphic.

Continuously metric dependent

$$z \to \frac{n^{(I)}}{2}\,\zeta(\tau,\mathcal{R}) \qquad \zeta(\tau,\mathcal{R}) \sim \frac{\partial^2\mathcal{F}}{\partial a\,\partial m_{inst}}$$

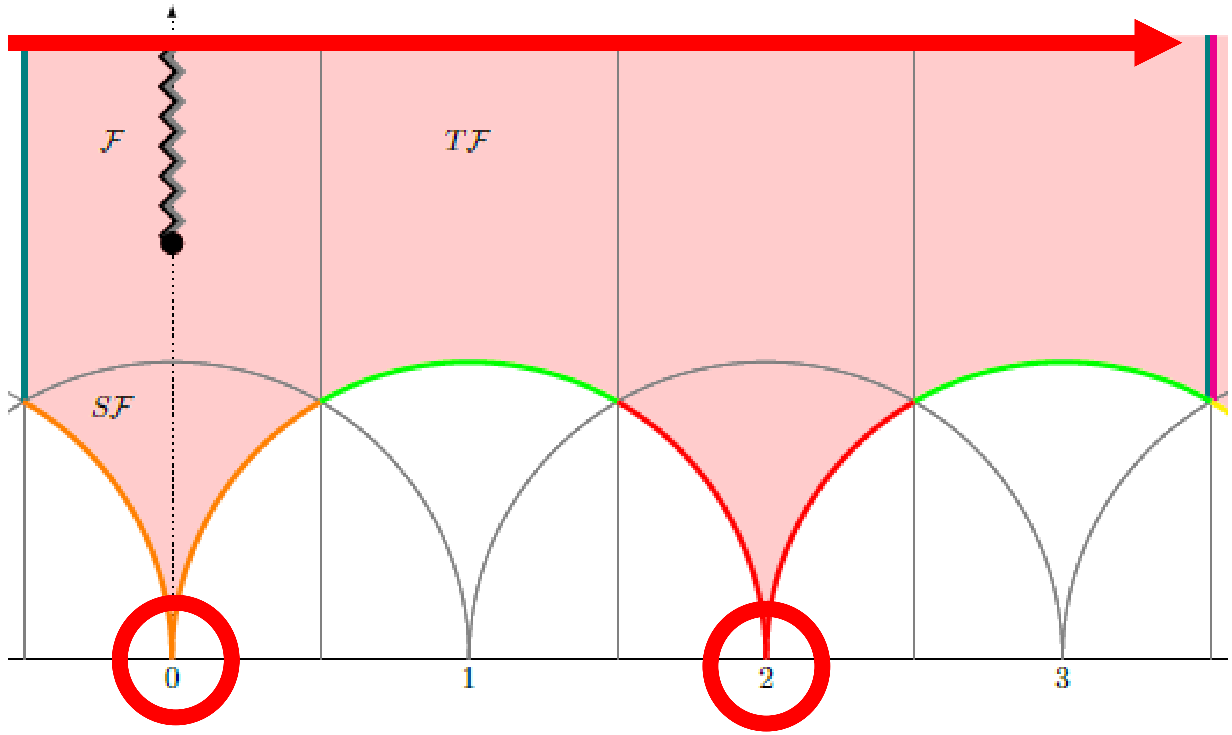
Measure As A Total Derivative

$$Z^{J}_{\text{Coul}}(\mathcal{R}, n^{(I)}) = \int_{\mathcal{F}} d\tau d\bar{\tau}\, \mathcal{H}(\tau)\ \ \Psi^{J}\left(\tau, \frac{n^{(I)}}{2}\zeta\right)$$

$\exists$ suitably modular invariant and nonsingular $\hat{G}^{J}(\tau, \bar{\tau})$

$$\frac{\partial}{\partial \bar{\tau}} \hat{G}^{J} = \Psi^{\mathrm{J}}$$

(It can be hard to find explicit formulae for $\hat{G}^{J}$ one needs the theory of mock modular forms….)



$$Z^{J}_{\mathrm{Coul}}(n,\mathcal{R}) = \sum_{i=1,2,3} \oint d\tau_i \mathcal{H}(\tau_i) G^{J}(\tau_i)$$

?????????

