d=4, $\mathcal{N}$=2, Field Theory and Physical Mathematics

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This is a review talk

For my recent research results see my talk at Strings-Math, Bonn, 2012:

http://www.physics.rutgers.edu/~gmoore/

1 Introduction

2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Defects in Quantum Field Theory

5 Wall Crossing 102

6 3D Reduction & Hyperkähler geometry

7 Theories of Class S

8 Spectral Networks

9 Conclusion

Two Important Problems In Mathematical Physics

1. Given a QFT what is the spectrum of the Hamiltonian? and how do we compute forces, scattering amplitudes, operator vev's ?

2. Find solutions of Einstein's equations,

and how can we solve Yang-Mills equations on Einstein manifolds?

Today, I will have something to say about each of these problems…

in the restricted case of d=4 quantum field theories with ``$\mathcal{N}$=2 supersymmetry.''

(Twice as much supersymmetry as in potentially realistic supersymmetric extensions of the standard model.)

What we can say about Problem 1

In the past 5 years there has been much progress in understanding a portion of the spectrum – the ``BPS spectrum'' – of these theories.

A corollary of this progress: many exact results have been obtained for ``line operator'' and ``surface operator'' vacuum expectation values.

What we can say about Problem 2

It turns out that understanding the BPS spectrum allows one to give very explicit constructions of ``hyperkähler metrics'' on certain manifolds associated to these d=4, $\mathcal{N}$=2 field theories.

Hyperkähler (HK) manifolds are Ricci flat, and hence are solutions to Einstein's equations.

Moreover, the results on ``surface operators'' lead to a construction of solutions to natural generalizations of the Yang-Mills equations on HK manifolds. (Hyperholomorphic connections.)

(On a 4-dimensional HK manifold a hyperholomorphic connection is the same thing as a self-dual Yang-Mills instanton.)

New Interrelations, Directions & Problems

A good development should open up new questions and directions of research and provide interesting links to other lines of enquiry.

It turns out that solving the above problems leads to interesting relations to …

Hitchin systems, integrable systems, moduli spaces of flat connections on surfaces, cluster algebras, Teichműller theory and the ``higher Teichműller theory'' of Fock & Goncharov, ….

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d=4,$\mathcal{N}$=2 Superalgebra

Poincare superalgebra $\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1$

$$\mathfrak{s}^0 = \mathrm{poin}(1,3) \oplus su(2)_R \oplus u(1)_R \oplus \mathbb{C} \cdot Z$$

$$\mathfrak{s}^1 = \left[(2,1;2)_{+1} \oplus (1,2;2)_{-1} \right]$$

$$Q_\alpha^A, \, \bar{Q}_{\dot{\alpha}}^A, \quad \{ Q_\alpha{}^A, \bar{Q}_{\dot{\beta}B} \} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A{}_B$$

$$\{ Q_\alpha{}^A, Q_\beta{}^B \} = 2\epsilon_{\alpha\beta}\epsilon^{AB} \bar{Z}$$

Constraints on the Theory

Representation theory: Field and particle multiplets

Lagrangians: Typically depend on very few parameters for a given field content.

BPS Spectrum: Special subspace in the Hilbert space of states

Example: $\mathcal{N}$=2 Super-Yang-Mills

$$a = 1, \ldots, \dim G$$

Gauge fields:

$$A_\mu^a$$

Doublet of gluinos:

$$\psi_{\alpha,1}^a \qquad \psi_{\alpha,2}^a$$

Complex adjoint scalars:

$$\varphi^a$$

Hamiltonian & Classical Vacua

The renormalizable Lagrangian is completely determined up to a choice of Yang-Mills coupling g^2.

$$E = g^{-2} \int_{\mathbb{R}^3} d^3x \mathrm{Tr} \left(\vec{E}^2 + \vec{B}^2 + |D\varphi|^2 \right) + g^{-2} \int_{\mathbb{R}^3} d^3x \mathrm{Tr} \left([\varphi, \varphi^\dagger]^2 \right)$$

Classical Vacua: $\vec{E} = \vec{B} = 0$

$$\varphi = \mathrm{Diag}\{a^1, \dots, a^K\} \in \mathfrak{t} \otimes \mathbb{C}$$

Quantum Moduli Space of Vacua

Claim: The continuous vacuum degeneracy is an exact property of the quantum theory:

$$\exists \quad |\Omega(u)\rangle \quad u \in \mathcal{B} := \mathfrak{t} \otimes \mathbb{C}/W$$

$$\langle \Omega(u)| \mathrm{Tr}\varphi^s |\Omega(u)\rangle = u_s$$

Physical properties depend on the vacuum $|\Omega(u)\rangle$

Low Energy: Abelian Gauge Theory

$$\langle \Omega(u) | \varphi | \Omega(u) \rangle = \mathrm{Diag}\{a^1, \dots, a^K\}$$

Unbroken gauge symmetry: $U(1)^r$

(r= Rank = K-1)

Low energy theory is described by an $\mathcal{N}$=2 extension of Maxwell's theory:

Maxwell fields F^I, I=1,…, r. i.e.

$F \in \Omega^2(\mathbb{R}^{1,3}, \mathfrak{t})$ & their superpartners

Low-Energy Effective Action

$\mathcal{N}$=2 susy constrains the low energy effective action of the Maxwell theory to be of the form

$$S = \int \mathrm{Im}\tau_{IJ} F^I * F^J + \mathrm{Re}\tau_{IJ} F^I F^J + \cdots$$

$$\tau_{IJ} = \frac{\theta_{IJ}}{8\pi} + i\frac{4\pi}{e^2_{IJ}}$$

is a symmetric, holomorphic matrix function of the vacuum parameters u.

Electro-magnetic Charges

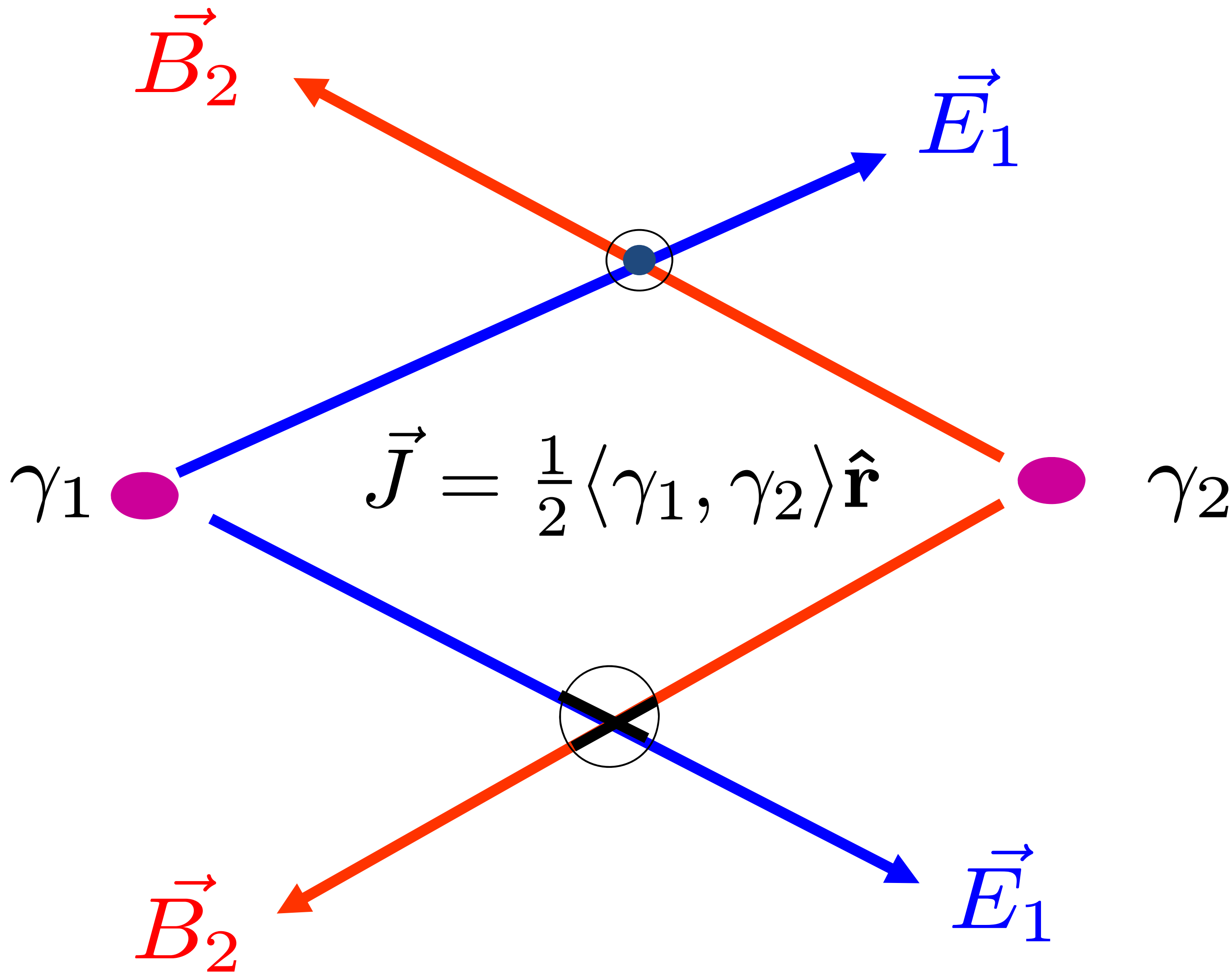
The theory will also contain ``dyonic particles'' – particles with electric and magnetic charges for the various Maxwell fields F^I, $I = 1,\ldots, r$.

(Magnetic, Electric) Charges:

$$\gamma = (p^I, q_I) \qquad I = 1, \ldots, r$$

On general principles they are in a <u>symplectic</u> lattice Γ_u.

Dirac Quantization:



$$\langle \gamma_1, \gamma_2 \rangle = p_1^I q_{2,I} - p_2^I q_{1,I} \in \mathbb{Z}$$

BPS States

Superselection sectors: $\mathcal{H} = \bigoplus_{\gamma\in\Gamma}\mathcal{H}_\gamma$

Taking the square of suitable Hermitian combinations of susy generators and using the algebra shows that in sector $\mathcal{H}_\gamma$

$$E \geq |Z_\gamma|$$

$$\boxed{\mathcal{H}_\gamma^{\rm BPS} := \{\psi : E\psi = |Z_\gamma|\psi\}}$$

The Central Charge Function

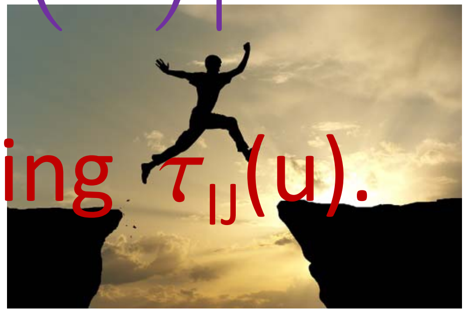
The central charge function is a linear function

$$Z : \Gamma \to \mathbb{C}$$

$$Z_{\gamma_1+\gamma_2} = Z_{\gamma_1} + Z_{\gamma_2}$$

This linear function depends holomorphically on the vacuum manifold $\mathcal{B}$. Denote it by Z(u).

On $\mathcal{H}_{\gamma}^{\mathrm{BPS}}$ $E = |Z_{\gamma}(u)|$



Knowing $Z_\gamma(u)$ is equivalent to knowing $\tau_{IJ}(u)$.

General d=4, $\mathcal{N}$=2 Theories

1. A moduli space $\mathcal{B}$ of quantum vacua,
   (a.k.a. the ``Coulomb branch’’).

The low energy dynamics are described by an effective $\mathcal{N}$=2 abelian gauge theory.

2. The Hilbert space is graded by an integral lattice of charges, Γ, with integral

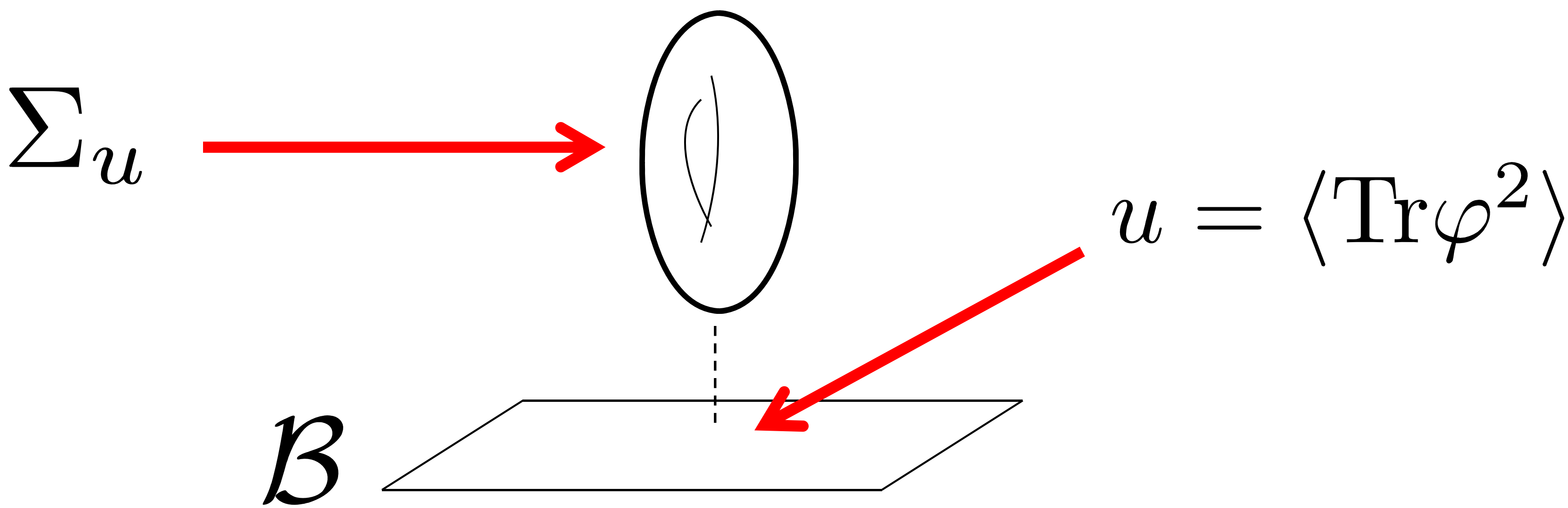
anti-symmetric form. There is a BPS subsector with masses given exactly by $|Z_{\gamma}(u)|$.

