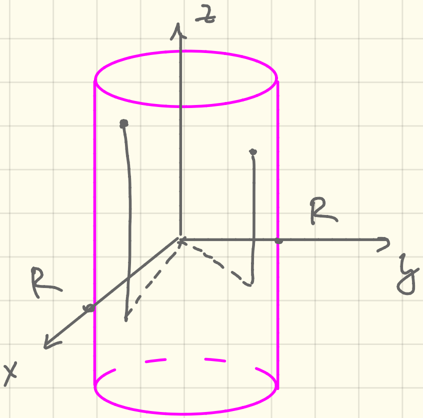
6.2

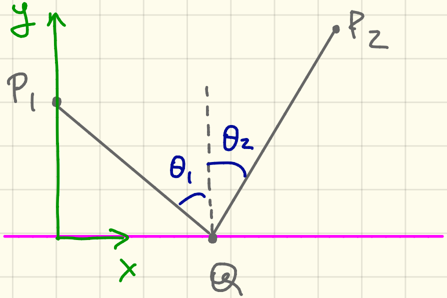


$\varphi_1, z_1 \longrightarrow \varphi_2, z_2$

$$dl^2 = (R\,d\varphi)^2 + dz^2$$

$$S = \int_{z_1}^{z_2} dl = \int_{z_1}^{z_2} \sqrt{R^2 \varphi'^2 + 1}\, dz$$

6.3



$P_1(0, y_1, 0)$

$P_2(x_2, y_2, 0)$

$Q(x, 0, z)$

$\tan\theta_1 = \frac{x}{y_1}$

$\tan\theta_2 = \frac{x_2 - x}{y_2}$

$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$

$$l_1^2 = y_1^2 + x^2 + z^2$$

$$l_2^2 = y_2^2 + (x_2 - x)^2 + z^2$$

$$l = \sqrt{y_1^2 + x^2 + z^2} + \sqrt{y_2^2 + (x_2 - x)^2 + z^2}$$

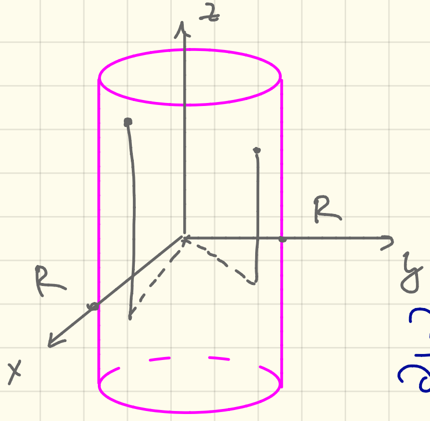
$$\frac{\partial l}{\partial t} = \frac{2z}{\sqrt{y_1^2 + x^2 + z^2}} + \frac{2z}{\sqrt{y_2^2 + (x_2 - x)^2 + z^2}} = 0 \Rightarrow z = 0.$$

$$\frac{\partial l}{\partial x} = \frac{x}{\sqrt{y_1^2 + x^2}} - \frac{(x_2 - x)}{\sqrt{y_2^2 + (x_2 - x)^2}} = 0$$

$$\frac{x/y_1}{\sqrt{1 + (x/y_1)^2}} = \frac{(x_2 - x)/y_2}{\sqrt{1 + \left(\frac{x_2 - x}{y_2}\right)^2}} \Rightarrow \cos\theta_1 \cdot \tan\theta_1 = \cos\theta_2 \tan\theta_2$$

$$\sin\theta_1 = \sin\theta_2 \Rightarrow \theta_1 = \theta_2.$$

6.7



$$S = \int_{z_1}^{z_2} \sqrt{R^2\varphi'^2 + 1}\, dz \qquad \varphi(z) - ?$$

$$f(\varphi, \varphi', z) = \sqrt{R^2\varphi'^2 + 1}$$

$$\frac{\partial f}{\partial \varphi} = 0 \qquad \frac{\partial f}{\partial \varphi'} = \frac{R^2\varphi'}{\sqrt{1 + R^2\varphi'^2}}$$

$$R^2\varphi' = A\sqrt{1 + R^2\varphi'^2}$$

$$R^4\varphi'^2 = A^2 + A^2R^2\varphi'^2$$

$$\varphi'^2 = \frac{A^2}{R^2(R^2 - A^2)} = \text{const}$$

$$\underline{\varphi = \varphi_0 + kz} \quad \underline{\text{a spiral}}$$

$\varphi_0 + kz$ passing through (φ_1, z_1) and (φ_2, z_2) is <u>not</u> unique: can have many loops:

$$k(z_2 - z_1) = \varphi_2 - \varphi_1 + 2\pi \cdot n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Shortest is $k = \dfrac{\varphi_2 - \varphi_1}{z_2 - z_1}$ or $\dfrac{\varphi_2 - \varphi_1}{z_2 - z_1} \pm \pi$ depending on φ_1 and φ_2

