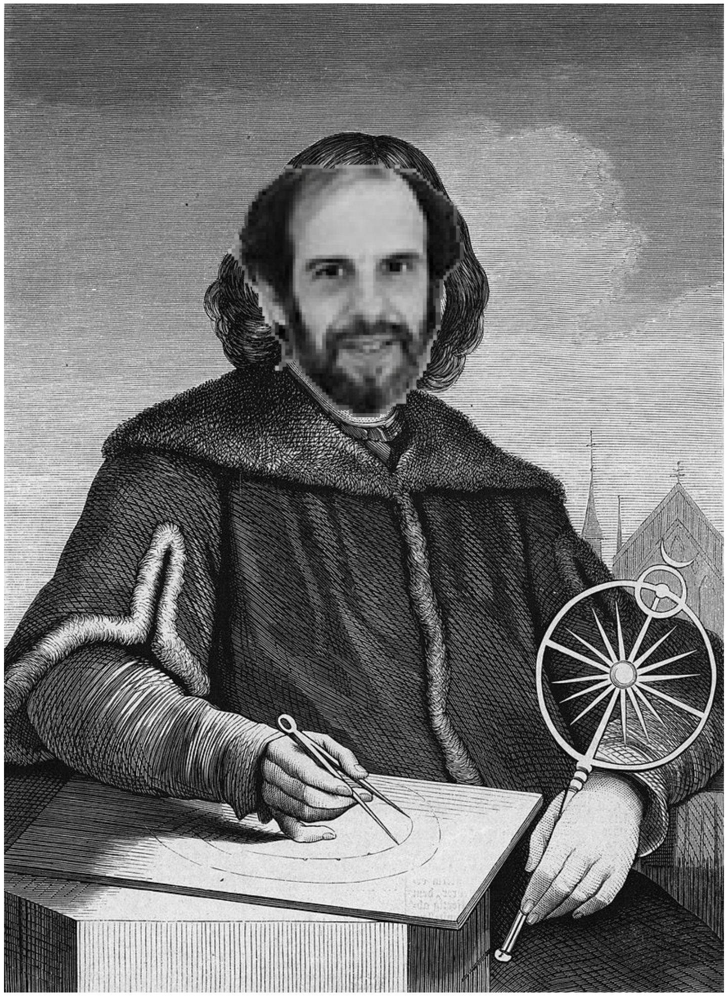
# Measuring
# The
# Elliptic Genus

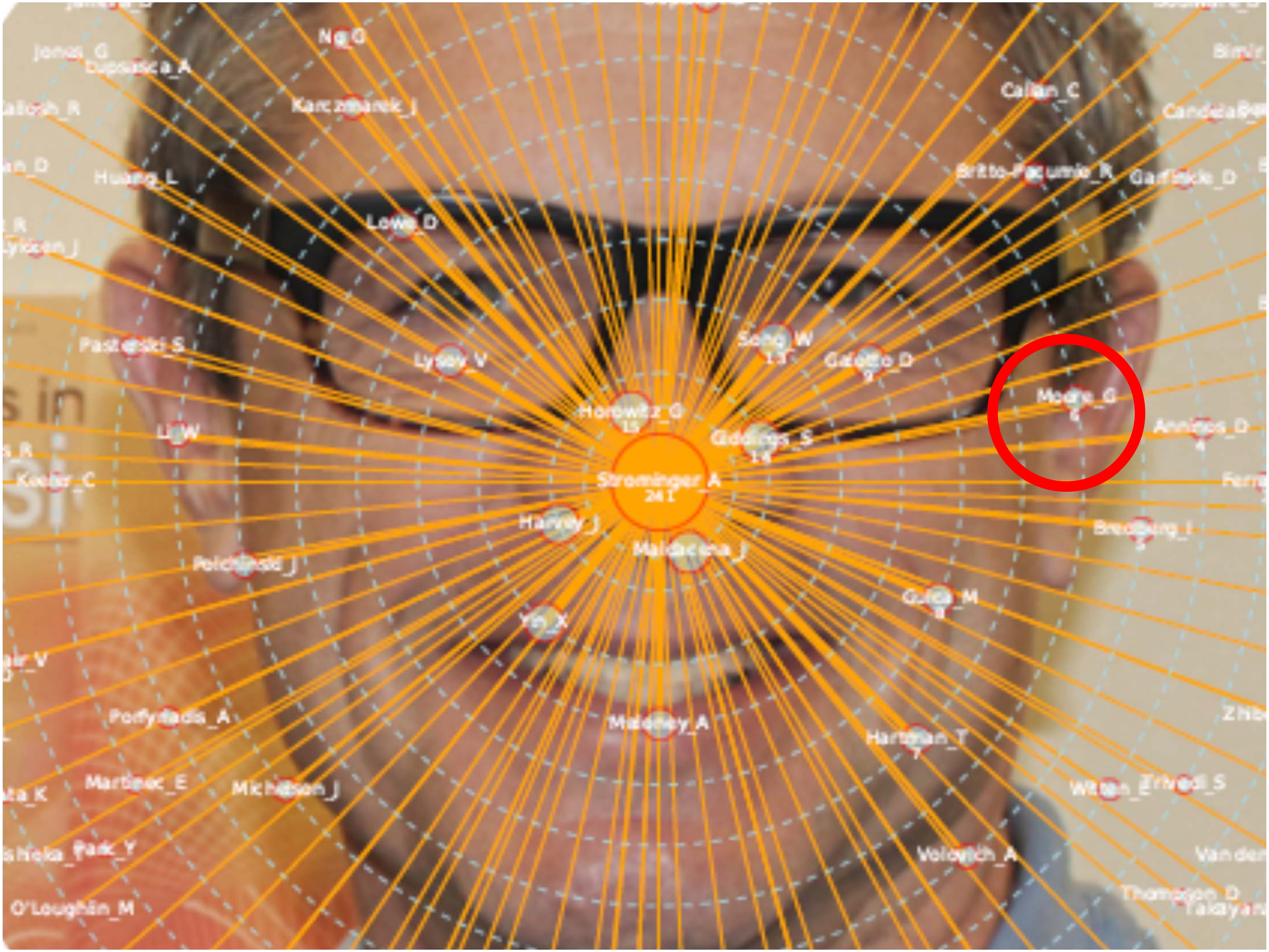
Gregory Moore

Rutgers

AndyFest, Harvard, July  31, 2015



# Nicolaus Ginsparnicus of Ithaka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Karczmarek_J
Callan_C
Huang_L
Britto-Pacumio_R
Garfinkle_D
Lowe_D
Pasterski_S
Lysov_V
Song_W
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Gaiotto_D
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Moore_G
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Horowitz_G
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Li_W
Giddings_S
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Anninos_D
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Keeler_C
Strominger_A
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Harvey_J
Maldacena_J
Polchinski_J
Yin_X
Porfyriadis_A
Maloney_A
Hartman_T
7
Martinec_E
Michelson_J
Witten_E
Trivedi_S
Park_Y
Volovich_A
