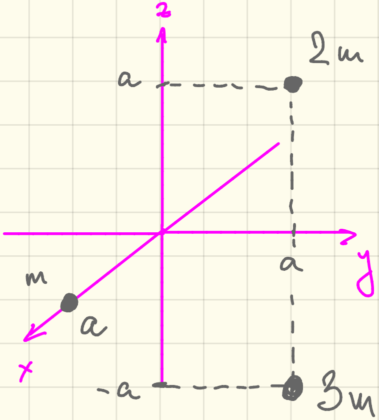
10.35



m @ $(a,0,0)$

$2m$ @ $(0,a,a)$

$3m$ @ $(0,a,-a)$

a)

$$I_{xx} = m\left[0 + 2\cdot(a^2+a^2) + 3(a^2+a^2)\right] = 10\,ma^2$$

$$I_{xy} = -m\left[0 + 0 + 0\right] = 0$$

$$I_{xz} = -m\left[0 + 0 + 0\right] = 0$$

$$I_{yy} = m\left[a^2 + 2\cdot a^2 + 3\cdot a^2\right] = 6\,ma^2$$

$$I_{yz} = -m\left[0 + 2\cdot a^2 - 3\,a^2\right] = ma^2$$

$$I_{zz} = m\left[a^2 + 2\,a^2 + 3\,a^2\right] = 6\,ma^2$$

$$\overleftrightarrow{I} = ma^2\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

i) $a \neq 0 \rightarrow \lambda = 10$

$$c = 4b$$
$$6c = 9b \rightarrow c, b = 0$$

$$\hat{e}_1 = (1,0,0), \quad \lambda_1 = 10$$

block diagonal matrix:

$\det = \det_1 \cdot \det_2$

$$\begin{cases} 10a = \lambda a \\ 6b + c = \lambda b \\ b + 6c = \lambda c \end{cases}$$

ii) $a = 0$

$$\begin{cases} (\lambda - 6) b - c = 0 \\ -b + (\lambda - 6) c = 0 \end{cases} \leftarrow \det = 0$$

$$(\lambda - 6)^2 - 1 = 0$$

$$\lambda - 6 = \pm 1 \quad \begin{matrix} \nearrow \lambda_2 = 5 \\ \searrow \lambda_3 = 7 \end{matrix}$$

$$\begin{cases} 6b + c = 5b \Rightarrow b = -c \\ 6b + c = 7b \Rightarrow b = c \end{cases}$$

$$\begin{cases} \lambda_2 = 5, \ \hat{e}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\ \lambda_3 = 7, \ \hat{e}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{cases}$$

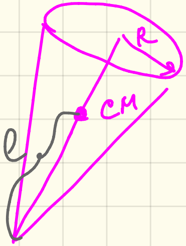
Final answer:

$\hat{e}_1 = (1,0,0) \rightarrow$ toward m, $\lambda_1 = 10ma^2$

$\hat{e}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \rightarrow$ toward $3m$, $\lambda_2 = 5ma^2$

$\hat{e}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \rightarrow$ toward $2m$, $\lambda_3 = 7ma^2$

10.39



$R = 2.5\ \text{cm}$

$h = 10\ \text{cm}$

$\omega = 1800\ \text{rpm}$

$= \frac{1800 \cdot 2\pi}{60} = 188\ \text{rad/s}$

$$\Omega = \frac{mg\ell}{\lambda_3 \omega}$$

$$\lambda_3 = \frac{3}{10} M \cdot R^2$$

(derived in lecture and Example 10.3)

$$\ell = \frac{1}{M}\int_0^h z \cdot dm = \frac{\rho}{M}\int_0^h z\, \pi \left(\frac{R}{h}\cdot z\right)^2 \cdot dz =$$

$$= \frac{\pi \rho R^2}{M \cdot h^2}\int_0^h z^3\, dz = \frac{\pi \rho R^2}{M \cdot h^2}\,\frac{1}{4} h^4 = \frac{1}{4M}\pi \rho R^2 h^2 =$$

$$= \frac{\rho}{M} \cdot \frac{\pi R^2 h^2}{4} = \frac{1}{V_{cone}}\,\frac{\pi R^2 h^2}{4} = \frac{3}{\pi R^2 h} \cdot \frac{\pi R^2 h^2}{4} = \frac{3}{4} h$$

$$\Omega = \frac{M g \frac{3}{4} h}{\frac{3}{10} M R^2 \omega} = \frac{5}{2}\,\frac{g h}{R^2 \omega} \approx \frac{5}{2} \cdot \frac{9.8 \cdot 0.1}{6.25 \cdot 10^{-4} \cdot 188} \approx 21\ \text{rad/s}$$

10.40

$$\begin{cases} \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2\omega_3 = \Gamma_1 \\ \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 = \Gamma_2 \\ \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1\omega_2 = \Gamma_3 \end{cases} \quad \left| \begin{array}{l} \times\ \lambda_1\omega_1 \\ \times\ \lambda_2\omega_2 \\ \times\ \lambda_3\omega_3 \end{array} \right.$$

if $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$:

$$\lambda_1^2\omega_1\dot{\omega}_1 + \lambda_2^2\omega_2\dot{\omega}_2 + \lambda_3^2\omega_3\dot{\omega}_3 = \omega_1\omega_2\omega_3\Big[\lambda_1(\cancel{\lambda_2} - \cancel{\lambda_3}) + \lambda_2(\cancel{\lambda_3} - \cancel{\lambda_1}) + \lambda_3(\cancel{\lambda_1} - \cancel{\lambda_2})\Big]$$

