Rutgers Physics 382 Mechanics II (Spring'16/Gershtein)

Class Exam - March 24, 2016

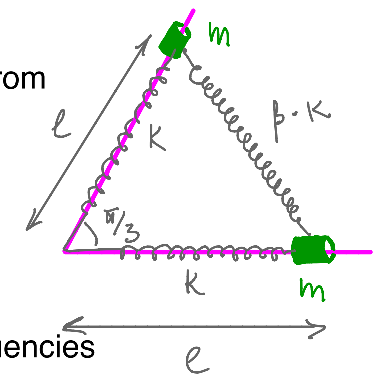
This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please attach the sheet to your solutions. Calculators are not needed. Exam duration - 1hr 20min. Solve all problems (30 points total)

1. Two masses ***m*** can slide without friction on two rods, connected as shown in the figure below, with opening angle $\pi/3$ between them. The masses are attached to the point where the rods are joined with springs (both of length ***l*** and coefficient ***k***).The masses are also connected to each other with the third spring, of the same length ***l*** and different constant, $\boldsymbol{k_1 = \beta \cdot k}$

   a) (3 pts) prove that for small deviations of the masses from equilibrium positions (***x, y << l***) the distance between the masses $\boldsymbol{a = l + (x+y)/2}$

   b) (5 pts) write down Hamiltonian equations for the system, assuming small deviations from equilibrium

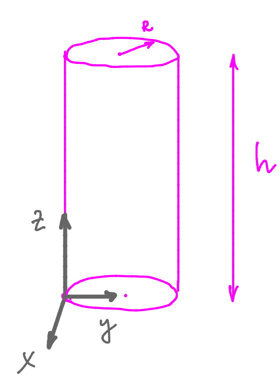
   c) (7 pts) Find and describe the normal modes and frequencies



2. **(5 pts)** Differential cross section of a scattering process is given by $\frac{d\sigma}{d\Omega} = A^2 \cdot (\sin\varphi)^2$. Calculate the total cross section.

3. **(5 pts)** A thin tube has radius R, mass M, and height h. Origin is on the rim, with z axis parallel to the cylinder and y axis pointing towards the cylinder axis (see figure)

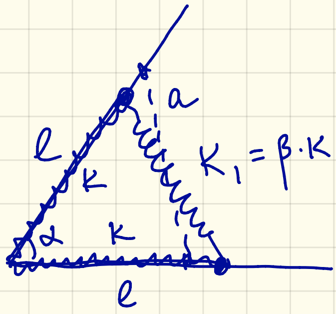
   Find all non-diagonal elements of the inertia tensor (i.e. products of inertia)



4. **(5 pts)** Euler equations with zero torque can be written as $\dot{\vec{L}} + \vec{\omega}\times\vec{L} = 0$. Prove that the magnitude of L is constant.

(1)

a)



$$a^2 = (l+x)^2 + (l+y)^2 - 2(l+x)(l+y)\cdot\frac{1}{2} =$$

$$= l^2 + 2lx + l^2 + 2ly - l^2 - lx - ly =$$

$$= l^2 + l(x+y)$$ (neglect x^2, y^2 terms)

$$a = l\sqrt{1+\frac{x+y}{l}} = l\left(1+\frac{x+y}{2l}\right) = l + \frac{x+y}{2}$$

b) $$U = \frac{1}{2}Kx^2 + \frac{1}{2}Ky^2 + \frac{1}{2}K_1\left(\frac{x+y}{2}\right)^2$$

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}Kx^2 + \frac{1}{2}Ky^2 + \frac{1}{8}K_1(x+y)^2$$

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} k x^2 + \frac{1}{2} k y^2 + \frac{1}{8} k_1 (x+y)^2$$

$$\frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} = \dot{x} \qquad \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} = \dot{y}$$

$$\frac{\partial \mathcal{H}}{\partial x} = kx + \frac{1}{4} \beta k (x+y) = xk \left(1 + \frac{1}{4}\beta\right) + \frac{1}{4}\beta k \cdot y = -\dot{p}_x = -m\ddot{x}$$

$$\frac{\partial \mathcal{H}}{\partial y} = ky + \frac{1}{4} \beta k (x+y) = \frac{1}{4}\beta k \cdot x + ky \left(1 + \frac{1}{4}\beta\right) = -\dot{p}_y = -m\ddot{y}$$