So far, everything I've said follows fairly straightforwardly from general principles.

But how do we compute $Z_{\gamma}(u)$ and $\tau_{IJ}(u)$ as functions of u ?

Seiberg-Witten Curve

Seiberg & Witten showed (for SU(2) SYM) that τ(u) can be computed in terms of the periods of a meromorphic differential form λ on a Riemann surface Σ both of which depend on u.



The Promise of Seiberg-Witten Theory

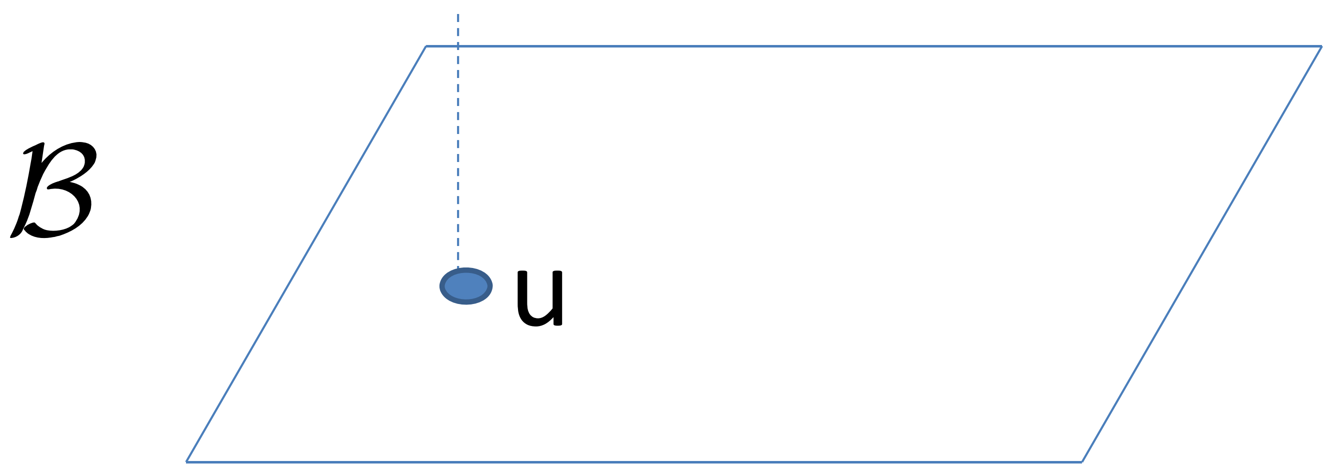
So Seiberg & Witten showed how to determine the LEEA exactly as a function of u, at least for G=SU(2) SYM.

They also gave cogent arguments for the exact BPS spectrum of this theory.

So it was natural to try to find the LEEA and the BPS spectrum for other d=4 $\mathcal{N}$=2 theories.

Extensive subsequent work showed that this picture indeed generalizes to all known solutions for the LEEA of $\mathcal{N}$=2 field theory:

The family of Riemann surfaces is usually called the ``Seiberg-Witten curve'' and the meromorphic differential thereupon is the ``Seiberg-Witten differential.''

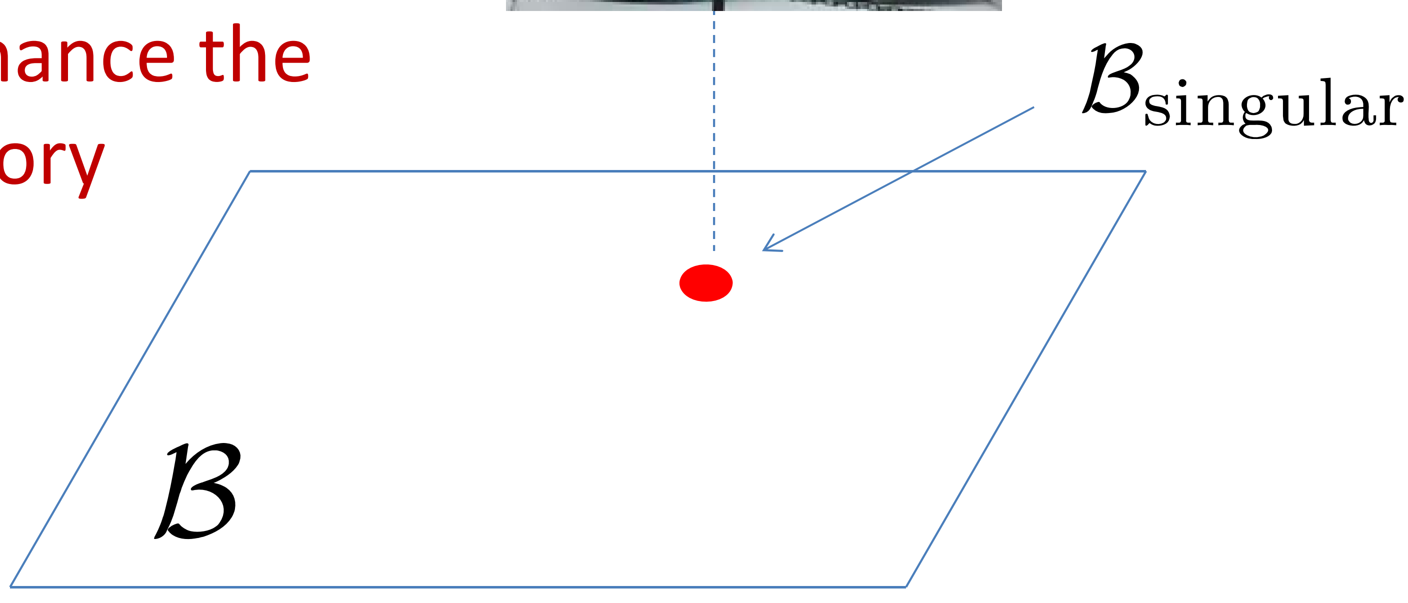


But, to this day, there is no general algorithm for computing the Seiberg-Witten curve and differential for a given $\mathcal{N}$=2 field theory.

Singular Locus

On a special complex codimension one sublocus $\mathcal{B}_{\text{singular}}$ the curve degenerates

new massless degrees of freedom enhance the Maxwell theory



But what about the BPS spectrum?

In the 1990's the BPS spectrum was only determined in a handful of cases…

(SU(2) with (N=2 supersymmetric) quarks flavors: N_f = 1,2,3,4, for special masses: Bilal & Ferrari)

In the past 5 years there has been a great deal of progress in understanding the BPS spectra in these and infinitely many other $\mathcal{N}=2$ theories.

One key element of this progress has been a much-improved understanding of the ``wall-crossing phenomenon.''

Recall the space of BPS states is:

$$\mathcal{H}_{\gamma}^{\mathrm{BPS}} = \{\psi : E\psi = |Z_{\gamma}(u)|\psi\}$$

It is finite dimensional.

It is a representation of so(3) $\oplus$ su(2)$_R$

It depends on u, since $Z_{\gamma}(u)$ depends on u.

But even the dimension can depend on u !

BPS Index

As in the index theory of Atiyah & Singer, $\mathcal{H}^{\mathrm{BPS}}$ is $\mathbb{Z}_2$ graded by $(-1)^F$ so there is an index, in this case a Witten index, which behaves much better (piecewise constant in u):

$$\Omega(\gamma) = \mathrm{Tr}_{\mathcal{H}_\gamma^{\mathrm{BPS}}} (2J_3)^2 (-1)^{2J_3}$$

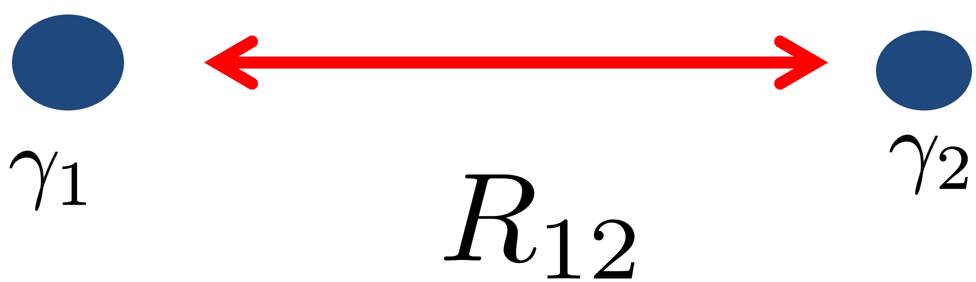
J_3 is any generator of so(3)

The Wall-Crossing Phenomenon

But even the ***index*** can depend on u !

$$\Omega(\gamma) \to \Omega(\gamma; u)$$

BPS particles can form bound states which are themselves BPS!



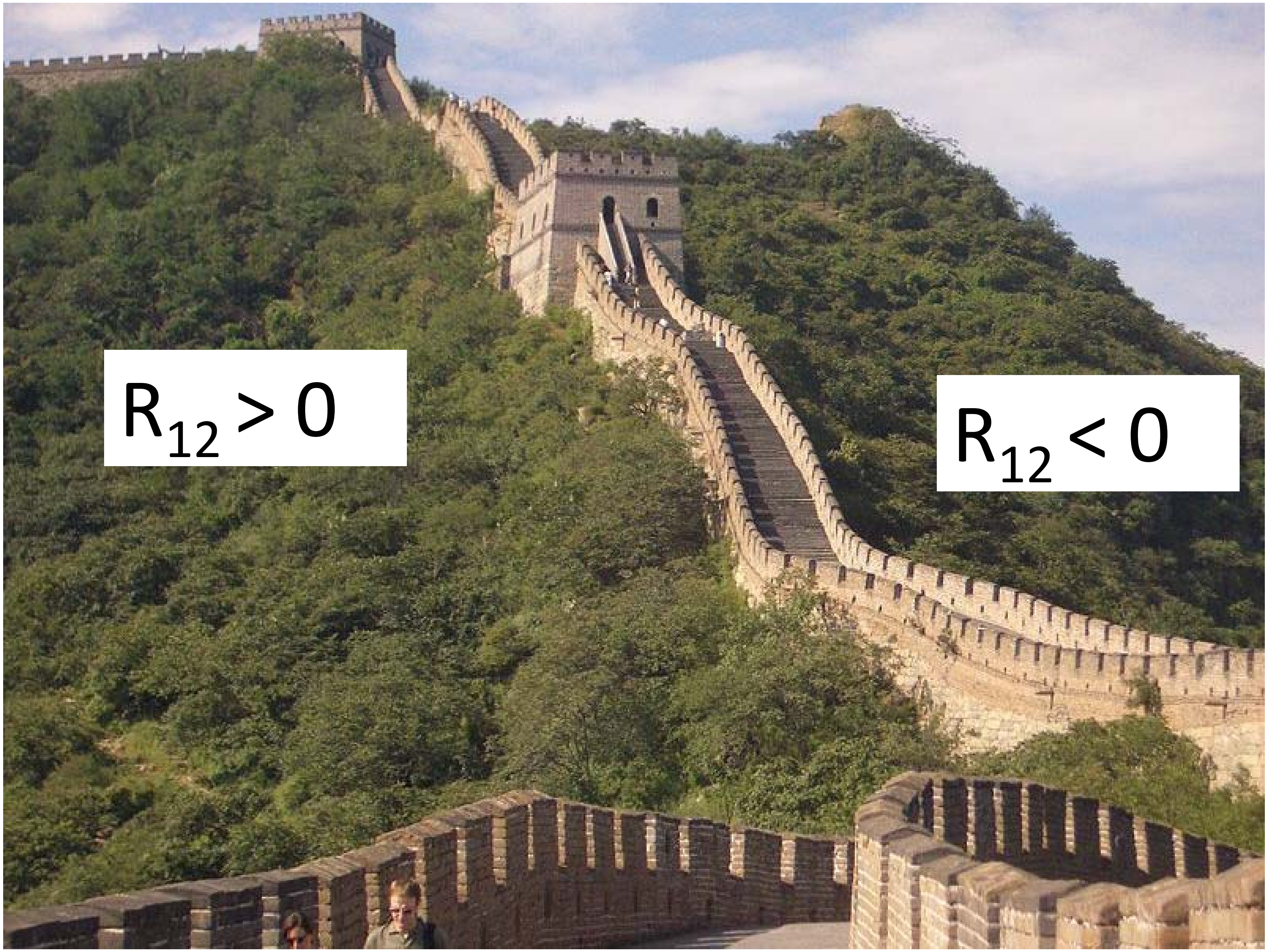
Denef's Boundstate Radius Formula

$$R_{12}(u) = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{2\mathrm{Im}\left(Z_{\gamma_1}(u) Z_{\gamma_2}(u)^*\right)}$$

