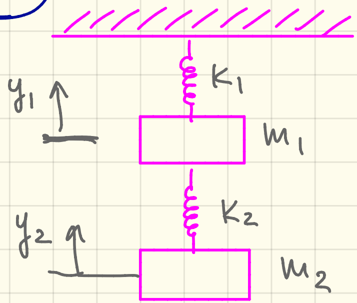
11.2



$$\begin{cases} m_2 \ddot{y}_2 = -k_2(y_2 - y_1) - m_2 g \\ m_1 \ddot{y}_1 = k_2(y_2 - y_1) - k_1 y_1 - m_1 g \end{cases}$$

y_1, y_2: relaxed spring positions.

$$\begin{cases} m_1 \ddot{y}_1 = -(k_1+k_2) y_1 + k_2 y_2 - m_1 g \\ m_2 \ddot{y}_2 = k_2 y_1 - k_2 y_2 - m_2 g \end{cases}$$

These would be absorbed into equilibrium positions.

$z_1 = y_1 - y_{10}, \quad z_2 = y_2 - y_{20}$

$$\left.\begin{aligned} k_2 y_{10} - k_2 y_{20} &= -m_2 g \\ -(k_1+k_2) y_{10} + k_2 y_{20} &= -m_1 g \end{aligned}\right\} (+): \; -k_1 y_{10} = -(m_1+m_2) g$$

$$y_{10} = \frac{m_1+m_2}{k_1} g$$

$$y_{20} = y_{10} + \frac{m_2 g}{k_2} = \frac{m_1+m_2}{k_1} g + \frac{m_2 g}{k_2}$$

$$-(k_1+k_2) y_{10} + k_2 y_{20} = -(m_1 + \cancel{m_2}) g - \cancel{\frac{k_2}{k_1}(m_1+m_2) g} + \cancel{m_2 g} + \cancel{\frac{k_2}{k_1}(m_1+m_2)} g$$

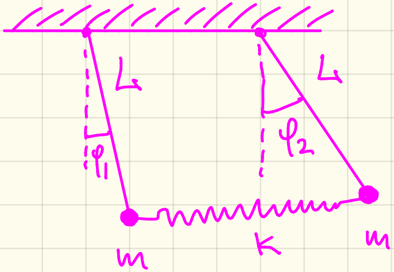
$$k_2 y_{10} - k_2 y_{20} = \cancel{\frac{k_2}{k_1}(m_1+m_2)} g - m_2 g - \cancel{\frac{k_2}{k_1}(m_1+m_2)} g$$

$$\begin{cases} m_1 \ddot{z}_1 = -(k_1+k_2) z_1 + k_2 z_2 \\ m_2 \ddot{z}_2 = k_2 z_1 - k_2 z_2 \end{cases}$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

$$M \ddot{z} = - K z$$

11.14



$\varphi_1, \varphi_2 \ll 1$

$$T = \frac{1}{2} m (L \dot{\varphi}_1)^2 + \frac{1}{2} m (L \dot{\varphi}_2)^2$$

$$U = \frac{1}{2} k \left[L(\varphi_1 - \varphi_2) \right]^2 - mgL\cos\varphi_1 - mgL\cos\varphi_2 \approx$$

$$\approx \frac{1}{2} k L^2 \left(\varphi_1^2 + \varphi_2^2 - 2\varphi_1\varphi_2 \right) + \frac{1}{2} mgL \left(\varphi_1^2 + \varphi_2^2 \right)$$

$$-\frac{\partial \mathcal{L}}{\partial \varphi_1} = kL^2\varphi_1 + mgL\varphi_1 - kL^2\varphi_2 \; ; \; -\frac{\partial \mathcal{L}}{\partial \varphi_2} = kL^2\varphi_2 - kL^2\varphi_1 + mgL\varphi_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = mL^2\dot{\varphi}_1 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = mL^2\dot{\varphi}_2$$

$$mL^2\ddot{\varphi}_1 = -(kL^2 + mgL)\varphi_1 + kL^2\varphi_2$$

$$mL^2\ddot{\varphi}_2 = +kL^2\varphi_1 - (kL^2 + mgL)\varphi_2$$

$$\ddot{\varphi} = -\begin{pmatrix} \frac{k}{m} + \frac{g}{L} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} + \frac{g}{L} \end{pmatrix}\varphi$$