$$\begin{cases} m\ddot{x} = -kx\left(1 + \beta/4\right) - ky \cdot \beta/4 \\ m\ddot{y} = -kx\, \beta/4 - ky\left(1 + \beta/4\right) \end{cases}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ddot{Q} = -\omega_0^2 \begin{pmatrix} 1+\beta/4 & \beta/4 \\ \beta/4 & 1+\beta/4 \end{pmatrix}, \quad Q = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \omega_0^2 = \frac{k}{m}$$

c) $M\ddot{Q} = -KQ \longrightarrow \det(K - \omega^2 M) = 0 \qquad \omega_0^2 = \frac{K}{m}$

$$\begin{vmatrix} (1+\beta/4)\omega_0^2 - \omega^2 & \frac{\beta}{4}\omega_0^2 \\ \frac{\beta}{4}\omega_0^2 & (1+\frac{\beta}{4})\omega_0^2 - \omega^2 \end{vmatrix} = 0 = \left[(1+\frac{\beta}{4})\omega_0^2 - \omega^2\right]^2 - \left[\frac{\beta}{4}\omega_0^2\right]^2 =$$

$$= \left[(1+\frac{\beta}{4})\omega_0^2 - \omega^2 - \beta/4\,\omega_0^2\right]\left[(1+\frac{\beta}{4})\omega_0^2 - \omega^2 + \beta/4\,\omega_0^2\right] =$$

$$= (\omega_0^2 - \omega^2)\left((1+\frac{\beta}{2})\omega_0^2 - \omega^2\right) = 0$$

$\omega_1 = \omega_0$

$\omega_2 = \omega_0\sqrt{1+\frac{\beta}{2}}$

$\omega_1^2 = \omega_0^2$:

$$\omega_0^2 \frac{\beta}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \quad \Rightarrow \quad x_1 = -y_1$$

→ opposite phases, the third spring is <u>irrelevant</u>

$$\omega_2^2 = \omega_0^2 \left(1 + \frac{\beta}{2}\right)$$

$$\omega_0^2 \frac{\beta}{4} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0 \quad \Rightarrow \quad x_2 = y_2$$

→ same phases.

(2)

$$d\sigma = A^2 \sin^2\varphi \, d\Omega$$

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$\sin^2\varphi = \frac{1}{2} - \cos 2\varphi$$

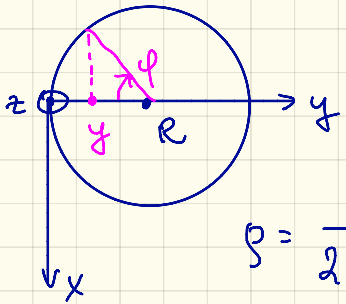
$$\sigma = A^2 \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, \sin^2\varphi \, d\varphi =$$

$$= A^2 \cdot 2 \cdot \int_0^{2\pi} \sin^2\varphi \, d\varphi = A^2 \cdot 2 \cdot 2\pi \cdot \frac{1}{2} = 2\pi A^2$$

(3)

$I_{xy} = 0 \leftarrow$ symmetry

$I_{xz} = 0 \leftarrow$ symmetry



$$\rho = \frac{m}{2\pi R \cdot h}$$

$$I_{yz} = -\int dm \cdot y \cdot z = -\int_0^h \int_0^{2\pi} \rho R(1-\cos\varphi)\, z \cdot R\, d\varphi\, dz$$

$$y = R(1-\cos\varphi)$$

$$dm = \rho \cdot R\, d\varphi \cdot dz$$

$$= -\rho R^2 \frac{h^2}{2} \int_0^{2\pi} (1-\cos\varphi)\, d\varphi = \boxed{-\frac{1}{2} m R h = I_{yz}}$$

Another way to solve (suggested by one of the students)

at the cylinder center, all non-diagonals are zero

Generalized parallel axis theorem (problem **10.24**)

$$I_{yz} = I_{yz}^{CM} - m \cdot \frac{h}{2} \cdot R = -\frac{1}{2} m R h$$, same as above.

(4) $\dot{\vec{L}} + \vec{\omega} \times \vec{L} = 0$

multiply by $\vec{L}$

$$\vec{L}\dot{\vec{L}} + \vec{L} \cdot (\vec{\omega} \times \vec{L}) = 0$$

$\hookrightarrow = 0$, since $\vec{\omega} \times \vec{L}$ is $\perp$ $\vec{L}$

$$\vec{L}\dot{\vec{L}} = \frac{1}{2}\frac{d}{dt}(\vec{L} \cdot \vec{L}) = 0$$

$$\vec{L} \cdot \vec{L} = \text{const.}$$

$$|\vec{L}| = \text{const.}$$