O'Loughlin_M
Thompson_D

IASSNS-HEP-87/??

# Infinite Genus Swallows The Dilaton!

Greg Moore and Andy Strominger
School of Natural Sciences
Institute for Advanced Study
Princeton, N.J. 08540 USA

4/87

This talk has its origins in the D1D5-system, where Andy has done such great work.

It was an important milestone on the road to AdS/CFT

$$\mathcal{C}_M = \mathrm{Sym}^M(X)$$

$$X = K3, T4$$

Holographically dual to IIB strings on

$$AdS_3 \times S^3 \times X$$

(we won't say anything about X = $S^3$ x $S^1$ .... )

Some recent activity has centered on question:

``Do more general sequences $\{\mathcal{C}_M\}$ have holographic duals with weakly coupled gravity? "

This talk: AdS3/CFT2. CFT's are unitary and (for simplicity) have (4,4) supersymmetry:

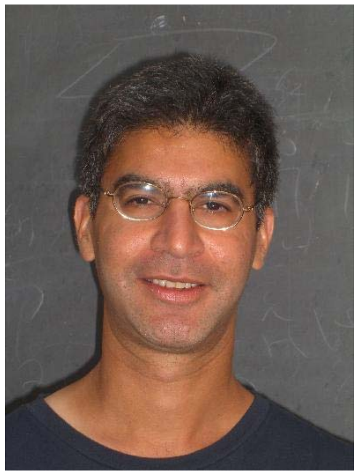
$$c = 6M$$

Put necessary conditions on partition functions $Z(\mathcal{C}_M)$ for a holographic dual of an appropriate type to exist.