6.9

$$S = \int_{(0,0)}^{(1,1)} (y'^2 + yy' + y^2)\,dx \qquad y = y(x) - ?$$

$$f(y, y', x) = y'^2 + yy' + y^2$$

$$\left.\begin{aligned} \frac{\partial f}{\partial y} &= y' + 2y \\ \frac{\partial f}{\partial y'} &= 2y' + y \end{aligned}\right\} \rightarrow y' + 2y = \frac{d}{dx}(2y' + y) = 2y'' + y'$$

$$y'' = y$$

$$y(x) = Ae^{x} + Be^{-x}$$

$$y(0) = 0 \rightarrow A = -B, \quad y(x) = A(e^{x} - e^{-x}) =$$

$$= 2A \cdot \sinh x$$

$$y(1) = 1 \rightarrow 2A \cdot \sinh 1 = 1$$

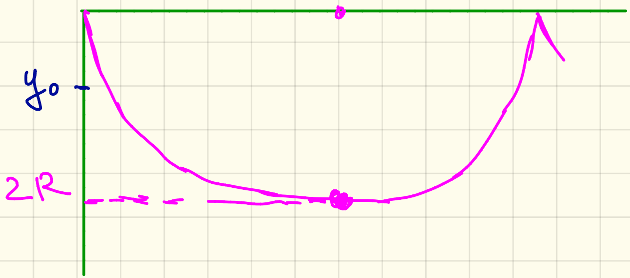
$$y(x) = \sinh(x)/\sinh(1)$$

6.25

$$y = R(1-\cos\theta)$$

$$x = -R\sin\theta + R\theta =$$

$$x = R(\theta - \sin\theta)$$



$$dy = R\sin\theta\, d\theta$$

$$dx = -R\cos\theta\, d\theta + R\, d\theta$$

$$\frac{1}{2}mv^2 = mg(y-y_0) = mg\left[R(1-\cos\theta) - R(1-\cos\theta_0)\right]$$

$$dl = \sqrt{R^2\sin^2\theta\, d\theta^2 + R^2(1-\cos\theta)^2 d\theta^2} =$$

$$v^2 = 2gR(\cos\theta_0 - \cos\theta)$$

$$= R\,d\theta\sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$

$$t = R\int_{\theta_0}^{\pi} \frac{\sqrt{(1-\cos\theta)^2+\sin^2\theta}}{\sqrt{2gR}\sqrt{\cos\theta_0 - \cos\theta}}\, d\theta = \sqrt{\frac{R}{g}}\int_{\theta_0}^{\pi}\frac{\sqrt{1-\cos\theta}}{\sqrt{\cos\theta_0-\cos\theta}}\, d\theta$$

I use slightly different path for integration:

$$\int \frac{\sqrt{1-\cos\theta}}{\sqrt{\cos\theta_0-\cos\theta}}\,d\theta$$

$$\begin{cases} \cos\theta_0 = \cos\left(\frac{\theta_0+\theta}{2}+\frac{\theta_0-\theta}{2}\right) = \cos\frac{\theta_0+\theta}{2}\cos\frac{\theta_0-\theta}{2} - \sin\frac{\theta_0+\theta}{2}\sin\frac{\theta_0-\theta}{2} \\ \cos\theta = \cos\left(\frac{\theta_0+\theta}{2}-\frac{\theta_0-\theta}{2}\right) = \cos\frac{\theta_0+\theta}{2}\cos\frac{\theta_0-\theta}{2} + \sin\frac{\theta_0+\theta}{2}\sin\frac{\theta_0-\theta}{2} \end{cases}$$

$$\begin{cases} \cos\theta_0-\cos\theta = 2\sin\frac{\theta+\theta_0}{2}\sin\frac{\theta-\theta_0}{2} = -(1-\cos\theta_0)+(1-\cos\theta) \\ \underset{\cos(0)}{1} - \cos\theta = 2\sin^2\frac{\theta}{2} = 2\left(1-\cos^2\frac{\theta}{2}\right) \end{cases}$$

$$\int \frac{\sqrt{1-\cos\theta}}{\sqrt{\cos\theta_0-\cos\theta}} = \int \frac{\sqrt{2}\sin\frac{\theta}{2}\,d\theta}{\sqrt{2-2\cos^2\frac{\theta}{2}-2+2\cos^2\frac{\theta_0}{2}}} = 2\int \frac{\sin\frac{\theta}{2}\,d\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_0}{2}-\cos^2\frac{\theta}{2}}} = \ldots$$

$$2\int_{\theta_0}^{\pi} \frac{\sin\frac{\theta}{2}\, d\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_0}{2} - \cos^2\frac{\theta}{2}}} \quad \left| \begin{array}{l} u = \cos\frac{\theta}{2} \\ du = -\sin\frac{\theta}{2}\, d\frac{\theta}{2} \\ a^2 = \cos^2\frac{\theta_0}{2} \end{array} \right| = -2\int_{a}^{0} \frac{du}{\sqrt{a^2 - u^2}} =$$

$$= -2\,\mathrm{atan}\, \frac{u}{\sqrt{a^2-u^2}} \Bigg|_a^0 = -2\,\mathrm{atan}\, 0 + 2\,\mathrm{atan}\, \frac{a}{0} = 0 + 2\cdot\frac{\pi}{2} = \pi$$

$$\boxed{t = \sqrt{\frac{R}{g}}\; \pi}$$ $\longrightarrow$ <u>does not depend on θ_0!</u>