The Z's are functions of the moduli $u \in \mathcal{B}$

So the moduli space of vacua $\mathcal{B}$ is divided into two regions:

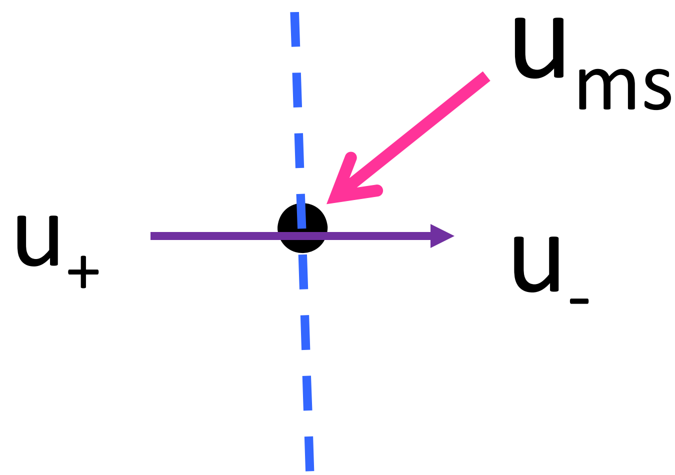
$\langle \gamma_1, \gamma_2 \rangle \mathrm{Im}(Z_1 Z_2^*) > 0$ ***OR*** $\langle \gamma_1, \gamma_2 \rangle \mathrm{Im}(Z_1 Z_2^*) < 0$



$R_{12} > 0$
$R_{12} < 0$

Wall of Marginal Stability

Consider a path of vacua crossing the wall:



Exact binding energy:

$$|Z_{\gamma_1+\gamma_2}(u)| - (|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|) \leq 0$$

$$MS(\gamma_1, \gamma_2) := \{u | Z_{\gamma_1}(u) \parallel Z_{\gamma_2}(u)\}$$

The Primitive Wall-Crossing Formula

(Denef & Moore, 2007)

$$R_{12} = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_1 + Z_2|}{2\mathrm{Im}(Z_1 Z_2^*)}$$

Crossing the wall: $\mathrm{Im}(Z_1 Z_2^*) \to 0$

$$\boxed{\Delta \mathcal{H} = (J_{12}) \otimes \mathcal{H}^{\mathrm{BPS}}_{\gamma_1} \otimes \mathcal{H}^{\mathrm{BPS}}_{\gamma_2}}$$

$$2J_{12} + 1 = |\langle \gamma_1, \gamma_2 \rangle|$$

Non-Primitive Bound States

But this is not the full story, since the same marginal stability wall holds for charges $N_1 \gamma_1$ and $N_2 \gamma_2$

The full wall-crossing formula, which describes all possible bound states which can form is the ``Kontsevich-Soibelman wall-crossing formula''

Line Defects

There are now several physical derivations of this formula, but – in my view -- the best derivation uses ``line operators'' – or more properly - ``line defects.''

These are nonlocal objects associated with dimension one subsets of spacetime.

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9. Conclusion

Interlude: Defects in Local QFT

Extended ``operators'' or ``defects'' have been playing an increasingly important role in recent years in quantum field theory.

Pseudo-definition: Defects are local disturbances supported on positive codimension submanifolds of spacetime.

Examples of Defects

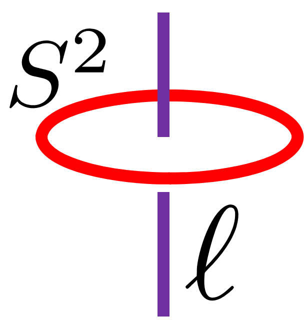
Example 1: d=0: Local Operators

Example 2: d=1: ``Line operators''

Gauge theory
Wilson line:

$$W(\ell) = P \exp \int_\ell A$$

4d Gauge theory
't Hooft loop:



$$F \sim m \sin\theta d\theta d\phi + \cdots$$

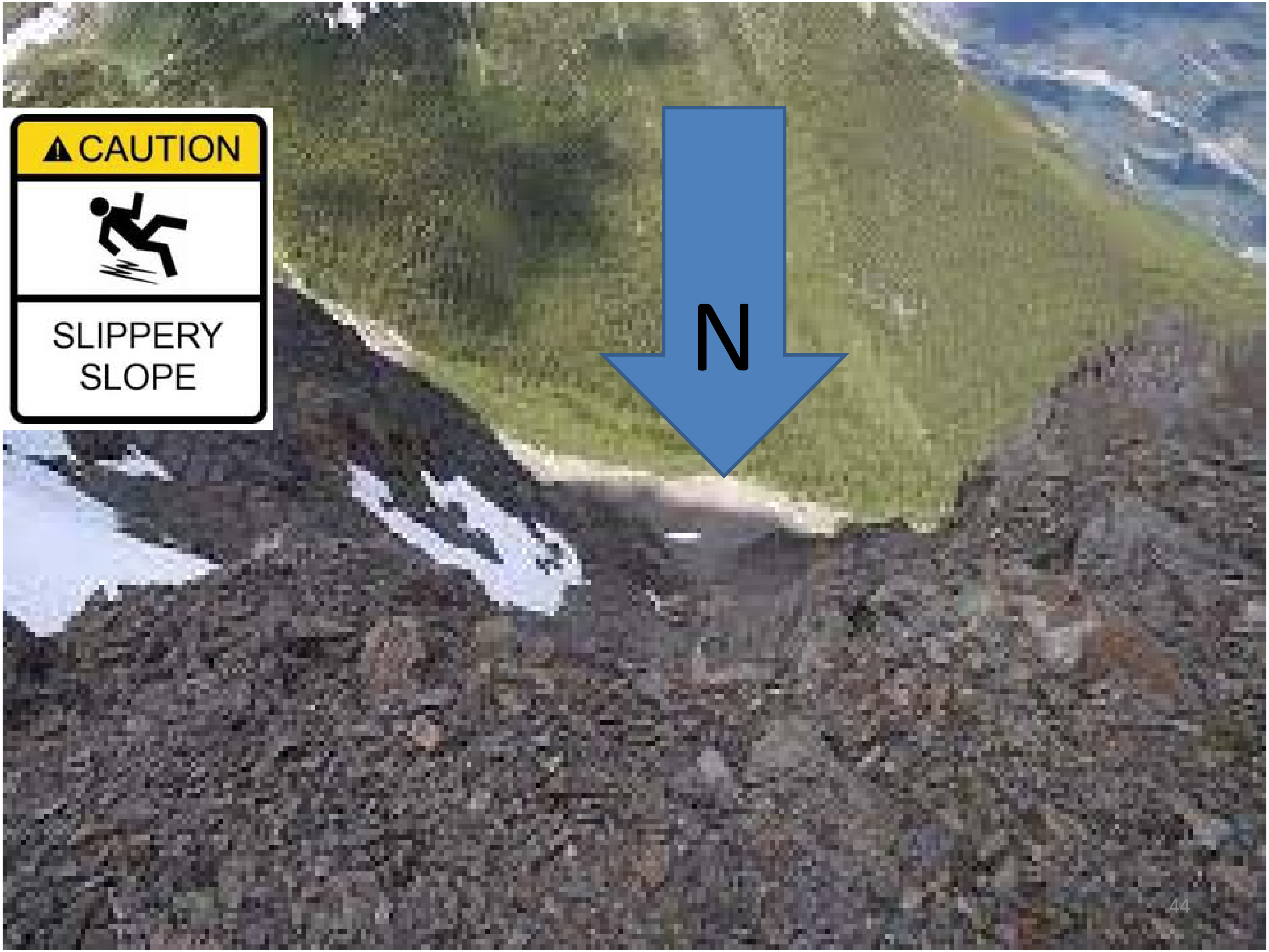
Example 3: Surface defects: Couple a 2-dimensional field theory to an ambient theory. These 2d4d systems play an important role later.

Extended QFT and N-Categories

The inclusion of these extended objects enriches the notion of quantum field theory.

Even in the case of topological field theory, the usual formulation of Atiyah and Segal is enhanced to ``extended TQFT's''   leading to beautiful relations to N-categories and the ``cobordism hypothesis'' …

D. Freed; D. Kazhdan; N. Reshetikhin; V. Turaev; L. Crane; Yetter; M. Kapranov; Voevodsky; R. Lawrence; J. Baez + J. Dolan ; G. Segal;   M. Hopkins, J. Lurie, C. Teleman,L. Rozansky, K. Walker, A. Kapustin, N. Saulina,…


CAUTION
SLIPPERY SLOPE
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1. Introduction
2. Review: d=4, $\mathcal{N}$=2 field theory
3. Wall Crossing 101
4. Defects in Quantum Field Theory
5. Wall Crossing 102
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We will now use these line defects to produce a physical derivation of the Kontsevich-Soibelman wall-crossing formula.

Gaiotto, Moore, Neitzke; Andriyash, Denef, Jafferis, Moore

Supersymmetric Line Defects

Our line defects will be at $\mathbb{R}_t \times \{ 0 \} \subset \mathbb{R}^{1,3}$

A ***line defect*** L is ***of type $\zeta = e^{i\vartheta}$*** if it preserves:

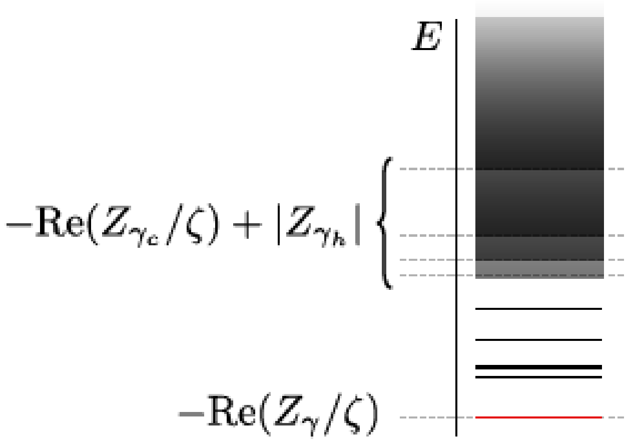
$$Q_\alpha^A + \zeta \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta} A}$$

Example: $$L_\zeta = \exp \int_{\mathbb{R}_t \times \vec{0}} \left(\frac{\varphi}{2\zeta} + A + \frac{\zeta}{2}\overline{\varphi} \right)$$

$$\mathcal{H}_L = \bigoplus_{\gamma \in \Gamma + \gamma_0} \mathcal{H}_{L,\gamma}$$

Physical picture for charge sector γ: As if we inserted an infinitely heavy BPS particle of charge γ

Framed BPS Index



$$E \geq -\mathrm{Re}(Z_\gamma/\zeta)$$

Framed BPS States are states in $\mathcal{H}_{\mathrm{L},\gamma}$ which saturate the bound.

$$\overline{\underline{\Omega}}(L;\gamma) = \mathrm{Tr}_{\mathcal{H}_{L,\gamma}}(-1)^{2J_3}$$

Framed BPS Wall-Crossing

Piecewise constant in ζ and u, but has wall-crossing across ``***BPS walls***'' (only defined for $\Omega(\gamma) \neq 0$):

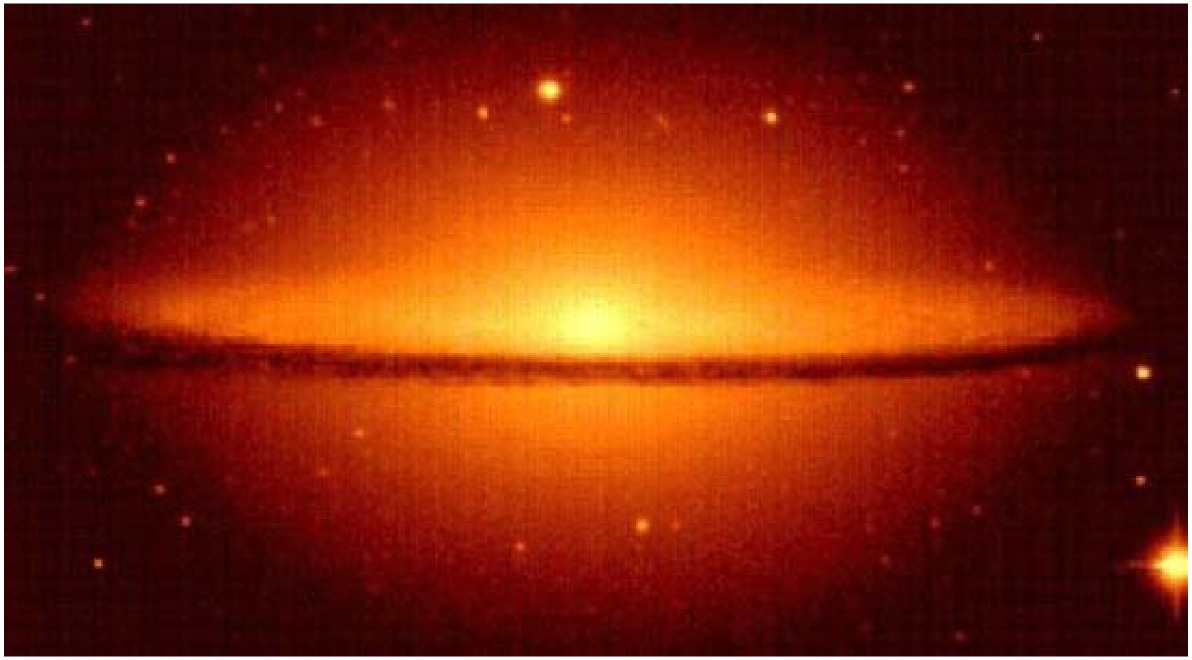
$$W_\gamma := \{(u,\zeta) : Z_\gamma(u)/\zeta \in \mathbb{R}_-\}$$

BPS particle of charge γ binds to the defect states in charge sector γ_c to make a new framed BPS state:

$$r_\gamma = \frac{\langle \gamma, \gamma_c \rangle}{2\mathrm{Im} Z_\gamma(u)/\zeta}$$

Halo Picture

But, particles of charge γ, and indeed n γ for any n>0 can bind in arbitrary numbers: they feel no relative force, and hence there is an entire Fock space of boundstates with halo particles of charges n γ.