Keller; Hartman, Keller, Stoica; Haehl, Rangamani; Belin, Keller, Maloney; ….

Miranda
Cheng

Nathan
Benjamin



Shamit
Kachru

Natalie
Paquette

Our paper: Apply criterion of existence of a Hawking-Page phase transition to the elliptic genus.

Criterion goes back to Witten and a paper of Maldacena-Strominger.

# AdS$_3$ Black Holes and a Stringy Exclusion Principle



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**Abstract**

The duality relating near-horizon microstates of black holes obtained as orbifolds of a subset of $AdS_3$ to the states of a conformal field theory is analyzed in detail. The

In each modular region of the $\tau$ plane there is a unique lowest (negative) action instanton. Defining

$$S_{min}(\tau) \equiv \min\left[\frac{i\pi k}{2}\left(\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right)\right], \tag{3.23}$$

the leading semiclassical approximation to the partition function is

$$Z(\tau) = e^{-S_{min}(\tau)}. \tag{3.24}$$

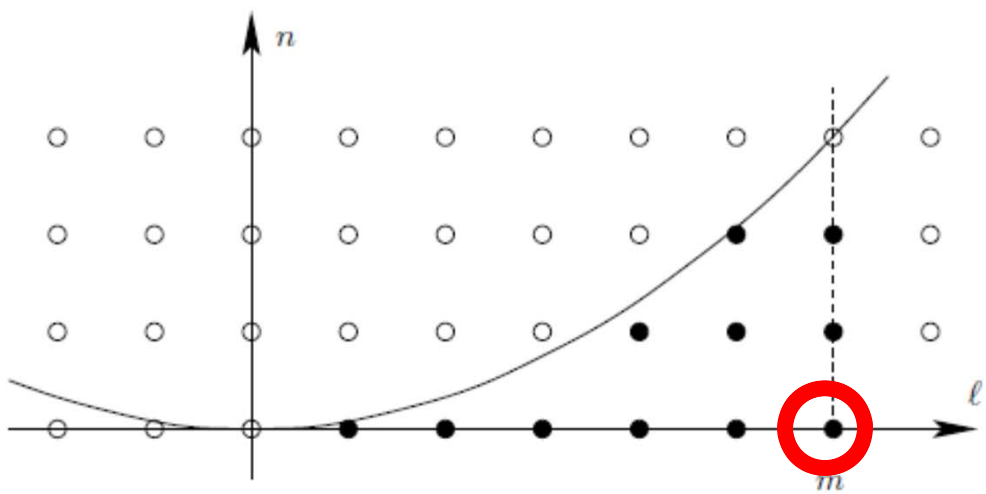
At low temperatures ( large $\beta$ ) the partition function is dominated by (3.20) corresponding to a thermal gas in $AdS_3$. At higher temperatures there is a transition to the black hole phase in which (3.16) dominates. This is a sharp first order phase transition in the limit $k \to \infty$ [20].

# Reminder On Elliptic Genera

$$\mathcal{E}(\tau, z; \mathcal{C}) = \mathrm{Tr}_{\mathcal{H}_{RR}} q^{L_0 - c/24} e^{2\pi i z J_0} \bar{q}^{\bar{L}_0 - c/24} e^{i\pi(J_0 - \bar{J}_0)}$$

$$\mathcal{E}(\tau, z; \mathcal{C}) = \sum_{n,\ell \in \mathbb{Z}} c(n, \ell; \mathcal{C}) q^n y^\ell$$

Modular object: Weak Jacobi form of weight zero and index m.