$$\frac{k}{m} = \omega_2^2$$

$$\frac{g}{L} = \omega_1^2$$

$$0 = \begin{vmatrix} \omega_1^2+\omega_2^2-\Omega^2 & -\omega_2^2 \\ -\omega_2^2 & \omega_1^2+\omega_2^2-\Omega^2 \end{vmatrix} = (\omega_1^2+\omega_2^2-\Omega^2)^2 - \omega_2^4$$

$$\omega_1^2+\omega_2^2-\Omega^2 = \pm\,\omega_2^2 \longrightarrow \begin{cases} \Omega_1^2 = \omega_1^2 \\ \Omega_2^2 = \omega_1^2 + 2\omega_2^2 \end{cases}$$

$$\Omega_1^2 = \omega_1^2 : \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0 \qquad \varphi_1 = \varphi_2$$

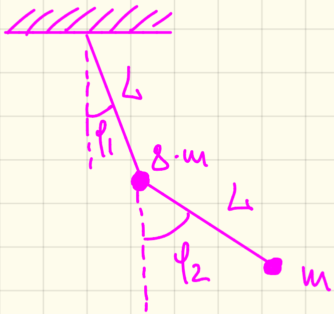
↳ two pendulums swing in unison,
spring is always relaxed

$$\Omega_1^2 = \omega_1^2 + 2\omega_2^2 : \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0 \qquad \varphi_1 = -\varphi_2$$

↳ two pendulums swing in counterphase
with same amplitude

→ very similar to two block problem we did
in class.

(11.17)



For small oscillations:

$$T = \frac{1}{2} 8m \cdot L^2 \dot{\varphi}_1^2 + \frac{1}{2} m \left(L\dot{\varphi}_1 + L\dot{\varphi}_2 \right)^2$$

$$U = 8m \cdot gL \, \frac{1}{2} \varphi_1^2 + mg \left(\frac{1}{2} L \varphi_1^2 + \frac{1}{2} L \varphi_2^2 \right)$$

$$\mathcal{L} = \frac{9}{2} m L^2 \dot{\varphi}_1^2 + \frac{1}{2} m L^2 \dot{\varphi}_2^2 + m L^2 \dot{\varphi}_1 \dot{\varphi}_2 - \frac{9}{2} mgL \varphi_1^2 - \frac{1}{2} mgL \varphi_2^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = 9 m L^2 \dot{\varphi}_1 + m L^2 \dot{\varphi}_2 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = m L^2 \dot{\varphi}_2 + m L^2 \dot{\varphi}_1$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} = -9 mgL \varphi_1 \qquad \frac{\partial \mathcal{L}}{\partial \varphi_2} = -mgL \varphi_2$$

a)

$$9 \ddot{\varphi}_1 + \ddot{\varphi}_2 = -9 \frac{g}{L} \varphi_1$$

$$\ddot{\varphi}_1 + \ddot{\varphi}_2 = -\frac{g}{L} \varphi_2$$

$$\boxed{\underbrace{\begin{pmatrix} 9 & 1 \\ 1 & 1 \end{pmatrix}}_{= M} \ddot{\varphi} = -\underbrace{\omega_0^2}_{\omega_0^2 = \frac{g}{L}} \underbrace{\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}}_{= K} \varphi}$$

$$0 = \begin{vmatrix} 9\omega_0^2 - 9\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 9\left(\omega^2 - \omega_0^2\right)^2 - \omega^4 =$$

$$= \left[3\left(\omega^2 - \omega_0^2\right) - \omega^2\right]\left[3\left(\omega^2 - \omega_0^2\right) + \omega^2\right] = \left(2\omega^2 - 3\omega_0^2\right)\left(4\omega^2 - 3\omega_0^2\right)$$

$$\boxed{\omega_1 = \sqrt{\frac{3}{2}}\,\omega_0 = \sqrt{\frac{3g}{2L}}} \qquad \boxed{\omega_2 = \sqrt{\frac{3}{4}\frac{g}{L}}}$$

$$9\left(\omega_0^2 - \frac{3}{2}\omega_0^2\right)a_1 - \frac{3}{2}\omega_0^2 a_2 = 0 \qquad 9\left(\omega_0^2 - \frac{3}{4}\omega_0^2\right)a_1 - \frac{3}{4}\omega_0^2 a_2 = 0$$

$$-\frac{9}{2}a_1 - \frac{3}{2}a_2 = 0 \qquad \frac{9}{4}a_1 - \frac{3}{4}a_2 = 0$$

$$a_2 = -3a_1 \qquad a_2 = 3a_1$$

$$\varphi = A \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \varphi = A \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

b) $\varphi_0 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$ $\qquad \varphi_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \cos \omega_1 t$

