Rutgers Physics 382 Mechanics II (Spring'17/Gershtein)

Final Exam – May 5, 2017

This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please attach the sheet to your solutions. Calculators are not needed. Exam duration – 3 hours.

PART A (16 pts total)
Solve all problems.

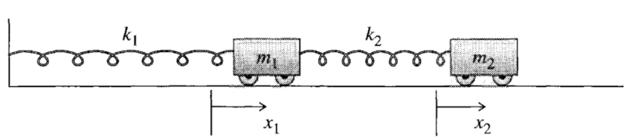
A1. A tensor of inertia for a body is $\overleftrightarrow{I} = I_0 \cdot \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Write the coordinates of a unit vector along **<u>one</u>** of the principle axes and the moment of inertia and the value of the principle moment along it. *(hint: there is no need for lengthy calculations)*

A2. The differential cross section of a scattering process is $d\sigma/d\Omega = 10^{-28}\ m^2/sr$. Target material has number density $n=10^{27}\ m^{-3}$, and the target is $t=10\ \mu m$ thick. Find the probability to scatter into a detector of area $A=1\ mm^2$ which is located distance d=1 cm away from the target.

A3. Two spaceships, A and B approach each other head on. In some inertial coordinate system their speeds are measured to be both equal 0.5c. What is the speed of spaceship B as observed from A? Is it different from the speed of spaceship A as observed from B?

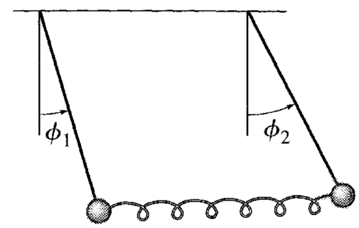
A4. Write down (but do not solve) the Hamilton's equations for the system shown below, using generalized coordinates x_1 and x_2 ($x_1=x_2=0$ when springs are relaxed).



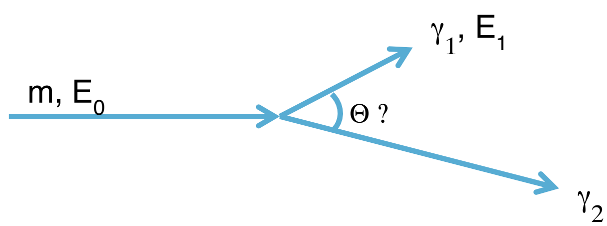
PART B (24 points total)
Solve 3 problems

B1. Two pendulums of equal masses *M* and lengths *L* are connected by a spring with coefficient *k* (see sketch below). The length of the relaxed spring is equal to the distance between the pendulums' supports, so that the spring is relaxed when $\phi_1=\phi_2=0$.

Find the normal frequencies for <u>small</u> oscillations of the system, and describe the motion of normal modes.



B2. A particle of known mass *m* has energy E_0 in the lab frame. It decays into two photons, and the energy of one of the photons was measured to be E_1 (also in the lab frame). What is the opening angle between the photons in the lab frame?



B3. A 4-speed of a particle in inertial frame S is $u=(u_0,u_x,u_y,u_z)=(1.25c,\ 0.75c,\ 0,\ 0)$. System S' moves along the x axis with the speed v=0.6c towards positive x. Find the components of the 4-speed u' of the particle in frame S'

B4. The displacement matrix of an elastic body is given below:

$$\overleftrightarrow{D} = \frac{\partial u_i}{\partial r_j} = \begin{bmatrix} 0.02 & -0.03 & 0.02 \\ 0.03 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

a) separate $\overleftrightarrow{D}$ into rotation and strain $\overleftrightarrow{E}$
b) what is the rotation angle?
c) what is the change of volume of the body?

A1

$I_{xz} = I_{yz} = 0$, so z is a principal axis!

$$\overleftrightarrow{I} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = I_0 \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = I_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ $\rightarrow$ corresponding principle moment is I_0

A2

$\frac{d\sigma}{d\Omega} = 10^{-28}\ m^2/sr$, target number density $n = 10^{27}\ m^{-3}$, thickness $\ell = 10\ \mu m$

detector of $A = 1\ mm^2$ @ $D = 1\ cm$ from target

$$P = \frac{d\sigma}{d\Omega} \cdot \frac{A}{D^2} \cdot n \cdot \ell = 10^{-28} \cdot \frac{10^{-6}}{10^{-4}}\, 10^{27} \cdot 10^{-5} = 10^{-8}$$

A3

$v = 0.5 \longrightarrow$ $\qquad$ $\longleftarrow v = 0.5$

$$v_{B\,\text{from}\,A} = \frac{0.5 + 0.5}{1 + 0.25} = \frac{1}{1.25} = 0.8\,c$$

$\rightarrow$ same as $v_{A\,\text{from}\,B}$

A4

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad p_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 \qquad p_2 = \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m \dot{x}_2$$

$$\mathcal{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$\frac{\partial \mathcal{H}}{\partial x_1} = k_1 x_1 - k_2 (x_2 - x_1)$$

$$\frac{\partial \mathcal{H}}{\partial x_2} = k_2 (x_2 - x_1)$$

$$\Rightarrow \boxed{\begin{aligned} m_1 \ddot{x}_1 &= -(k_1 + k_2) x_1 + k_2 x_2 \\ m_2 \ddot{x}_2 &= k_2 x_1 - k_2 x_2 \end{aligned}}$$