# Extreme Polar Coefficient

$$\mathcal{E}(\tau, z; \mathcal{C}) = \mathfrak{e}(\mathcal{C}) y^m + \cdots$$

Benjamin et. al. put constraints on coefficients of elliptic genera of a sequence $\{\mathcal{C}_M\}$ so that it exhibits HP transition. A corollary:

A ***necessary*** condition for $\{\mathcal{C}_M\}$ to exhibit a HP transition is that

$\mathfrak{e}(\mathcal{C}_M)$ has at most polynomial growth in M for $M \longrightarrow \infty$

Just a necessary condition.

# Shamit's Question

``How *likely* is it for a sequence of CFT's $\{ \mathcal{C}_M \}$ to have a holographic dual with weakly coupled gravity? "

We'll now make that more precise, and give an answer.

# Zamolodchikov Metric

Space of CFT's is thought to have a topology. So we can speak of continuous families and connected components.

At smooth points the space is thought to be a manifold and there is a canonical isomorphism:

$$\Psi : V^{1,1}(\mathcal{C}) \to T_{\mathcal{C}}\mathcal{M}$$

$$\frac{\partial}{\partial t}|_0 S[t] = \int \mathcal{O} \qquad \Psi(\mathcal{O}) = \frac{\partial}{\partial t}|_0 = v \in T_{\mathcal{C}}\mathcal{M}$$

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle := g_{\mathrm{Z}}(v,v)\frac{d^2z_1 d^2z_2}{|z_1 - z_2|^4}$$

# Strategy

Suppose we have an ensemble $\mathcal{E}$ of (4,4) CFTs:

$$\mathcal{E} = \amalg_M \mathcal{E}_M \qquad \mathcal{E}_M = \amalg_\alpha \mathcal{E}_{M,\alpha}$$

$$\sum_\alpha \mathrm{vol}_Z(\mathcal{E}_{M,\alpha}) < \infty$$

Then use the Z-measure to define a probability density on $\mathcal{E}_M$ for fixed M.

# Strategy – 2/2

Now suppose { $\mathscr{C}_M$ } is a sequence drawn from $\mathcal{E}$.

$$p_M(\kappa, \ell) := \sum_{\mathfrak{e} \leq \kappa M^{\ell}} \frac{\mathrm{vol}(\mathfrak{e}; \mathcal{E}_M)}{\mathrm{vol}(\mathcal{E}_M)}$$

$$\wp(\ell) := \lim_{M \to \infty} p_M(\kappa, \ell)$$

$\wp(\ell)$ probability that a sequence drawn from $\mathcal{E}$ has extremal polar coefficient growing at most like a power $M^{\ell}$

# Multiplicative Ensembles

$\mathfrak{e}(\mathcal{C})$ is constant on each component: $\mathcal{C} \in \mathcal{E}_{M,\alpha}$

$$\mathfrak{e}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathfrak{e}(\mathcal{C}_1)\mathfrak{e}(\mathcal{C}_2)$$

Definition: A ***multiplicative ensemble*** satisfies:

$$\mathrm{vol}(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{vol}(\mathcal{C}_1)\mathrm{vol}(\mathcal{C}_2)$$

Definition: A CFT $\mathcal{C}$ in a multiplicative ensemble is ***prime*** if it is not a product of CFT's (even up to deformation) each of which has m>0.

# A Generating Function

$\mathcal{C}_{m,\alpha}$ prime CFT's with c= 6m, $\alpha = 1, \dots, f_m$

$$\mathfrak{e}(m,\alpha) = \mathfrak{e}(\mathcal{C}_{m,\alpha})$$

$$v(m,\alpha) = \mathrm{vol}_Z(\mathcal{C}(m,\alpha))$$

$$\prod_{m=1}^{\infty} \prod_{\alpha=1}^{f_m} \frac{1}{1 - v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s;M) q^M$$

$$\xi(s;M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathfrak{e}^s}$$

# Some Representative (?) Ensembles

We do not know what the space of (4,4) CFT's is

We do not even know how to classify compact hyperkähler manifolds !

$$S^m K3 := \mathrm{Hilb}^m(K3)$$

$$S^m T4 := \mathrm{Hilb}^{m+1}(T4)/T4$$

$$\mathcal{E} = \{(S^1 X)^{n_1} \times \cdots \times (S^r X)^{n_r}\}$$

$$X = K3 \qquad X = T4$$

$$X \in \{K3, T4\}$$

# Moduli Spaces Of The Prime CFTs -1/2

These ensembles are multiplicative.

