PHYSICS 385 SOMALWAR

ELECTROMAGNETISM

EXAM I ①

SOLUTIONS

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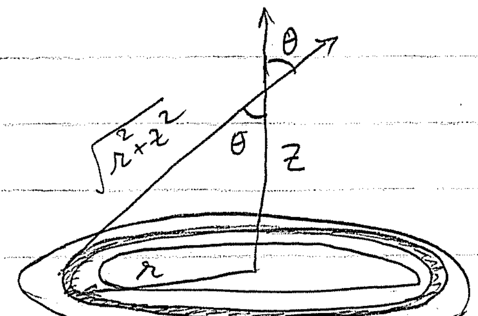

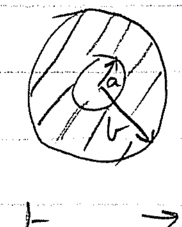

$$\cos\theta = \frac{z}{\sqrt{r^2+z^2}}$$

By Symmetry, $\vec{E}$ will be along z axis

Consider a ring of charge at radius r $(a<r<b)$ and width dr. Hence, the total charge in the thin ring is $dq = \sigma\, 2\pi r\, dr$

The field due to the ring follows from Coulomb's law and the angle factor $\cos\theta$ since the field from little elements along the ring all point along angle θ from the z axis

$$dE_z = \frac{(\sigma\, 2\pi r\, dr)}{4\pi\varepsilon_0 (r^2+z^2)} \cos\theta$$

$$= \frac{\sigma}{2\varepsilon_0} \cdot \frac{r\,dr}{r^2+z^2} \cdot \frac{z}{\sqrt{r^2+z^2}} = \frac{\sigma z}{2\varepsilon_0} \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

$$\therefore E_z = \frac{\sigma z}{2\varepsilon_0} \int_{r=a}^{r=b} \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

use $t = r^2+z^2$

$\therefore dt = 2r\,dr$

$$= \frac{\sigma z}{2\varepsilon_0} \cdot \frac{1}{2} \cdot \int_{a^2+z^2}^{b^2+z^2} \frac{dt}{t^{3/2}} = \frac{\sigma z}{2\varepsilon_0} \left[\frac{-1}{\sqrt{t}}\right]_{a^2+z^2}^{b^2+z^2}$$

$$\therefore \quad E_z = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]$$

Limiting Cases:

(i) $a \to 0$, $b \to \infty$ $\quad E_z \to \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{z} - 0\right] = \frac{\sigma}{2\varepsilon_0}$

This is precisely the field of an infinite charged plane

(ii) $z \to \infty$ $\quad \frac{1}{\sqrt{a^2+z^2}} = \frac{1}{z\sqrt{1+\frac{a^2}{z^2}}} = \frac{\left(1+\frac{a^2}{z^2}\right)^{-\frac{1}{2}}}{z}$

$$\approx \frac{1-\frac{a^2}{2z^2}}{z}$$

& similarly $\frac{1}{\sqrt{b^2+z^2}} \approx \frac{1-\frac{b^2}{2z^2}}{z}$

$$\therefore \quad E_z \longrightarrow \frac{\sigma z}{2\varepsilon_0}\left[\frac{1-a^2/2z^2}{z} - \frac{1-b^2/2z^2}{z}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\,\frac{(b^2-a^2)}{2z^2}$$

but area $= \pi(b^2-a^2)$ $\quad \therefore \; q = \sigma\pi(b^2-a^2)$

$$\therefore \quad E_z \longrightarrow \frac{q/\pi}{2\varepsilon_0 \cdot 2z^2} = \frac{q}{4\pi\varepsilon_0 z^2}$$

i.e. a point charge field.

(The annulus looks like a point charge (from very far)

(2) Using a cylindrical Gaussian surface and symmetry (as in class)

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$$

$$\therefore \quad V(r) = -\int_{r_0}^{r} \frac{\lambda}{2\pi\varepsilon_0 r}\, dr = \frac{-\lambda}{2\pi\varepsilon_0} \ln(r/r_0)$$

$$-\frac{\partial V}{\partial r} = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{r/r_0} \cdot \frac{1}{r_0} = \frac{\lambda}{2\pi\varepsilon_0 r}$$

(3) The E field is non-zero only for $a < r < b$ (Consider three Gaussian spheres: $r < a$, $a < r < b$, $r > b$, only $a < r < b$ encloses nonzero charge)

The field is simply Coulomb field $E_r = \frac{q}{4\pi\varepsilon_0 r^2}$

$$\therefore \quad W = \frac{\varepsilon_0}{2} \int_{\substack{\text{all} \\ \text{space}}} E^2 d\tau = \frac{\varepsilon_0}{2} \int_a^b \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{\varepsilon_0}{2} \cdot \frac{q^2}{(4\pi\varepsilon_0)^2} 4\pi \int_a^b \frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

If $b \to \infty$, $W = \frac{q^2}{8\pi\varepsilon_0 a}$ is simply the energy of assembly for a spherical shell carrying charge q. If $a = b$, $W = 0$ since the charges neutralize. If $a \to 0$, $W \to \infty$ since one is trying to assemble a point charge, which has infinite self-energy.