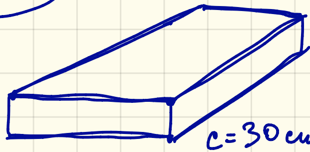
$$\frac{d}{dt}\big(\underbrace{\lambda_1^2\omega_1^2 + \lambda_2^2\omega_2^2 + \lambda_3^2\omega_3^2}_{L^2}\big) = 0$$

$$L = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

if $\frac{d}{dt}L^2 = 0 \Rightarrow \underline{\underline{L = \text{const}}}$

(10.42)

Inertia tensor of the book



$a = 3$ cm (see 10.25, previous HW)

$c = 30$ cm

$b = 20$ cm

$$\mathbb{I} = \frac{1}{3} M \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{3} M (b^2 + c^2)$$

$$\lambda_2 = \frac{1}{3} M (a^2 + c^2)$$

$$\lambda_3 = \frac{1}{3} M (b^2 + a^2)$$

$\lambda_1 > \lambda_2 > \lambda_3$

spin around $\hat{e}_1$

$\omega_1 = \omega \approx$ const $\gg \omega_2, \omega_3$

$$\begin{cases} \lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\, \omega\, \omega_3 \\ \lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\, \omega\, \omega_2 \end{cases}$$

$$\begin{cases} \dot{\omega}_2 = \dfrac{a^2 - c^2}{a^2 + c^2}\, \omega\, \omega_3 \\[2ex] \dot{\omega}_3 = \dfrac{b^2 - a^2}{b^2 + a^2}\, \omega\, \omega_2 \end{cases}$$

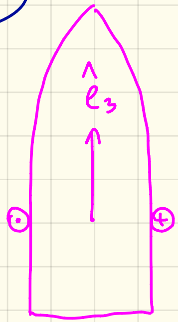
$$\ddot{\omega}_2 = - \frac{c^2 - a^2}{c^2 + a^2}\, \omega\, \dot{\omega}_3 = - \frac{(c^2 - a^2)(b^2 - a^2)}{(a^2 + c^2)(a^2 + b^2)}\, \omega^2 \omega_2$$

$$\Omega = \omega \cdot \sqrt{\frac{(c^2 - a^2)(b^2 - a^2)}{(a^2 + c^2)(a^2 + b^2)}} \approx 0.97\,\omega \approx 174 \text{ rpm}$$

if spinning around $\hat{e}_3$, $\Omega = \omega \sqrt{\dfrac{(a^2 - c^2)(b^2 - c^2)}{(a^2 + c^2)(b^2 + c^2)}} \approx 0.61 \cdot \omega \approx 110$ rpm

10.44

$\lambda_2 = \lambda_1 = \lambda$



$$\begin{cases} \lambda \dot{\omega}_1 - (\lambda - \lambda_3)\,\omega_2\omega_3 = 0 \\ \lambda \dot{\omega}_2 - (\lambda_3 - \lambda)\,\omega_3\omega_1 = 0 \\ \lambda_3 \dot{\omega}_3 = \Gamma \end{cases}$$

$$\frac{\lambda - \lambda_3}{\lambda} = \alpha$$

$$\omega_3 = \omega_{30} + \Gamma/\lambda_3 \cdot t$$

$$\begin{cases} \dot{\omega}_1 = \alpha\,\omega_3\,\omega_2 \\ \dot{\omega}_2 = -\alpha\,\omega_3\,\omega_1 \end{cases}$$

$$\eta = \omega_1 + i\omega_2$$

$$\dot{\eta} = \dot{\omega}_1 + i\,\dot{\omega}_2$$

$$\dot{\eta} = \alpha\,\omega_3\left(\omega_2 - i\omega_1\right) = -i\alpha\omega_3\left(i\omega_2 + \omega_1\right) = -i\alpha\omega_3\eta$$

$$\dot{\eta} = -i\alpha\left(\omega_{30} + \frac{\Gamma}{\lambda_3}t\right)\cdot\eta$$

$$\eta(t=0) = \omega_{10}$$

$$\dot{\eta} = -i\alpha\left(\omega_{30} + \frac{\Gamma}{\lambda_3} t\right) \cdot \eta$$

$$\eta = A \cdot e^{at + bt^2}$$

$$\dot{\eta} = A e^{at + bt^2} (a + 2bt)$$

$$a + 2bt = -i\alpha\omega_{30} - i\alpha \frac{\Gamma}{\lambda_3} t$$

$$a = -i\alpha\omega_{30}, \quad b = -i\alpha \frac{\Gamma}{2\lambda_3}$$

$$\eta = A \cdot e^{-i\alpha\left(\omega_{30} + \frac{\Gamma}{2\lambda_3} t\right) t} = A\left(\cos\left[\alpha\left(\omega_{30} + \frac{\Gamma}{2\lambda_3} t\right) \cdot t\right] + i \sin\left[\ \right]\right)$$

$$\frac{d\eta}{dt} = a(b + ct) \cdot \eta$$

$$\ln \frac{\eta}{\eta_0} = a(b + ct) \cdot t$$

$$\frac{\eta}{\eta_0} = e^{a(b + ct) \cdot t}$$

$$\begin{cases} \omega_1 = \omega_{10} \cdot \cos\left[\frac{\lambda - \lambda_3}{\lambda}\left(\omega_{30} + \frac{\Gamma}{2\lambda_3} t\right) t\right] \\ \omega_2 = \omega_{10} \cdot \sin\left[\frac{\lambda - \lambda_3}{\lambda}\left(\omega_{30} + \frac{\Gamma}{2\lambda_3} t\right) t\right] \\ \omega_3 = \omega_{30} + \frac{\Gamma}{\lambda_3} t \end{cases}$$

$\to$ $\vec{\omega}$ component in xy plane has fixed magnitude, and rotates with frequency increasing with time

$\to$ ω_3 uniformly increases