Primes: $S^m X$

$$Q_{r,s} = II^{r+8s,r}$$

$$\mathcal{N}_{r+8s,r} = O_{\mathbb{Z}}(Q_{r,s})\backslash O_{\mathbb{R}}(Q_{r,s})/O(r+8s)\times O(r)$$

$$\mathcal{M}(X) = \begin{cases} \mathcal{N}_{4,4} & X = T4 \\ \mathcal{N}_{20,4} & X = K3 \end{cases}$$

# Moduli Spaces Of The Prime CFTs -2/2

One can derive $\mathcal{M}(S^m(X))$ using the *ATTRACTOR MECHANISM*:

Dijkgraaf;
Seiberg & Witten

Begin with O(5,21) (or O(5,5)) moduli space of supergravity

Consider the subgroup fixing a primitive vector $u \in II^{r+8s,r}$

The conjugacy class only depends on $u^2 = 2m$

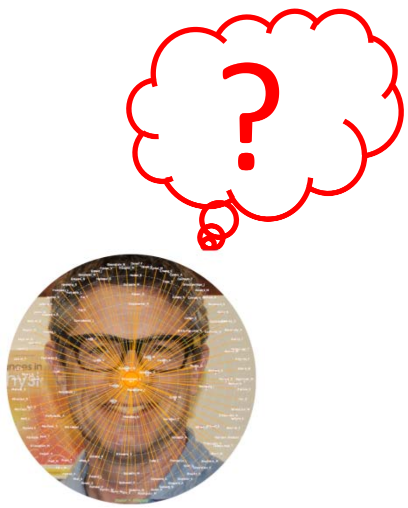
$$O_{\mathbb{Z}}(Q_{r,s}, m)\backslash O_{\mathbb{R}}(Q_{r,s}, m)/O(r+8s) \times O(r-1)$$

$S^m(X)$ for m > 1 has ``extra'' hypermultiplet of blowup modes

$$(r,s) = \begin{cases} (5,0) & X = T4 \\ (5,2) & X = K3 \end{cases}$$

# A Generating Function

$\mathcal{C}_{m,\alpha}$ prime CFT's with c= 6m, $\alpha = 1, \dots, f_m$




$$\mathfrak{e}(m,\alpha) = \mathfrak{e}(\mathcal{C}_{m,\alpha})$$

$$\prod_{m=1}^{\infty} \prod_{\alpha=1}^{f_m} \frac{1}{1 - v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s;M) q^M$$

$$\xi(s;M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathfrak{e}^s}$$

# Extreme Polar Coefficient

From the formula for the partition functions of symmetric product orbifolds we easily find

$$\mathfrak{e}(S^m K3) = m + 1$$

For the torus we have to work harder, since the elliptic genus vanishes.

Fortunately, that work has been done, thanks to

# Counting BPS Blackholes
# in
# Toroidal Type II String Theory

Juan Maldacena,[1,2] Gregory Moore,[1,3] and Andrew Strominger[2]

[1] School of Natural Sciences
Institute for Advanced Study
Princeton, NJ 08540

(Never let Andy submit papers! )

$$\mathfrak{e}(S^m T4) = m + 1$$

# A Generating Function

$\mathcal{C}_{m,\alpha}$ prime CFT's with c= 6m, $\alpha = 1, \dots, f_m$


$$v(m,\alpha) = \mathrm{vol}_Z(\mathcal{C}(m,\alpha))$$

$$\prod_{m=1}^{\infty} \prod_{\alpha=1}^{f_m} \frac{1}{1 - v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s} q^m} = 1 + \sum_{M=1}^{\infty} \xi(s;M) q^M$$

$$\xi(s;M) = \sum_{\mathfrak{e}=1}^{\infty} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathfrak{e}^s}$$