Framed BPS Generating Function

$$F(L) := \sum_{\gamma} \overline{\underline{\Omega}}(L;\gamma) X_{\gamma}$$

$$X_{\gamma_1} X_{\gamma_2} = (-1)^{\langle \gamma_1 , \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

(The sign takes account of the fact that some halo particles are bosonic or fermionic.)

When crossing a BPS wall W_{γ} the charge sector γ_c gains or loses a Fock space factor

$$X_{\gamma_c} \to (1 - (-1)^{\langle \gamma, \gamma_c \rangle} X_{\gamma})^{\langle \gamma, \gamma_c \rangle \Omega(\gamma)} X_{\gamma_c}$$

Description via Differential Operators

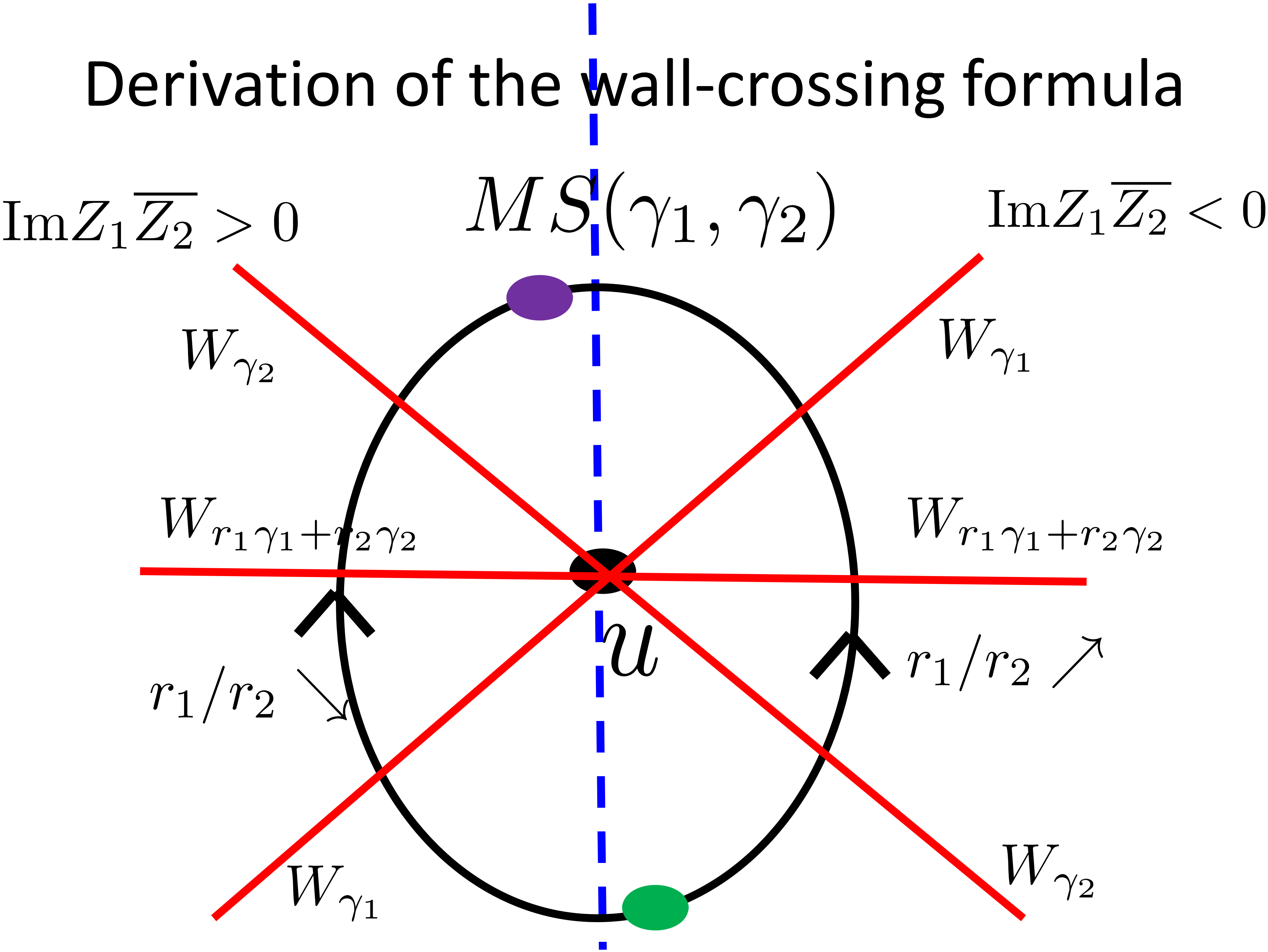
So the change of F(L) across a BPS wall W_γ is given by the action of a ***differential operator***:

$$F(L) \to K_\gamma^{\Omega(\gamma)} F(L)$$

$$K_\gamma = (1 - X_\gamma)^{D_\gamma}$$

$$D_\gamma X_\rho := \langle \gamma, \rho \rangle X_\rho$$

Derivation of the wall-crossing formula



The Kontsevich-Soibelman Formula

$$\prod_{\nearrow} K_{r_1\gamma_1+r_2\gamma_2}^{\Omega(r_1\gamma_1+r_2\gamma_2;-)} = \prod_{\searrow} K_{r_1\gamma_1+r_2\gamma_2}^{\Omega(r_1\gamma_2+r_2\gamma_2;+)}$$

$$K_{\gamma_2}^{\Omega(\gamma_2;u_-)} \cdots K_{\gamma_1}^{\Omega(\gamma_1;u_-)}$$

$$= \quad K_{\gamma_1}^{\Omega(\gamma_1;u_+)} \cdots K_{\gamma_2}^{\Omega(\gamma_2;u_+)}$$

Example 1: The Pentagon Identity

$$\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$$

$$\langle \gamma_1, \gamma_2 \rangle = +1$$

$$K_{\gamma_2} K_{\gamma_1} = K_{\gamma_1} K_{\gamma_1+\gamma_2} K_{\gamma_2}$$

Related to consistency of simple superconformal field theories ("Argyres-Douglas theories") coherence theorems in category theory & associahedra, 5-term dilogarithm identity, …

Example 2

$$\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$$

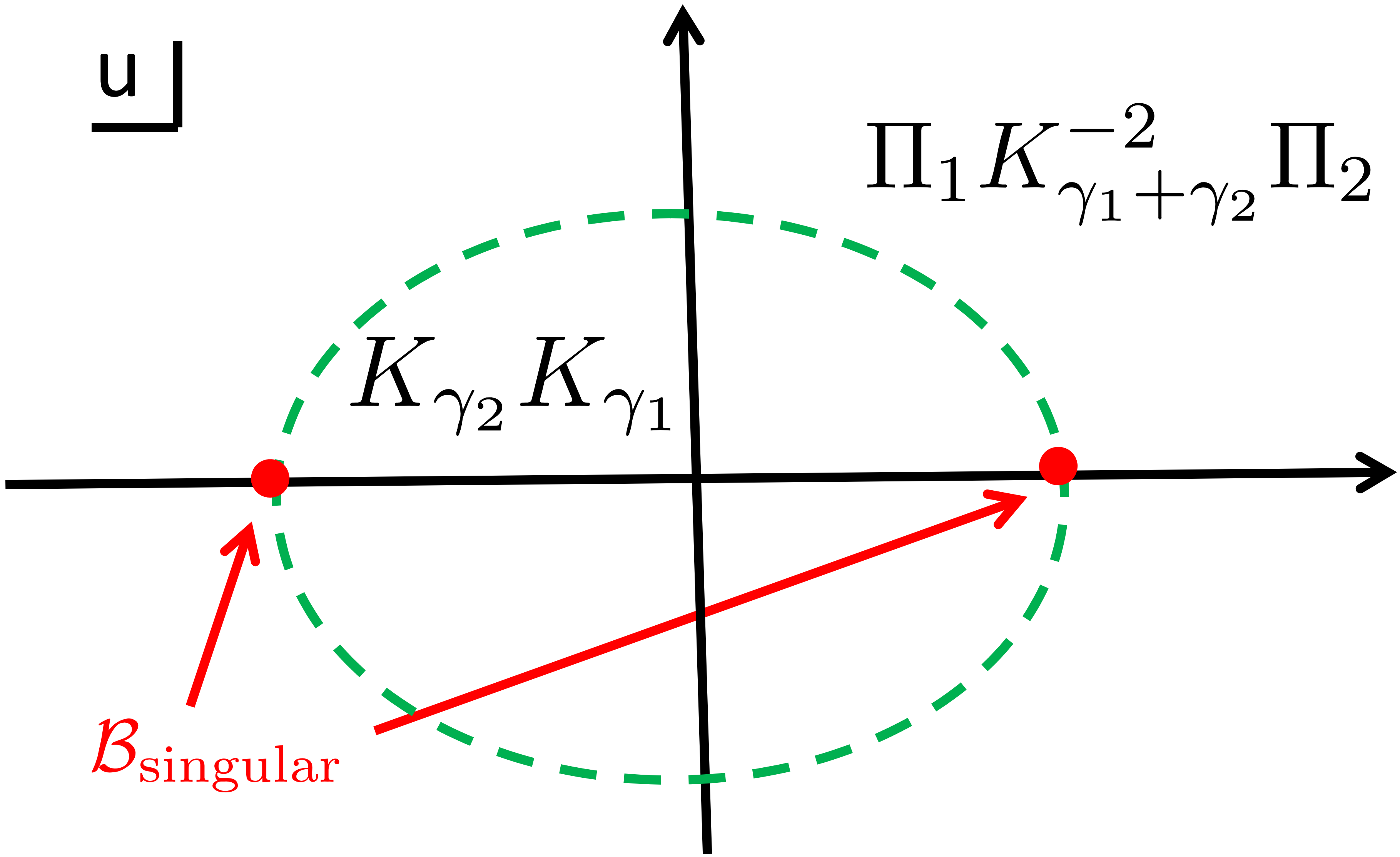
$$\langle \gamma_1, \gamma_2 \rangle = +2$$

$$K_{\gamma_2} K_{\gamma_1} = \Pi_1 K_{\gamma_1+\gamma_2}^{-2} \Pi_2$$

$$\Pi_1 = K_{\gamma_1} K_{2\gamma_1+\gamma_2} K_{3\gamma_1+2\gamma_2} \cdots$$

$$\Pi_2 = \cdots K_{2\gamma_1+3\gamma_2} K_{\gamma_1+2\gamma_2} K_{\gamma_2}$$

The SU(2) Spectrum



(No) Wild Wall Conjecture

For other values of $< \gamma_1, \gamma_2 >$ rearranging $K_1 K_2$ produces exponentially growing BPS degeneracies.

This is in conflict with basic thermodynamics of QFT, and hence for physical reasons we expect that there are never any such ``wild wall crossings''

This seems very nontrivial from the mathematical viewpoint.