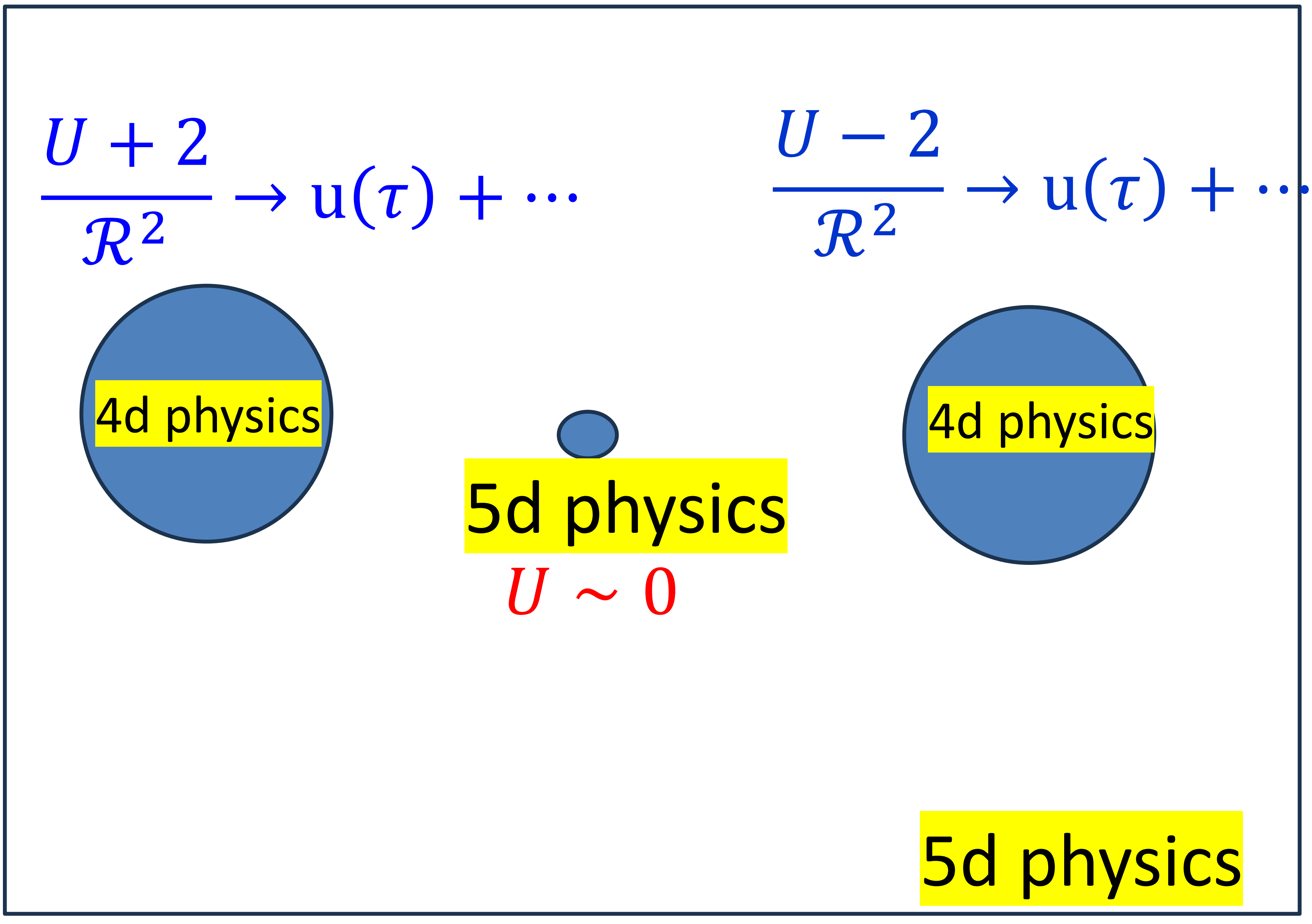
$U, C, \hat{G}^J$ are functions of τ and of $\mathcal{R}$