# Volumes

Using results from number theory, especially the ``mass formulae'' of Carl Ludwig Siegel, one can -- with some nontrivial work -- compute the Z-volumes of these spaces. For example:

$$\mathrm{vol}_Z(K3) =$$

$$\pi^{-40}\frac{(131)(283)(593)(617)(691)^2(3617)(43867)}{2^{40}\cdot 3^{34}\cdot 5^{15}\cdot 7^{9}\cdot 11^{5}\cdot 13^{4}\cdot 17^{3}\cdot 19^{3}\cdot 23}$$

$$\cong 1.66\times 10^{-61}$$

$$\mathrm{vol}_Z(S^M K3) = \rho M^{42} f_{13}(M)$$

$$f_{13}(M) = \prod_{p|M} \frac{1 - p^{-12-12e_p(M)}}{1 - p^{-12}}$$

$$\rho = \pi^{-42} \frac{(103)(131)(283)(593)(617)(691)(3617)(43867)(2294797)}{2^{51} \cdot 3^{35} \cdot 5^{15} \cdot 7^{10} \cdot 11^{5} \cdot 13^{4} \cdot 17^{3} \cdot 19^{3} \cdot 23^{2}}$$

$$\cong 5.815 \times 10^{-63}$$

# Result For Probabilities

$$H_\ell(s) := \lim_{M\to\infty} (M+1)^{\ell s} \frac{\xi(s;M)}{\xi(0;M)}$$

$$= \lim_{M\to\infty} \sum_{\mathfrak{e}=M+1}^{2^M} \frac{\mathrm{vol}(\mathfrak{e};M)}{\mathrm{vol}(M)} \left( \frac{(M+1)^\ell}{\mathfrak{e}} \right)^s$$

$$\geq \lim_{M\to\infty} \kappa^{-s} p_M(\kappa,\ell) \geq 0$$

# Result For Probabilities

$$H_\ell(s) := \lim_{M\to\infty} (M+1)^{\ell s} \frac{\xi(s;M)}{\xi(0;M)}$$

Claim: The limit exists for all nonnegative $\ell$ and

$$H_\ell(s) = \begin{cases} 1 & s = 0 \\ 0 & s > 0 \end{cases}$$

$$\Rightarrow \wp(\ell) = 0$$

Conclusion: Almost every sequence $\{\mathcal{C}_M\}$ does not have a holographic dual

# Proof – 1/3

$$\prod_{m=1}^{\infty}\prod_{\alpha=1}^{f_m}\frac{1}{1-v(m,\alpha)\mathfrak{e}(m,\alpha)^{-s}q^m}=1+\sum_{M=1}^{\infty}\xi(s;M)q^M$$

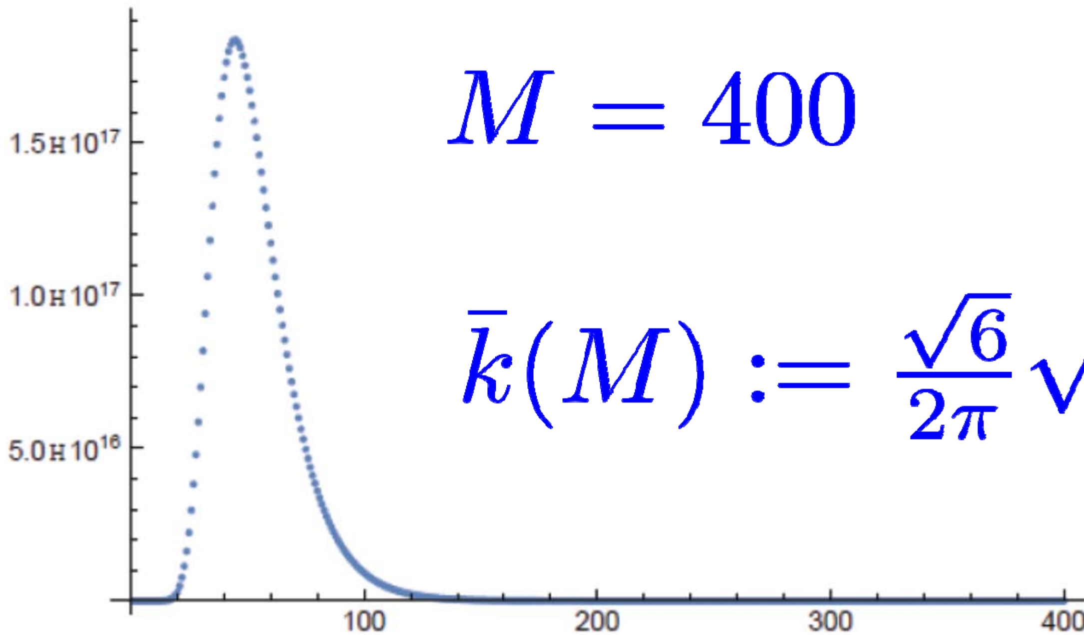
So $\xi$(s:M) is a sum over partitions:

$$M=\lambda_1+\cdots+\lambda_k$$

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k$$

# Statistics Of Partitions

For large M the distribution of partitions into k parts is sharply peaked:




$$M = 400$$

$$\bar{k}(M) := \frac{\sqrt{6}}{2\pi}\sqrt{M}\log M$$

Moreover the ``typical'' partition has ``most'' parts of order: $\lambda_j \cong \sqrt{M}$

# Proof – 3/3

$\xi(s;M)$ is dominated by:

$$\xi(s;M) \cong V(\sqrt{M})^{\sqrt{M}}(\sqrt{M})^{-s\sqrt{M}}e^{2\pi\sqrt{\frac{M}{6}}}$$

$$H_\ell(s) := \lim_{M\to\infty}(M+1)^{\ell s}\frac{\xi(s;M)}{\xi(0;M)}$$

$$= \lim_{M\to\infty}(M+1)^{\ell s}(\sqrt{M})^{-s\sqrt{M}}$$

$$= \begin{cases} 1 & s=0 \\ 0 & s>0 \end{cases}$$

# Some Wild Speculation:

(Discussions with Shamit Kachru and Alex Maloney)

***Siegel Mass Formula:***

Two lattices $\Gamma_1$ and $\Gamma_2$ are in the same genus if

$$\Gamma_1 \oplus S \cong \Gamma_2 \oplus S$$

Even unimodular lattices of rank 8n form a single genus, and:

$$\sum_\alpha \frac{1}{|\mathrm{Aut}\Gamma_\alpha|} = \prod_p \alpha_p$$

# A Natural Ensemble & Measure

Consider the ensemble of ***holomorphic*** CFT's.

(What would they be dual to? Presumably some version of chiral gravity in 3d! )

Holomorphic CFTs have c = 24n

They are completely rigid

$$Z = \sum_\alpha \frac{1}{|\mathrm{Aut}(\mathcal{C}_\alpha)|} < \infty$$

$$\mu(\mathcal{C}_\alpha) = \frac{1}{Z} \frac{1}{|\mathrm{Aut}(\mathcal{C}_\alpha)|}$$

# Some Wild Speculation – 3/4

$$\mathcal{E}_N = \{\mathcal{C}(\Gamma, G) | G \subset \mathrm{Aut}(\Gamma) \quad \& \quad \Gamma \in II^{24N}\}$$

(Speculation: This set exhausts the set of c=24N holomorphic CFTs.)