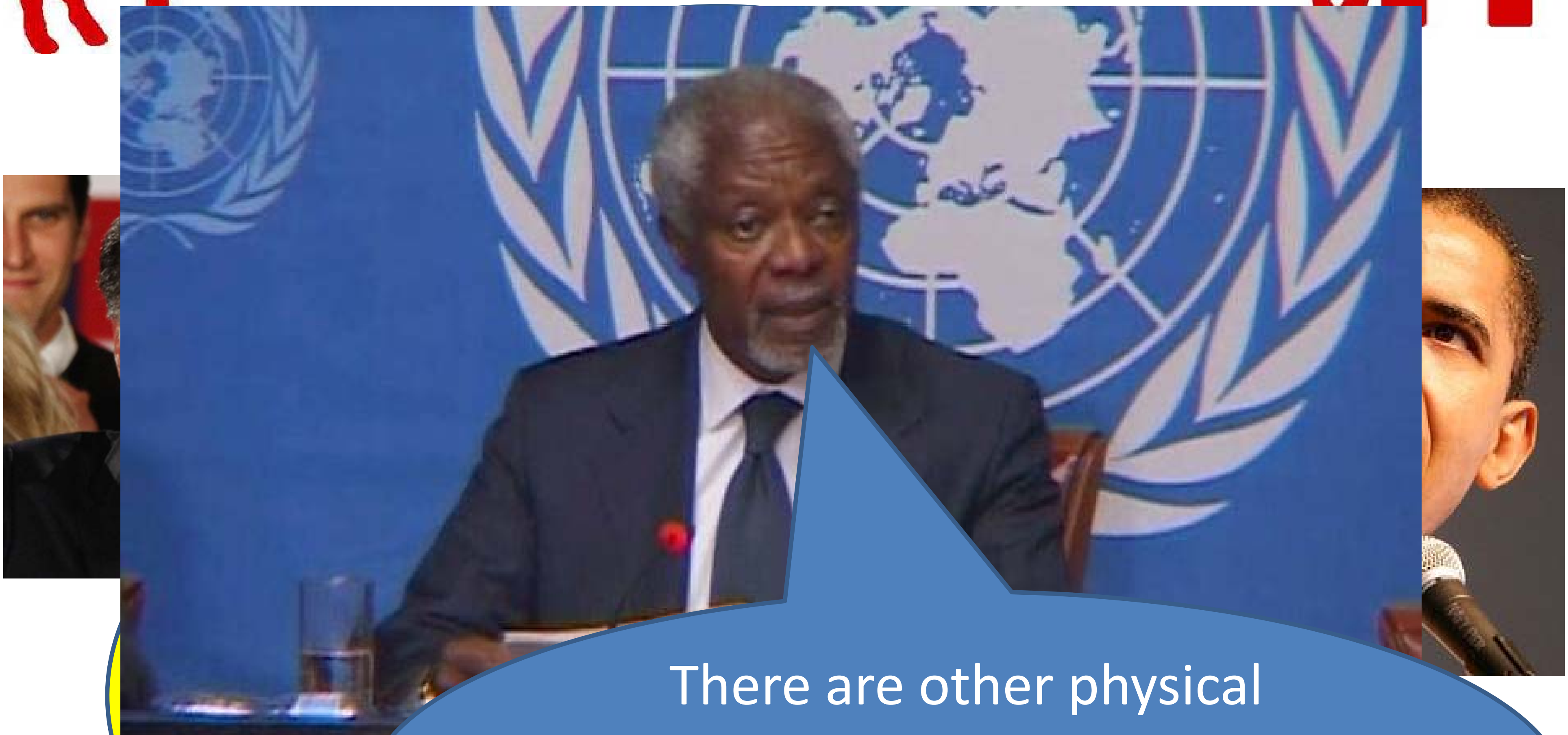
Only half the battle…

The wall crossing formula only describes the <u>CHANGE</u> of the BPS spectrum across a wall of marginal stability.

It does NOT determine the BPS spectrum!

We'll return to that in Part 8, for theories of class S.

Political Advertisement



There are other physical derivations of the KSWCF due to Cecotti & Vafa and Manschot, Pioline, & Sen.

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Strategy

Compactification on a circle of radius R leads to a 3-dimensional sigma model with target space $\mathcal{M}$, a hyperkähler manifold.

In the large R limit the metric can be solved for easily. At finite R there are mysterious instanton corrections.

Finding the HK metric is equivalent to finding a suitable set of functions on the twistor space of $\mathcal{M}$.

The required functions are solutions of an explicit integral equation (resembling Zamolodchikov's TBA).

Low Energy theory on $\mathbb{R}^3 \times S^1$

(Seiberg & Witten)

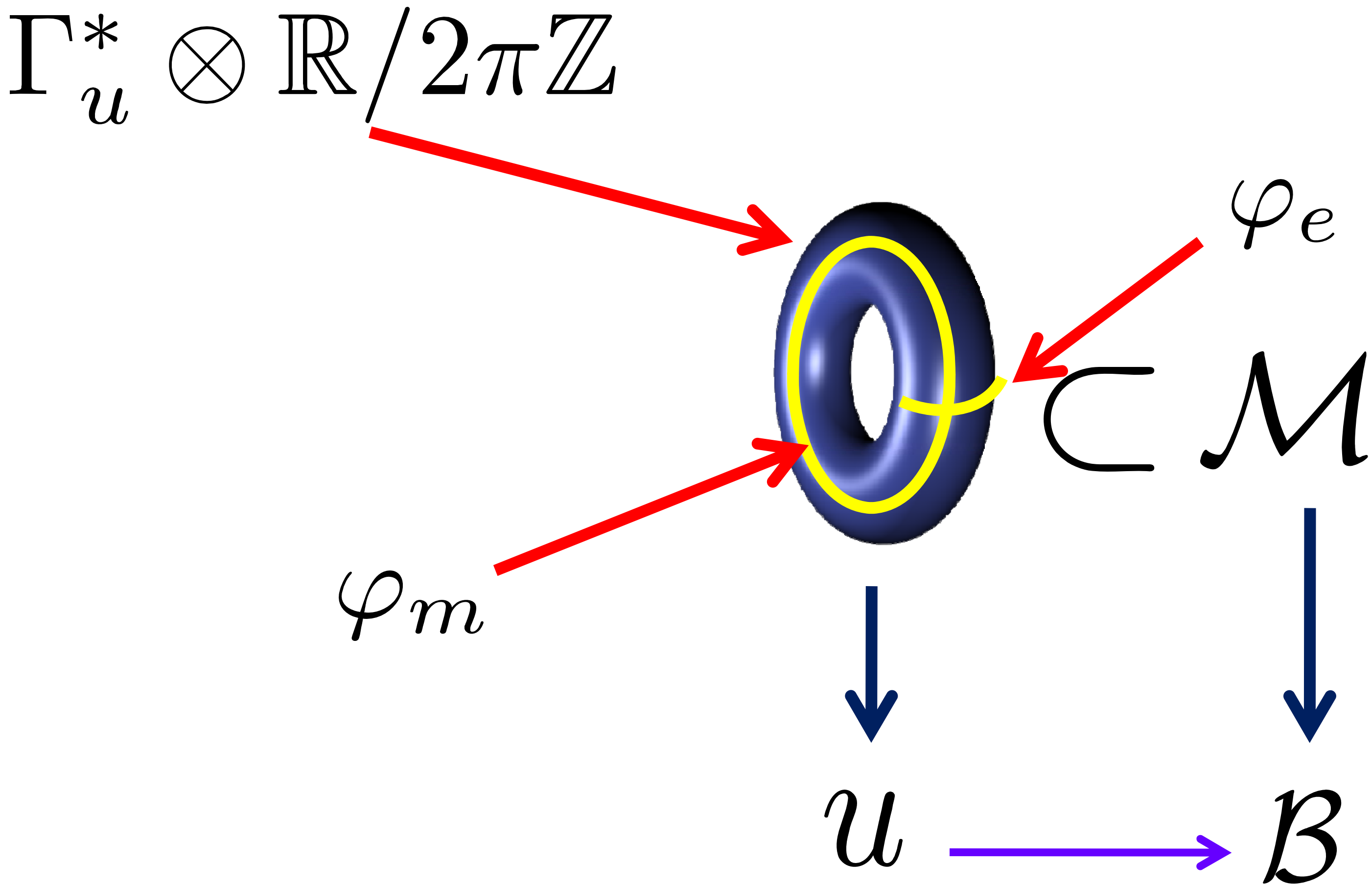
3D sigma model with target space $\mathcal{M}$

4D scalars reduce to 3d scalars: $a^I(x) \in \mathcal{B}$

$$\left.\begin{aligned} \varphi_e^I &= \oint_{S^1} A_4^I dx^4 \\ \varphi_{m,I} &= \oint_{S^1} (A_{D,I})_4 dx^4 \end{aligned}\right\}$$

Periodic
Wilson
scalars

Seiberg-Witten Moduli Space $\mathscr{M}$



($\Rightarrow$ Relation to integrable systems)

Semiflat Metric

The leading approximation in the R $\longrightarrow \infty$ limit is straightforward to compute:

$$g^{\mathrm{sf}} = da^I R \mathrm{Im}\tau_{IJ} d\bar{a}^J + dz_I \frac{1}{R \mathrm{Im}\tau_{IJ}} d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ} d\varphi_e^J$$

Singular on $\mathcal{B}_{\mathrm{sing}}$

Twistor Space

$$\mathcal{Z} := \mathcal{M} \times \mathbb{C}P^1 \xrightarrow{p} \mathbb{C}P^1$$

Fiber above ζ is $\mathcal{M}$ in complex structure ζ

Hitchin Theorem: A HK metric g is equivalent to a fiberwise holomorphic symplectic form

$$\varpi \in \Omega^2_{\mathcal{Z}/\mathbb{C}P^1} \otimes \mathcal{O}(2)$$

$$\varpi_\zeta = \zeta^{-1}\omega_+ + \omega_3 + \zeta\omega_- \qquad \zeta \in \mathbb{C}^*$$

Local Charts

$\mathcal{M}$ has a coordinate atlas $\{\mathcal{U}\}$ with charts of the form

$$\mathcal{U} \cong \Gamma^* \otimes \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{r}$$

Contraction with γ defines canonical ``Darboux functions'' Y_γ

$$Y_{\gamma_1} Y_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} Y_{\gamma_1+\gamma_2}$$

Canonical holomorphic symplectic form:

$$\varpi_T = \epsilon^{ij} d\log Y_{\gamma_i} \wedge d\log Y_{\gamma_j}$$

The ``Darboux functions''

So we seek a ``suitable'' holomorphic maps

$$\mathcal{Y} : \mathcal{U} \times \mathbb{C}^* \to \Gamma^* \otimes \mathbb{C}^*$$

such that

$$\varpi_\zeta = \mathcal{Y}^*(\varpi_T) = \epsilon^{ij} d \log \mathcal{Y}_{\gamma_i} \wedge d \log \mathcal{Y}_{\gamma_j}$$

solves the problem.

$$\mathcal{Y}_\gamma(u, \varphi_e, \varphi_m; \zeta) := \mathcal{Y}^*(Y_\gamma)$$

Darboux Functions for the Semiflat Metric

For the semiflat metric one can solve for the Darboux functions in a straightforward way:

$$\mathcal{Y}_\gamma^{\rm sf} = \exp\left[\pi R \zeta^{-1} Z_\gamma(u) + i\varphi_\gamma + \pi R \zeta \overline{Z}_\gamma\right]$$

(Neitzke, Pioline, Vandoren)

Strategy: Find the quantum corrections to the metric from the quantum corrections to the Darboux functions:

$$\mathcal{Y}_\gamma = \mathcal{Y}_\gamma^{\rm sf} \cdot \mathcal{Y}_\gamma^{\rm quantum correction}$$

Riemann-Hilbert Problem

The desired properties of the exact functions

$$\mathcal{Y}_\gamma(u, \varphi_e, \varphi_m; \zeta) := \mathcal{Y}^*(Y_\gamma)$$

lead to a list of conditions which correspond to a Riemann-Hilbert problem for $\mathcal{Y}_\gamma$ on the ζ-plane.

Solution Via Integral Equation

(Gaiotto, Moore, Neitzke: 2008)

$$\log \mathcal{Y}_\gamma = \log \mathcal{Y}_\gamma^{\mathrm{sf}} +$$

$$+ \sum_{\gamma' \in \Gamma} \langle \gamma, \gamma' \rangle \Omega(\gamma') \mathbb{K}_{\gamma'} * \log(1 - \mathcal{Y}_{\gamma'})$$

$$\mathbb{K}_\gamma * f = \int_{W_\gamma} d\zeta' k(\zeta; \zeta') f(\zeta')$$

$$W_\gamma := \{\zeta | Z_\gamma / \zeta < 0\}$$

Remarks

1. Solving by iteration converges for large R for sufficiently tame BPS spectrum.

(A typical field theory spectrum will be tame; a typical black hole spectrum will NOT be tame!)

2. The HK metric carries an ``imprint'' of the BPS spectrum, and indeed the metric is smooth iff the KSWCF holds!

3.   The coordinates $\mathcal{Y}_\gamma$ are cluster coordinates.

Other Applications of the Darboux Functions

The same functions allow us to write explicit formulae for the vev's of line defects:

$$\langle L_\zeta \rangle_{m \in \mathcal{M}} = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta, \gamma) \mathcal{Y}_\gamma(m)$$

- Exact results on line defect vevs. (Example below).
- Deformation quantization of the algebra of holomorphic functions on $\mathcal{M}$

Generalized Darboux Functions & Generalized Yang-Mills Equations

In a similar way, surface defects lead to a generalization of Darboux functions.

These functions also satisfy an integral equation strongly reminiscent of those used in inverse scattering theory.

Geometrically, these functions can be used to construct hyperholomorphic connections on $\mathcal{M}$

(A hyperholomorphic connection is one whose fieldstrength is of type (1,1) in all complex structures.)

We now turn to a rich set of examples of d=4, $\mathcal{N}$=2 theories,

the theories of class S.

("S" is for six)

In these theories many physical quantities have elegant descriptions in terms of Riemann surfaces and flat connections.

The six-dimensional theories

Claim, based on string theory constructions:

There is a family of stable interacting field theories, S[$\mathfrak{g}$] ,
with six-dimensional (2,0) superconformal symmetry.
(Witten; Strominger; Seiberg).

These theories have not been constructed – even by physical standards - but some characteristic properties of these hypothetical theories can be deduced from their relation to string theory and M-theory.

These properties will be treated as axiomatic. Later they should be theorems.

Theories of Class S

Consider nonabelian (2,0) theory S[$\mathfrak{g}$] for ``gauge algebra'' $\mathfrak{g}$

The theory has half-BPS codimension two defects D

Compactify on a Riemann surface C with D_a inserted at punctures z_a

$$so(5)_R \to so(3)_R \oplus \underbrace{so(2)_R}_{\text{Twist to preserve d=4,N=2}}$$

Witten, 1997
GMN, 2009
Gaiotto, 2009

$$S[\mathfrak{g}, C, D]$$

Type II duals via ``geometric engineering'' KLMVW 1996

Most ``natural'' theories are of class S:

For example, SU(K) N=2 SYM coupled to ``quark flavors''.