Subtle order of limits: $\mathcal{R} \to 0$ vs. $\mathrm{I}m\,\tau \to \infty$

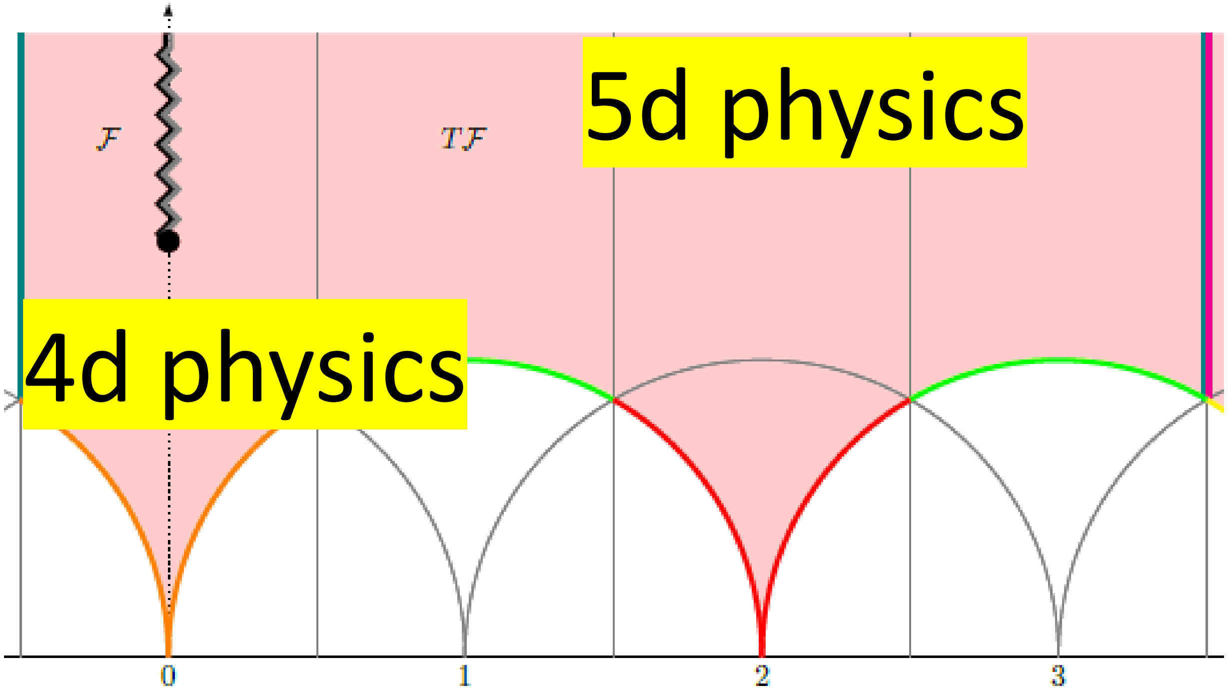
Example: $u(\tau) \sim \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} - \frac{31}{4} q^{\frac{3}{4}} + \mathcal{O}\left(q^{\frac{5}{4}}\right)$

$$U^2 = 4\,(\mathcal{R}^4 - 2\,\mathcal{R}^2\,u(\tau) + 1)$$

$U \to \pm 2$ **vs** $U \to \infty$



$\frac{U+2}{\mathcal{R}^2} \to \mathrm{u}(\tau) + \cdots$
$\frac{U-2}{\mathcal{R}^2} \to \mathrm{u}(\tau) + \cdots$
4d physics
5d physics
$U \sim 0$
4d physics
5d physics



$$\frac{\pi \tau_{bp}}{4} = i \log(1/\mathcal{R}) + \cdots$$

$$Z_{\text{Coul}}(\mathcal{R}, n^{(I)})$$
$$= 2\sum_i \left[\ \nu(\tau,\mathcal{R})\ \ C(\tau,\mathcal{R})^{n^2}\ \ G(\tau,\mathcal{R})\right]_{q_i^0}$$

If we *first* expand the expressions in [....] in $\mathcal{R}$ around $\mathcal{R} = 0$ *then* take the constant q^0 term at each order in $\mathcal{R}$ we find remarkable and nontrivial agreement with GNY.

Examples Of Explicit Results

$$X = \mathbb{CP}^2$$

$$Z_{Coul}(n, \mathcal{R}) = \left[\nu(\tau, \mathcal{R}) \; C(\tau, \mathcal{R})^{n^2} \; G(\tau, \mathcal{R}) \right]_{q^0}$$

$$G(\tau, \mathcal{R}) = -\frac{e^{i\pi n \frac{\zeta(\tau,\mathcal{R})}{2}}}{\vartheta_4(\tau)} \sum_{\ell \in \mathbb{Z}} (-1)^{\ell} \; \frac{q^{\frac{\ell^2}{2} - \frac{1}{8}}}{1 - e^{i\pi n\, \zeta(\tau,\mathcal{R})} q^{\ell - \frac{1}{2}}}$$

$$\zeta(\tau, \mathcal{R}) = \left(\vartheta_2(\tau)\vartheta_3(\tau)\right)^{-1} \int_0^{\mathcal{R}} \frac{dx}{\sqrt{1 - 2u(\tau)x^2 + x^4}}$$

Examples Of Explicit Results

Wall Crossing Formula:

$$Z^{J}_{\mathrm{Coul}} - Z^{J'}_{\mathrm{Coul}} = \left[\nu \, C^{n^2} \Theta^{J,J'}(\tau, \mathcal{R}) \right]_{q^0}$$