$\varphi_0 = \frac{\alpha}{6} \left[\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right]$ $\qquad \varphi_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos \omega_2 t$

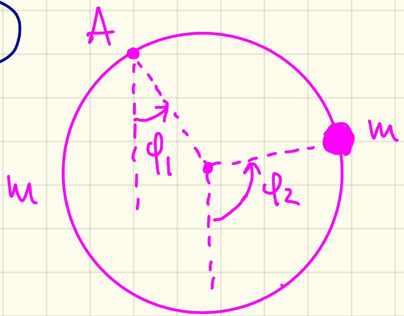
$$\varphi(t) = \frac{\alpha}{6} \left[\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos \omega_2 t - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cos \omega_1 t \right]$$

→ but is it periodic?

$$\omega_2 = \frac{1}{2} \sqrt{\frac{3g}{4}} \qquad \omega_1 = \frac{1}{\sqrt{2}} \sqrt{\frac{3g}{4}}$$

$\frac{\omega_1}{\omega_2} = \sqrt{2}$ → irrational number $\therefore$ $\varphi(t)$ is <u>not</u> periodic

11.26



$$T_1 = \frac{1}{2}\left(mR^2 + mR^2\right)\cdot\dot{\varphi}_1^2 = mR^2\dot{\varphi}_1^2$$

parallell axis theorem

$$T_2 = \frac{1}{2} m v^2$$

$\varphi_1, \varphi_2 \ll 1$

$$x = R\sin\varphi_1 + R\sin\varphi_2 \approx R(\varphi_1 + \varphi_2)$$

$$y = -R\cos\varphi_1 - R\cos\varphi_2 = -2R + \frac{1}{2}R\varphi_1^2 + \frac{1}{2}R\varphi_2^2$$

$$v^2 \approx R^2(\dot{\varphi}_1 + \dot{\varphi}_2)^2 = R^2\dot{\varphi}_1^2 + R^2\dot{\varphi}_2^2 + 2R^2\dot{\varphi}_1\dot{\varphi}_2$$

$$\mathcal{L} = \frac{3}{2}mR^2\dot{\varphi}_1^2 + \frac{1}{2}mR^2\dot{\varphi}_2^2 + mR^2\dot{\varphi}_1\dot{\varphi}_2 - mgR\varphi_1^2 - \frac{1}{2}mgR\varphi_2^2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_1} = 3mR^2\dot{\varphi}_1 + mR^2\dot{\varphi}_2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_2} = mR^2\dot{\varphi}_2 + mR^2\dot{\varphi}_1$$

$$\frac{\partial\mathcal{L}}{\partial\varphi_1} = -2mgR\varphi_1$$

$$\frac{\partial\mathcal{L}}{\partial\varphi_2} = -mgR\varphi_2$$

$$mR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \ddot{\varphi} = -mgR \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \varphi \qquad \omega_0^2 = \frac{g}{R}$$

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \qquad K = \omega_0^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$0 = \begin{vmatrix} 2\omega_0^2 - 3\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} = (2\omega_0^2 - 3\omega^2)(\omega_0^2 - \omega^2) - \omega^4 =$$

$$= 2\omega^4 - 5\omega_0^2\omega^2 + 2\omega_0^2 = 0$$

$$D = 25\omega_0^4 - 16\omega_0^4 = 9\omega_0^4 \qquad \omega^2 = \frac{1}{4}\left(5\omega_0^2 \pm 3\omega_0^2\right) =$$

$$\omega_1^2 = 2\omega_0^2 : \qquad -4a_1 - 2a_2 = 0 \qquad a_2 = -2a_1$$

$$\omega_2^2 = \frac{1}{2}\omega_0^2 : \qquad \frac{1}{2}a_1 - \frac{1}{2}a_2 = 0 \qquad a_2 = a_1$$

$$\varphi_1 = A \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{i\omega_1 t} \qquad \varphi_2 = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_2 t}$$

$$\omega_1 = \sqrt{\frac{2g}{R}} \qquad \omega_2 = \sqrt{\frac{g}{2R}}$$

$\downarrow$ oscillate out of phase, bead with twice the amplitude of the loop.

$\downarrow$ oscillate together, bead does not move along the loop