B1 $T = \frac{1}{2} M \cdot L^2 \dot{\varphi}_1^2 + \frac{1}{2} M \cdot L^2 \dot{\varphi}_2^2$

$$U \approx \frac{1}{2} M g L \varphi_1^2 + \frac{1}{2} M g L \varphi_2^2 + \frac{1}{2} k L^2 (\varphi_1 - \varphi_2)^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = M L^2 \dot{\varphi}_1 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = M L^2 \dot{\varphi}_2$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} = -M g L \varphi_1 - k L^2 (\varphi_1 - \varphi_2) = M L^2 \ddot{\varphi}_1$$

$$\frac{g}{L} = \omega_1^2$$

$$\frac{k}{M} = \omega_2^2$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_2} = - M g L \varphi_2 + k L^2 (\varphi_1 - \varphi_2) = M L^2 \ddot{\varphi}_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{pmatrix} = - \begin{pmatrix} \omega_1^2 + \omega_2^2 & -\omega_2^2 \\ -\omega_2^2 & \omega_1^2 + \omega_2^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\det\begin{pmatrix} \omega_1^2+\omega_2^2-\omega^2 & -\omega_2^2 \\ -\omega_2^2 & \omega_1^2+\omega_2^2-\omega^2 \end{pmatrix} = 0$$

$$(\omega_1^2+\omega_2^2-\omega^2)^2 - \omega_2^4 = 0$$

$$(\omega_1^2+\omega_2^2-\omega^2-\omega_2^2)(\omega_1^2+\omega_2^2-\omega^2+\omega_2^2) = 0$$

$$\omega_A^2 = \omega_1^2 \rightarrow \omega_2^2 x - \omega_2^2 y = 0 \rightarrow \vec{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ pendula in phase, spring is not stretched

$$\omega_B^2 = \omega_1^2 + 2\omega_2^2 \rightarrow -\omega_2^2 x - \omega_2^2 y = 0 \rightarrow \vec{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ pendula in counter-phase

B2

$$\underline{P}_0^2 = \left(\underline{P}_{\gamma_1} + \underline{P}_{\gamma_2}\right)^2 = 2\,\underline{P}_{\gamma_1}\cdot\underline{P}_{\gamma_2}$$

$$\underline{P}_0^2 = m^2c^2 \qquad \underline{P}_{\gamma_1} = \left(\frac{E_1}{c}, \vec{p}_1\right) \qquad |\vec{p}_1| = \frac{E_1}{c}$$

$$\underline{P}_{\gamma_2} = \left(\frac{E_2}{c}, \vec{p}_2\right) \qquad |\vec{p}_2| = \frac{E_2}{c} = \frac{E_0 - E_1}{c}$$

$$m^2c^2 = 2\,\frac{E_1(E_0-E_1)}{c^2}(1-\cos\theta)$$

$$1-\cos\theta = \frac{m^2c^4}{2E_1(E_0-E_1)}$$

$$\boxed{\theta = \arccos\left(1 - \frac{m^2c^4}{2E_1(E_0-E_1)}\right)}$$

B3

4-speed of a particle in frame K is

$(1.25c, 0.75c, 0, 0)$

K' moves along x with speed $v = 0.6c$

$$u_0' = \gamma\left(u_0 - \frac{v}{c}u_x\right)$$

$$u_x' = \gamma\left(u_x - \frac{v}{c}u_0\right)$$

$$u_0' = 1.25(1.25 - 0.6\cdot 0.75)c =$$

$$= \frac{5}{4}\left(\frac{5}{4} - \frac{3}{5}\cdot\frac{3}{4}\right)c =$$

$$= \frac{\cancel{5}}{4}c\,\frac{25-9}{\cancel{5}\cdot 4} = \frac{16}{4\cdot 4} = 1\cdot c$$

$$u_1' = 1.25(0.75 - 0.6\cdot 1.25)c = 0$$

$$\frac{c^2}{1-\frac{v^2}{c^2}} - \frac{v^2}{1-\frac{v^2}{c^2}} = c^2$$

$v = 0.6\cdot c$

$\frac{v^2}{c^2} = 0.36 \quad \sqrt{1-\frac{v^2}{c^2}} = \sqrt{0.64} = 0.8$

$\gamma = 1.25$

$(1.25, 0.75, 0, 0)$

$$\boxed{u_i' = (c, 0, 0, 0) \leftarrow \text{at rest!}}$$

(B4)

$$\overleftrightarrow{D} = \begin{bmatrix} 0.02 & 0 & 0.01 \\ 0 & 0.01 & 0 \\ 0.01 & 0 & 0.01 \end{bmatrix} + \begin{bmatrix} 0 & -0.03 & 0.01 \\ 0.03 & 0 & 0 \\ -0.01 & 0 & 0 \end{bmatrix}$$

$\overleftrightarrow{E}$ (strain)

rotation, $\theta_z = -0.03$ $\theta_y = -0.01$

total rotation: $\theta = 0.01 \cdot \sqrt{9+1} =$

$$\boxed{\theta = \frac{1}{\sqrt{10}}}$$

Change in volume:

$$\boxed{\frac{\Delta V}{V} = 0.02 + 0.01 + 0.01 = 0.04}$$