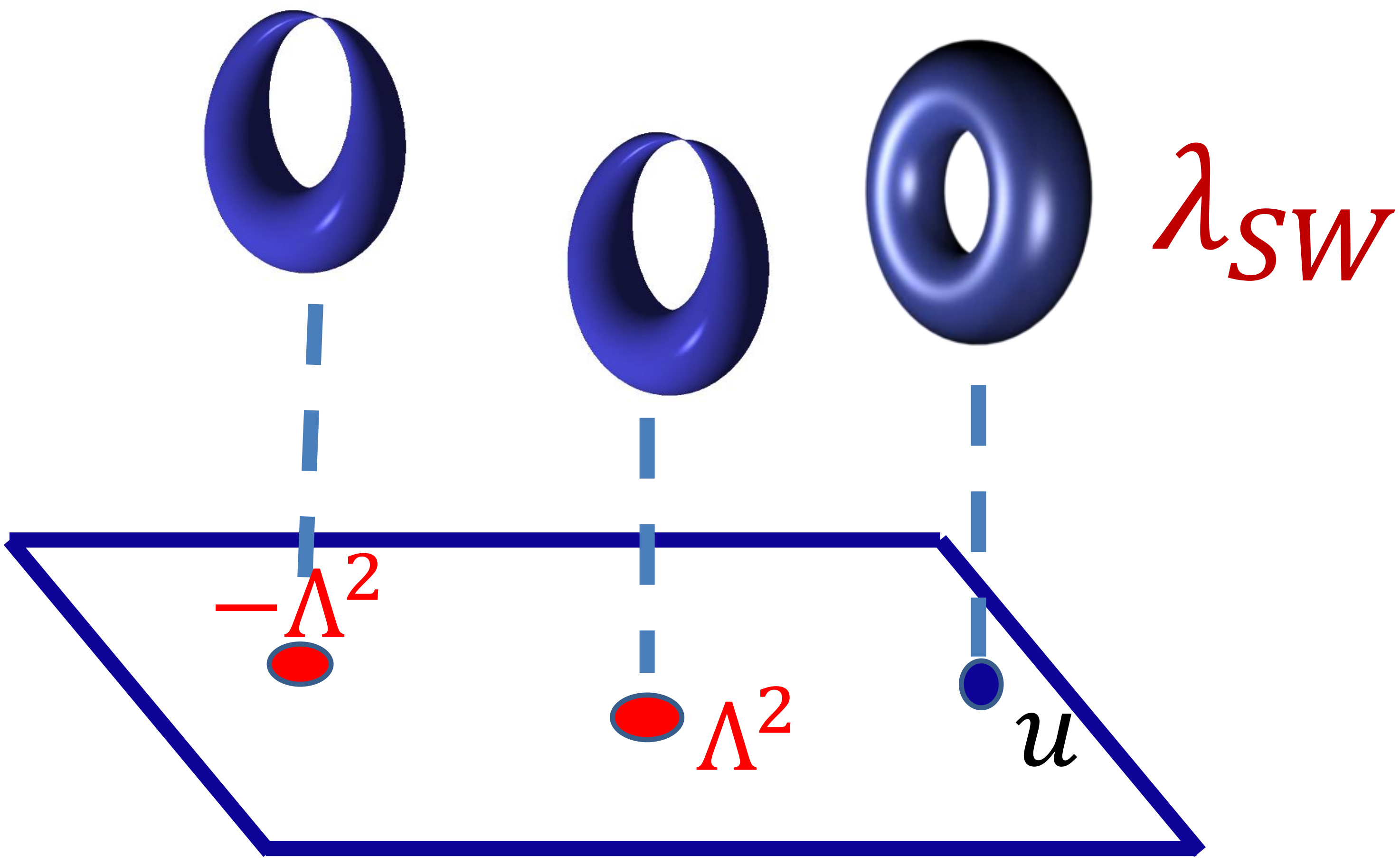
$$\Theta^{J,J'} = \sum_{k \in H^2(X,\mathbb{Z})} \left(s_k^J - s_k^{J'} \right) q^{-\frac{k^2}{2}} e^{-2\pi i \, k \cdot n \, \zeta(\tau,\mathcal{R})} \, (-1)^{k \cdot K}$$

$$s_k^J := sign\left(\sqrt{Im\tau} \, \left(k + \frac{Im \, \zeta(\tau, \mathcal{R})}{Im \, \tau} \right) \cdot J \right)$$