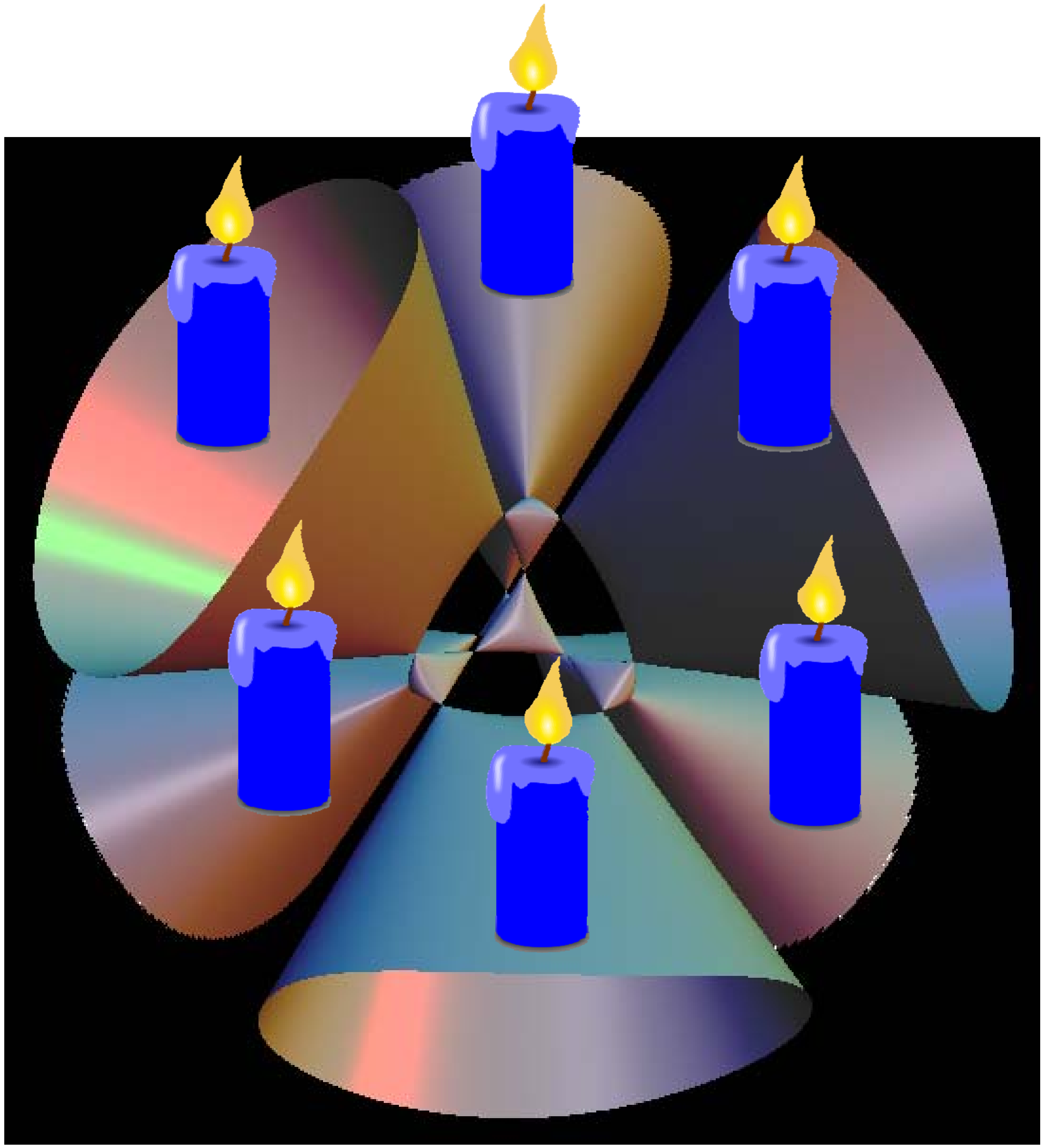
Speculation: Using results on the mass formula for lattices with nontrivial automorphism we can again prove that sequences $\{\mathcal{C}_M\}$ with a holographic dual are measure zero.

# Even Wilder Speculation – 4/4

Define a ``genus'' to be an equivalence class under tensoring with a lattice theory of chiral scalar fields.

$$\sum_\alpha \frac{1}{|\mathrm{Aut}\mathcal{C}(\Gamma,G)_\alpha|} = \prod_p \alpha_p^{CFT}$$

Where the local densities are computed by counting automorphisms of the vertex operator algebra localized at a prime p.



HAPPY BIRTHDAY ANDY!!