But there are also (infinitely many) theories of class S with no (known) Lagrangian, e.g. Argyres-Douglas theories, or the trinion theories of (Gaiotto, 2009).

Relation to Hitchin systems

$$\begin{tikzcd}
 & S[\mathfrak{g}] \arrow[dl, "C"'] \arrow[dr, "S^1"] & \\
\boxed{S[\mathfrak{g}, C, D]} \arrow[dr, "S^1"'] & & \boxed{\text{5D } \mathfrak{g} \text{ SYM}} \arrow[dl, "C"] \\
 & \mathbb{R}^{1,2} \to \mathcal{M} &
\end{tikzcd}$$

σ-Model: $\mathbb{R}^{1,2} \to \mathcal{M}$

$$F + R^2[\varphi, \bar{\varphi}] = 0$$
$$\bar{\partial}_A \varphi = 0$$

Effects of Defects

$$\varphi \sim \frac{dz}{(z-z_a)^{\ell_a}} \mathfrak{r}_a + \cdots \quad \ell_a \geq 1$$

Physics depends on choice of ℓ_a & $\mathfrak{r}_a$

Physics of these defects is still being understood: (Gaiotto, Moore, Tachikawa; Chacaltana, Distler, Tachikawa)

Relation to Flat Complex Gauge Fields

If (φ, A) solves the Hitchin equations then

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi}$$

is flat:

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

$\mathcal{M}_{\zeta} \cong$ a moduli space of flat SL(K,$\mathbb{C}$) connections.

We will now show how

Seiberg-Witten curve & differential λ

Charge lattice & Coulomb branch $\mathcal{B}$

BPS states

Line & surface defects

can all be formulated geometrically in terms of the geometry and topology of the UV curve C and its associated flat connection $\mathcal{A}$.

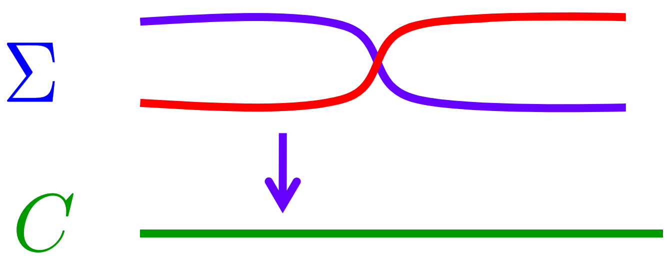
UV Curve

Seiberg-Witten Curve

$$\Sigma : \det(\lambda - \varphi(z,\bar{z})) = 0 \subset T^*C$$

$$\lambda = p\,dq \qquad \lambda|_{\Sigma}$$ SW differential

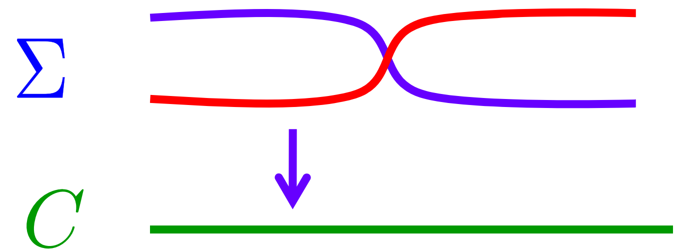
For $\mathfrak{g}$=su(K) $\quad \pi : \Sigma \to C$



is a K-fold branched cover

$$\lambda^K + \lambda^{K-2}\phi_2(z) + \cdots + \phi_K(z) = 0$$

Coulomb Branch & Charge Lattice



Coulomb branch $\mathcal{B} = \{u = (\phi_2, \dots, \phi_K)\}$

{ Meromorphic differential with prescribed singularities at z_a }

Local system of charges $\Gamma = H_1(\Sigma; \mathbb{Z})$

(Actually, Γ is a subquotient. Ignore that for this talk.)

BPS States: Geometrical Picture

Label the sheets of the covering $\Sigma \longrightarrow C$ by i,j,=1,…, K.

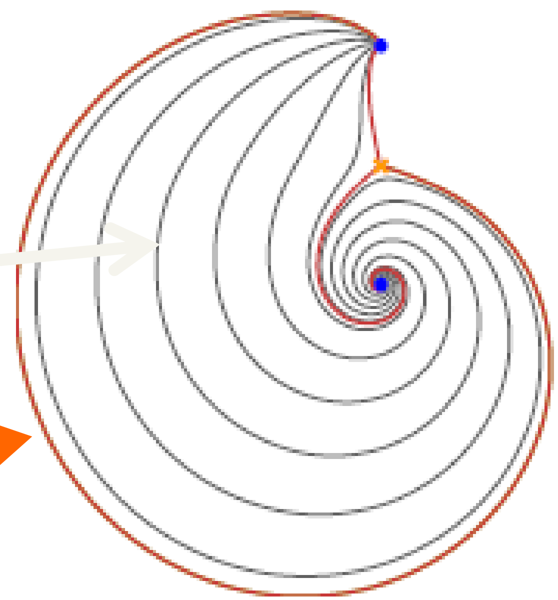
A ***WKB path*** of phase ϑ is an integral path on C

$$\langle \lambda_i - \lambda_j , \partial_t \rangle = e^{i\vartheta}$$

where i, j are two sheets of the covering.

Generic WKB paths have both ends on singular points z_a

Separating WKB paths begin on branch points, and for generic ϑ, end on singular points



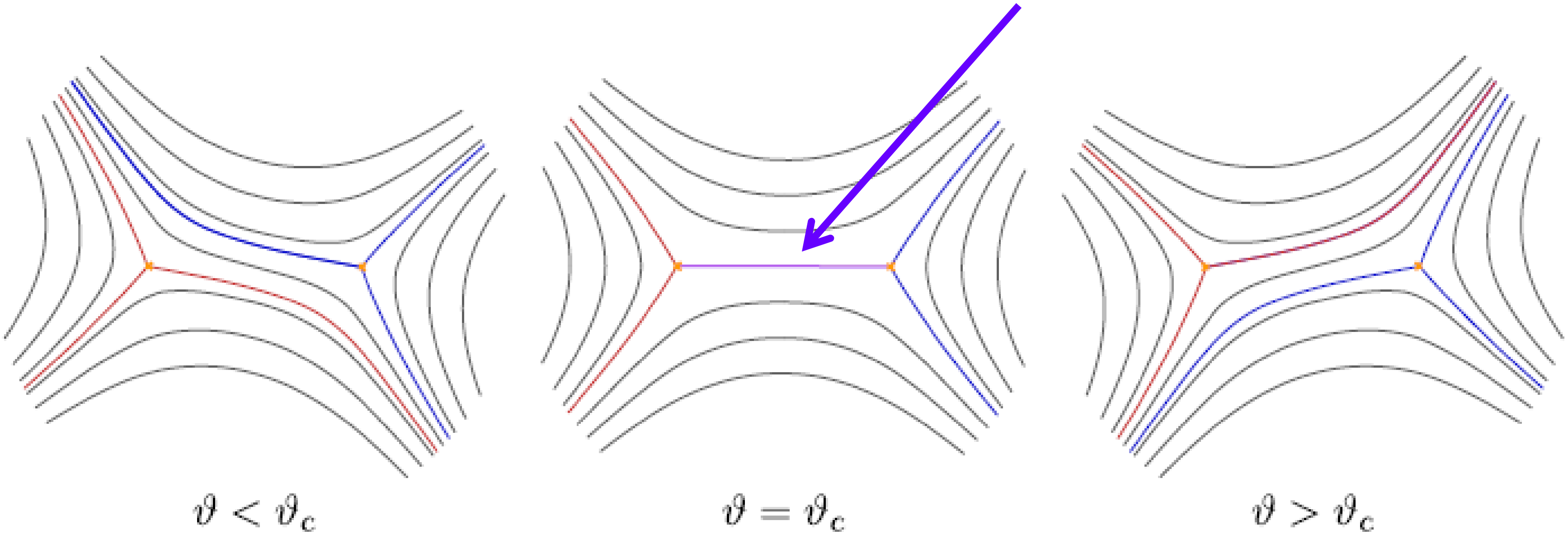
WKB paths generalize the trajectories of quadratic differentials, of importance in Teichmuller theory: (Thurston, Jenkins, Strebel,Zorich,....)

$$\lambda^2 + \phi_2 = 0$$

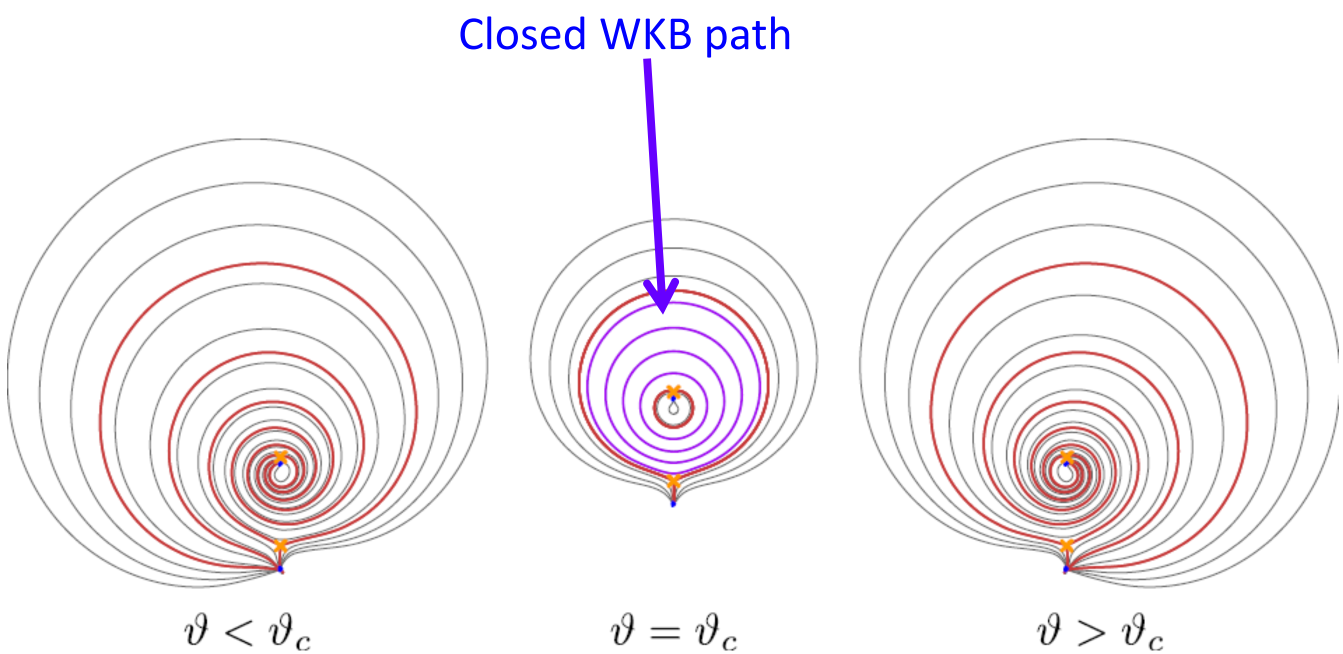
$$\lambda^K + \lambda^{K-2}\phi_2(z) + \cdots + \phi_K(z) = 0$$

String Webs – 1/4

But at critical values of $\vartheta=\vartheta_c$ ``string webs appear'':

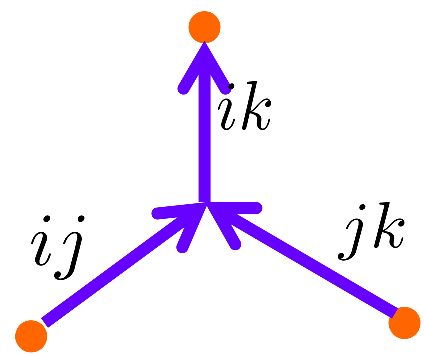


String Webs – 2/4



String Webs – 4/4

At higher rank, we get string *junctions* at critical values of ϑ:



A ``***string web***'' is a union of WKB paths with endpoints on branchpoints or such junctions.

These webs lift to closed cycles γ in Σ and represent BPS states with

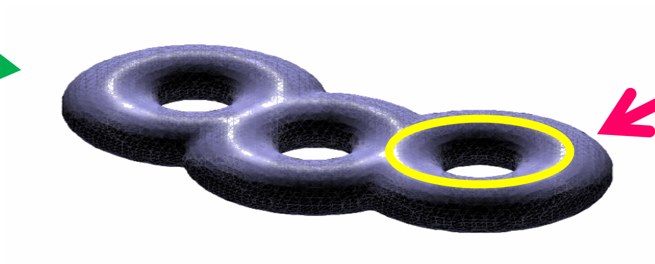
$$Z_\gamma = \oint_\gamma \lambda = e^{i\vartheta_c} |Z_\gamma|$$

Line defects in S[$\mathfrak{g}$,C,D]

6D theory S[$\mathfrak{g}$] has supersymmetric surface defects:

$$\sigma = \mathbb{R} \times \{\vec{0}\} \times \wp$$

C



$$L_{\wp,\zeta}$$

Line defect in 4d *labeled* by a ***closed*** path $\wp$.