Z^{J}_{Coul} Integral over the Coulomb branch of vacua on $\mathbb{R}^4$,

$G = SU(2)$: parametrized by $u = \langle Tr\phi^2 \rangle$,

Measure is computed using SW LEET for U(1) VM



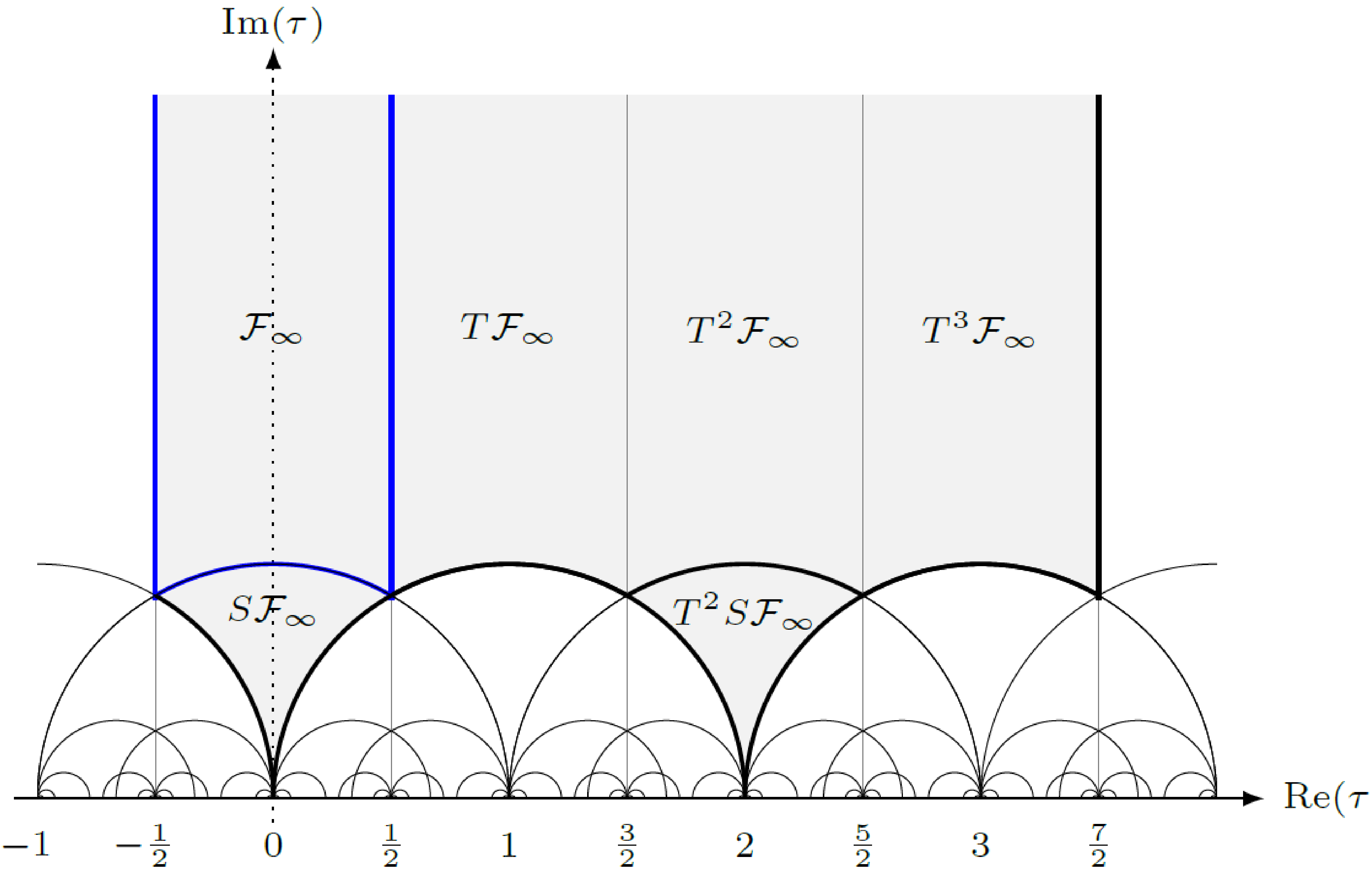
u-Plane Integral

SW94: Coulomb branch has a modular parametrization:

$$u(\tau) = \frac{\vartheta_2^4 + \vartheta_3^4}{2\,\vartheta_2^2\vartheta_3^2} = \frac{1}{8}q^{-\frac{1}{4}} + \frac{5}{2}q^{\frac{1}{4}} + \cdots$$

$$q = e^{2\pi i\tau}$$

Coulomb branch $\cong UHP/\Gamma^0(4)$



$$\Gamma^0(4) = \{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} mod 4\} \subset SL(2,\mathbb{Z})$$