Press-Schechter Formalism: Structure Formation by Self-Similar Condensation

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Based on Press & Schechter’s 1974 Paper
Structure Formation in Simulations

Images Courtesy of the Max Planck Institute of Astrophysics & Virgo Consortium (Top, Millennium-II; Right, Aquarius Project).
Basic Definitions of Dark Matter

- **Number Density**

- **Characteristic Particle Mass**

- **Deceleration Parameter**: Describes the ability of the universe to inhibit condensation.

- **Jean’s Number**: Describes the number of particles needed to begin condensation. See Eqn. (3) for an empirical formula.

\[ n_\ast \equiv \int_0^\infty n(m)dm \]

\[ m_\ast \equiv n_\ast^{-1} \int_0^\infty mn(m)dm = \frac{\rho}{n_\ast} \]

\[ q = \frac{4}{3} \pi m_\ast n_\ast G / H^2 \]

\[ N_J = n \left( \frac{v}{H} \right)^3 \]
Intuitions for Self-Similarity

- Gravitational Collapse: “Large correlations in the gas must be interpreted as changes in $n(m)$, by treating highly correlated groups of particles as single more massive particles.” (Pg. 427)

- Non-Linearity of Equations: If we expect $n(m)$ to depend only on the statistics of the current scale, then the evolutions of $n(m)$ will depend on the statistical properties of the non-linear differential equations. (Pg. 428)
Preparation for Derivation

- Define Mass Variance inside of Volume V.

\[ \Sigma^2(V) = \langle M \rangle^2_V - \langle M^2 \rangle_V \]

- An upper bound on the variance can be found by taking the dark matter particles to be distributed uniformly, so that the variance is linear in volume.

\[ \Sigma^2(V) = \sigma^2 V \]
Derivation

- We define the probability of finding a fractional mass deviation between $\delta$ and $\delta + d\delta$ in volume $V$ as $P(\delta, V)$.

\[
p(\delta, V) = \frac{1}{\sqrt{2\pi}\delta_*^2} e^{-\frac{\delta^2}{2\delta_*^2}}
\]

\[
\delta = \frac{(M - \langle M \rangle)}{M} = \frac{\sqrt{\Sigma^2(V)}}{M} = \frac{\sigma V^{-\frac{1}{2}}}{\rho}
\]

\[
\delta_* \equiv \frac{\sqrt{\Sigma^2(V)}}{M} = \frac{\sigma V^{-\frac{1}{2}}}{\rho}
\]
Derivation (Cont’d)

\[
P = \int_{\delta = \frac{R_1}{R_2}}^\infty p(\delta, V) d\delta = \frac{erfc\left(\frac{R_1 \rho \sqrt{V}}{\sqrt{2} R_2 \sigma}\right)}{2}
\]
\[R_2 = \frac{R_1}{\delta}\]

- The turn-around scale, \( R_2 \), for \( R_1 \) is found for spherical collapse in the Appendix.

- We can define the probability of having an overdensity \( \delta \) collapse before \( R_2 \) as \( P \).
The number density distribution is found by multiplying the percentage of collapsed mass with the mass density of the second scale and dividing by M.

The factor of 2 was added to take into account the underdensities around the collapsed objects (Improved explanation can be found in Bond et al. 1991).

\[
\frac{dP}{dM} = 2 \frac{3}{2} \pi^{\frac{1}{2}} \frac{R_1}{R_2} \rho_1^{\frac{1}{2}} M^{-\frac{1}{2}} e^{\left(\frac{-1R_1^2 \rho_1 M}{2R_2^2 \sigma_1^2}\right)}
\]

\[
n_2(m) = \frac{2}{M} \rho_1 \left(\frac{R_1}{R_2}\right)^3 \frac{dP}{dM}
\]

\[
\rho_2 = \rho_1 \left(\frac{R_1}{R_2}\right)^3
\]

\[
\sigma_2^2 = \sigma_1^2 \left(\frac{R_1}{R_2}\right)
\]
Summary

- Press-Schechter Formalism allows us to analytically recreate the linear evolution of dark matter perturbations without the need of resource exhaustive computer simulations.

- Press-Schechter Formalism describes a universe where different scales collapse in a similar manner without a dependence on the scale size (Self-similar condensation).

- The Formalism has been extensively used and compared to simulations to confirm its accuracy.

- An improved Sheth-Tormen Formalism, which uses ellipsoidal collapse and excursion set theory, reproduces the main results of Press-Schechter while correcting the discrepancies at the high and low mass extremes of the halo mass function.