Line Defect VEVs

$$\langle \mathrm{Tr} L_{\wp,\zeta} \rangle = \sum_\gamma \overline{\underline{\Omega}}(L_{\wp,\zeta},\gamma) \mathcal{Y}_\gamma = \mathrm{Tr} \mathcal{P} \exp \int_\wp \mathcal{A}$$

Example: SU(2) SYM Wilson line

$$L_\zeta = \exp \int_{\mathbb{R}_t \times \vec{0}} \left(\frac{\varphi}{2\zeta} + A + \frac{\zeta}{2} \overline{\varphi} \right)$$

Surprising nonperturbative correction

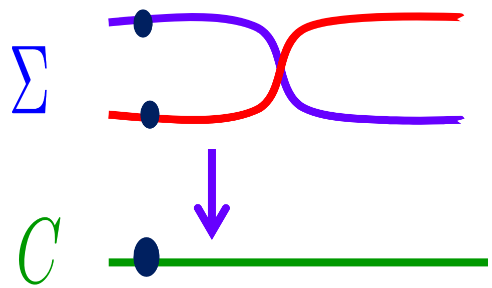
Large R limit gives expected terms

$$\langle \mathrm{Tr}_2 L_\zeta \rangle = \sqrt{\mathcal{Y}_{\gamma_e}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_e}}} + \sqrt{\mathcal{Y}_{\gamma_m + \gamma_e}}$$

Canonical Surface Defect in S[$\mathfrak{g}$,C,D]

For z $\in$ C we have a ***canonical surface defect*** $\mathbb{S}_z$

$$\sigma = \mathbb{R}^{1,1} \times \{\vec{0}\} \times \{z\}$$



This is a 2d-4d system. The QFT on the surface $\mathbb{S}_z$ is a d=2 susy theory whose massive vacua are naturally identified with the points on the SW curve covering z.

There are many exact results for $\mathbb{S}_z$. As an example we turn to spectral networks…

1 Introduction

2 Review: d=4, $\mathcal{N}$=2 field theory

3 Wall Crossing 101

4 Defects in Quantum Field Theory

5 Wall Crossing 102

6 3D Reduction & Hyperkähler geometry

7 Theories of Class S

8 Spectral Networks

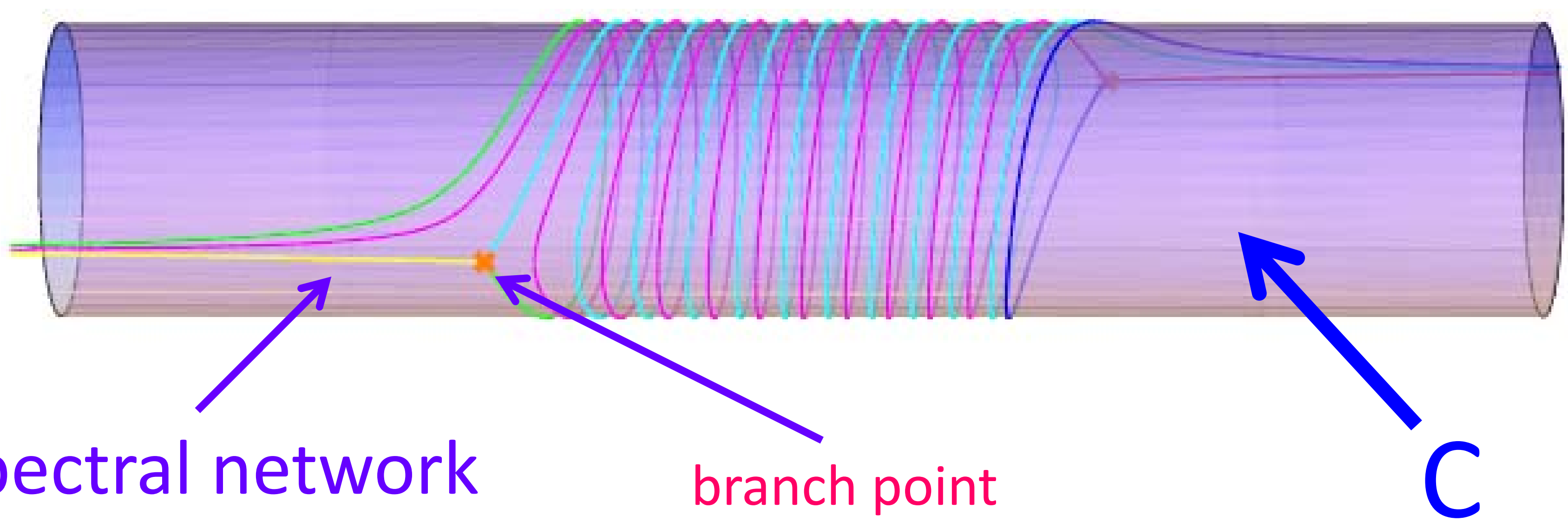
9 Conclusion

As we have emphasized, the WCF does not give us the BPS spectrum.

For theories of class S we can solve this problem – at least in principle – with the technique of spectral networks.

What are Spectral Networks ?

Spectral networks are combinatorial objects associated to a covering of Riemann surfaces $\Sigma \longrightarrow C$

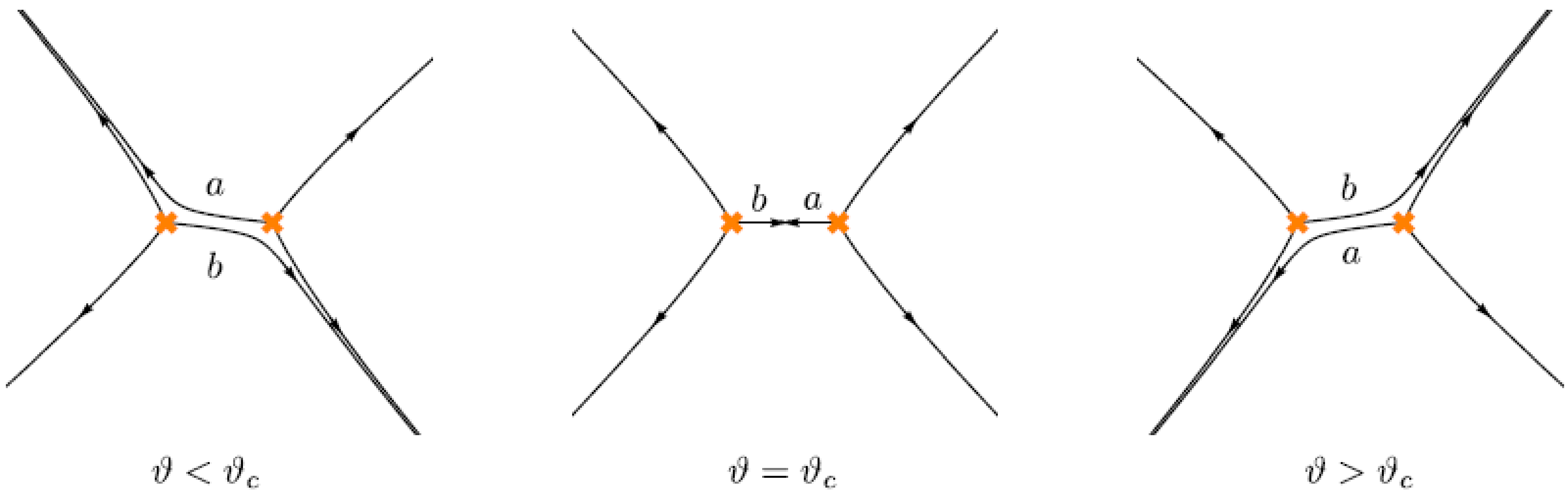


Spectral networks are defined by the physics of two-dimensional solitons on the surface defect $\mathbb{S}_z$

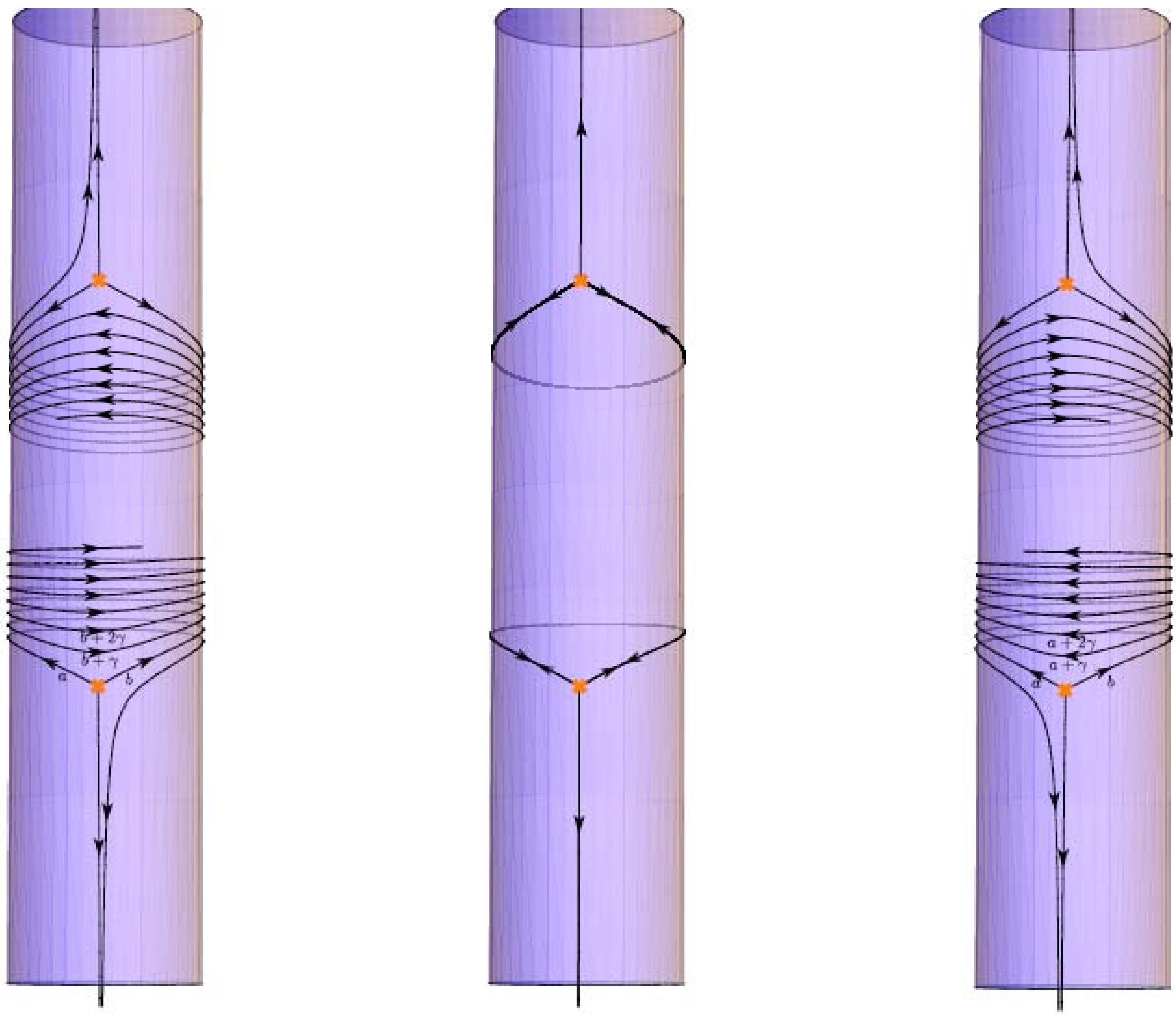
Paths in the network are constructed from WKB paths of phase ϑ according to known local rules

The combinatorial method for extracting the BPS spectrum in theories of class S is based on the behavior under variation of the phase ϑ

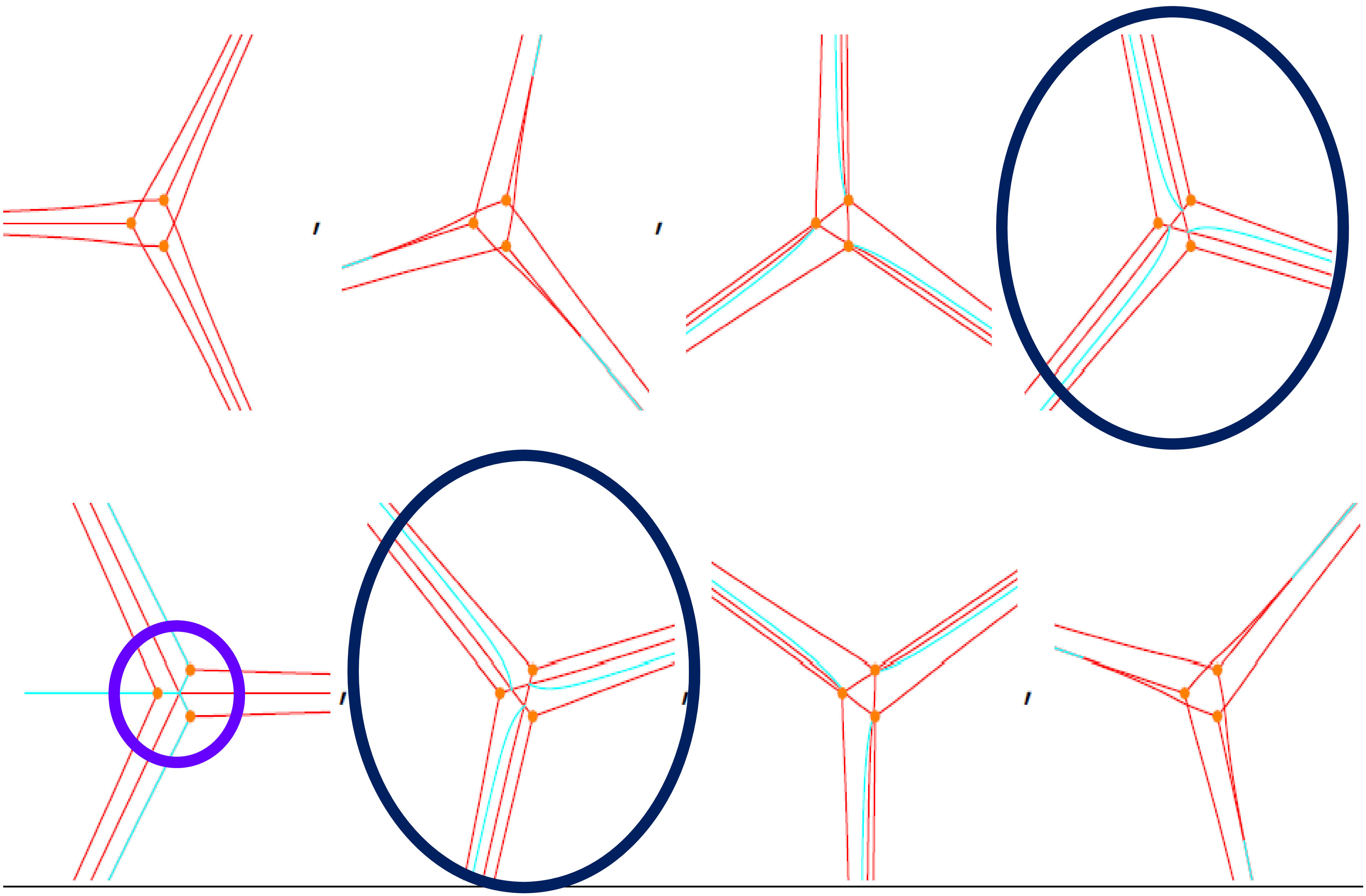
Movies:
http://www.ma.utexas.edu/users/neitzke/spectral-network-movies/



Movies: http://www.ma.utexas.edu/users/neitzke/spectral-network-movies/



$\vartheta < \vartheta_c$ $\qquad$ $\vartheta = \vartheta_c$ $\qquad$ $\vartheta > \vartheta_c$



Finding the BPS Spectrum

One can write very explicit formulae for the BPS degeneracies $\Omega(\gamma)$ in terms of the combinatorics of the change of the spectral network.

GMN, Spectral Networks, 1204.4824

Mathematical Applications of Spectral Networks

Spectral networks are the essential data to construct a symplectic ``nonabelianization map''

$$\Psi_{\mathcal{W}} : \mathcal{M}(\Sigma, GL(1); \mathfrak{m}) \to \mathcal{M}_F(C, GL(K); \mathfrak{m}).$$

$$\nabla^{\mathrm{abelian}} \to \nabla^{\mathrm{nonabelian}} = d + \mathcal{A}$$

They thereby construct a system of coordinates on moduli spaces of flat connections which generalize the cluster coordinates of Thurston, Penner, Fock, Fock and Goncharov.

Application to WKB Theory

The equation for the flat sections

$$\left(\frac{d}{dz} + \mathcal{A}\right)\Psi = 0$$

is an ODE generalizing the Schrodinger equation (K=2 cover)

The asymptotics for $\zeta \longrightarrow 0\,,\,\infty$ is a problem in WKB theory. K>2 is a nontrivial extension of the K=2 case.

The spectral network can be interpreted as the network of Stokes lines for the $\zeta \longrightarrow 0, \infty$ asymptotics of the differential equation.

1. Introduction
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3. Wall Crossing 101
4. Defects in Quantum Field Theory
5. Wall Crossing 102
6. 3D Reduction & Hyperkähler geometry
7. Theories of Class S
8. Spectral Networks
9. Conclusion

Conclusion: Main Results

1. A good, physical, understanding of wall crossing. Some understanding of the computation of the BPS spectrum, at least for class S.

2. A new construction of hyperkähler metrics and hyperholomorphic connections.

3. Nontrivial results on line and surface defects in theories of class S: Vev's and associated BPS states.

4. Theories of class S define a ``conformal field theory with values in d=4 $\mathcal{N}$=2 quantum field theories.''

1 2 3 S-Duality and the modular groupoid 4 5 6 7 8

9 10 11 12 13 14 AGT: Liouville & Toda theory

15 16 Higgs_branches

17 18 Cluster algebras

19 20 Holographic duals

21 N=4 scattering

22 23 Ω-backgrounds, Nekrasov partition functions, Pestun 24 localization.

25 26 $Z(S^3 \times S^1)$ Scfml indx

27 28 29 Nekrasov-Shatashvili: Quantum 30 Integrable systems 31

32 33 34 35 Three dimensions, Chern-Simons, and mirror symmetry

Rest rooms

Conclusion: Some Future Directions & Open Problems

1. Make the spectral network technique more effective. Spectrum Generator?

2. Geography problem: How extensive is the class S? Can we classify d=4 N=2 theories?

3. Can the method for producing HK metrics give an explicit nontrivial metric on K3 surfaces?

4. + many, many more.

Conclusion: 3 Main Messages

1. Seiberg and Witten's breakthrough in 1994, opened up many interesting problems. Some were quickly solved, but some remained stubbornly open.

But the past five years has witnessed a renaissance of the subject, with a much deeper understanding of the BPS spectrum and the line and surface defects in these theories.

Conclusions: Main Messages

2. This progress has involved nontrivial and surprising connections to other aspects of Physical Mathematics:

Hyperkahler geometry, cluster algebras, moduli spaces of flat connections, Hitchin systems, instantons, integrable systems, Teichműller theory, …

Conclusions: Main Messages

3. There are nontrivial superconformal fixed points in 6 dimensions.

(They were predicted many years ago from string theory.)

We have seen that the mere existence of these theories leads to a host of nontrivial results in quantum field theory.

Still, formulating 6-dimensional superconformal theories in a mathematically precise way remains an outstanding problem in Physical Mathematics.

A Central Unanswered Question

Can we construct $S[\mathfrak{g}]$?

MIND THE GAP


NOT
That's all Folks!

Some References

Denef and Moore, Split states, entropy enigmans, holes and halos, hep-th/0702146

Diaconescu and Moore, Crossing the wall: Branes vs. Bundles, hep-th/0702146

Gaiotto, Moore, & Neitzke:

Spectral Networks and Snakes, to appear

Spectral Networks, 1204.4824

Wall-crossing in Coupled 2d-4d Systems: 1103.2598

Framed BPS States: 1006.0146

Wall-crossing, Hitchin Systems, and the WKB Approximation: 0907.3987

Four-dimensional wall-crossing via three-dimensional field theory: 0807.4723

Andriyash, Denef, Jafferis & Moore, Wall-crossing from supersymmetric galaxies, 1008.0030

Kontsevich & Soibelman, Motivic Donaldson-Thomas Invariants: Summary of Results, 0910.4315

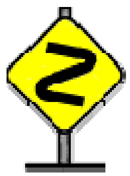
Pioline, Four ways across the wall, 1103.0261

Cecotti and Vafa, 0910.2615

Manschot, Pioline, & Sen, 1011.1258

Generalized Conformal Field Theory

Twisting $\Rightarrow$ $S[\mathfrak{g},C,D]$ only depends on the conformal structure of C.



For some C, D there are subtleties in the 4d limit.

``Conformal field theory valued in d=4 N=2 field theories''

(Moore & Tachikawa)

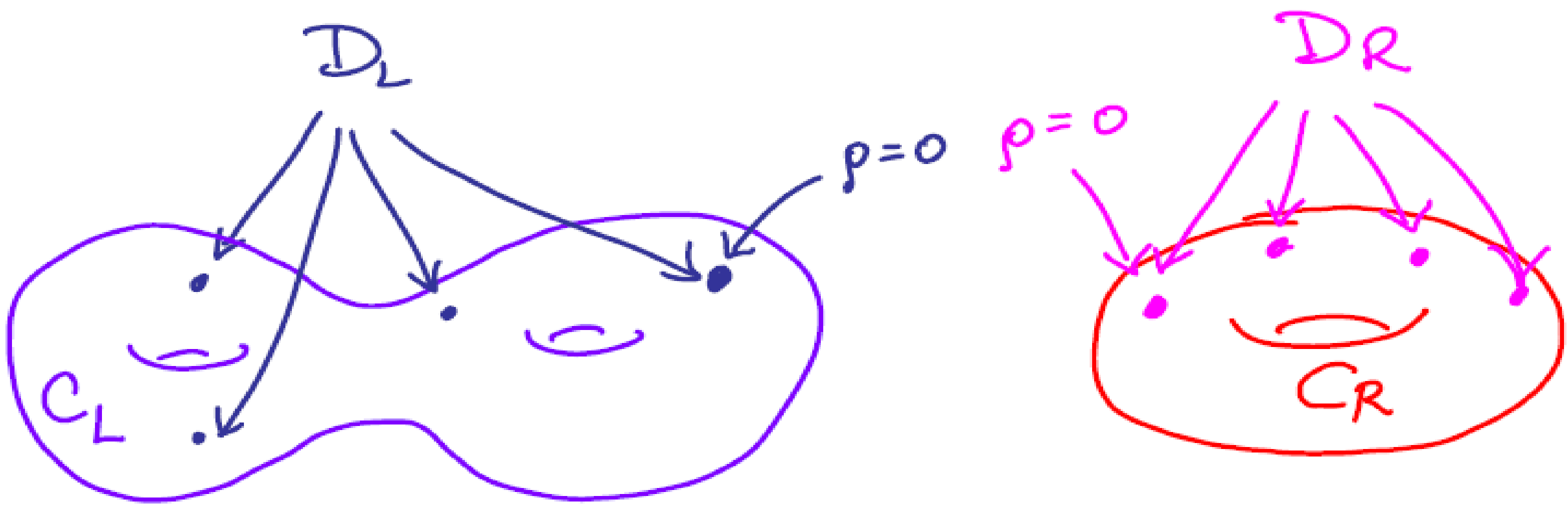
Space of coupling constants = $\mathcal{M}_{g,n}$

This is the essential fact behind the AGT conjecture, and other connections to 2d conformal field theory.

Gaiotto Gluing Conjecture -A

D. Gaiotto, ``N=2 Dualities''

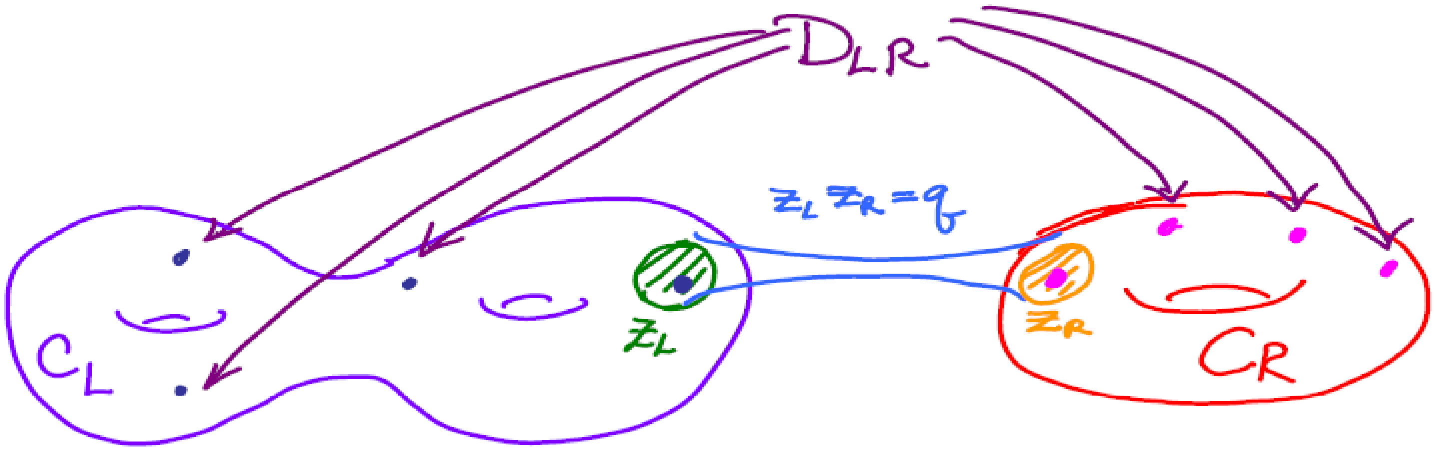
Slogan: Gauging = Gluing



Gauge the diagonal $G \subset G_L \times G_R$ symmetry with $q = e^{2\pi i\tau}$:

$$S[\mathfrak{g}, C_L, D_L] \times_{G,q} S[\mathfrak{g}, C_R, D_R]$$

Gaiotto Gluing Conjecture - B



Glued surface: $z_L z_R = q$ $\Rightarrow$ $C_L \times_q C_R$

$$S[\mathfrak{g}, C_L \times_q C_R, D_{LR}] = S_L \times_{G,q} S_R$$

Nevertheless, there are situations where one gauges just a subgroup – the physics here could be better understood. (Gaiotto, Moore, Tachikawa)