A loop of $SU(2)$ gauge fields on S^4 stable under the Yang-Mills flow

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The space of connections

G a compact Lie group	—	say $G = SU(2)$
M a riemannian 4-manifold	—	say $M = S^4$
B a principle G-bundle over M	—	say $B = S^4 \times SU(2)$

$\mathcal{A}$ = the space of connections in B

$D = d + A$ the covariant derivative

A an $su(2)$-valued 1-form on S^4

$F = D^2$ the curvature, an $su(2)$-valued 2-form

$\mathcal{G}$ = the group of automorphisms of B (gauge transformations)

$$\mathcal{G} = Maps(S^4 \to SU(2))$$

$\mathcal{A}/\mathcal{G}$ = the gauge equivalence classes of connections

The Y-M flow on the unparametrized loops of connections

The Yang-Mills action $$S_{YM}(A) = \frac{1}{2\pi^2} \int_M \frac{1}{4} \mathrm{tr}(-F * F)$$
(normalized so that the instanton has $S_{YM} = 1$.)

The metric on $\mathcal{A}$ $$(ds)^2_{\mathcal{A}} = \frac{1}{2\pi^2} \int_M \frac{1}{2} \mathrm{tr}(-\delta A * \delta A)$$

The gradient flow $$\frac{\partial A}{\partial t} = -\nabla S_{YM} = * D * F(A)$$

$$\left(\partial_t A_\mu(x) = D^\nu F_{\nu\mu}(x) = \partial^\nu \partial_\nu A_\mu(x) + \cdots\right)$$

The Y-M flow is $\mathcal{G}$-invariant, so it acts:

- on the gauge equivalence classes, $\mathcal{A}/\mathcal{G}$
- pointwise on loops of connections, $Maps(S^1 \to \mathcal{A}/\mathcal{G})$
- on unparametrized loops, $\mathcal{L} = \ Maps(S^1 \to \mathcal{A}/\mathcal{G})/Diff(S^1)$

The problem

What is the generic long-time behavior of the Y-M flow on the space $\mathcal{L}$ of unparametrized loops?

A stable loop would be a parametrized loop $\sigma \mapsto A(\sigma)$ on which the flow acts by reparametrization: $\partial_t A(\sigma) = v(\sigma)\partial_\sigma A(\sigma)$.

Are there nontrivial stable loops for all the nontrivial elements in $\pi_0\mathcal{L} = \pi_1(\mathcal{A}/\mathcal{G})$?

$\mathcal{A}$ is contractible, so

$$\pi_1(\mathcal{A}/\mathcal{G}) = \pi_0\mathcal{G} = \pi_0 Maps(S^4 \to G) = \pi_4 G$$

$$\pi_4 SU(2) = \mathbb{Z}_2$$

$SU(2)$ is the only compact Lie group with nontrivial π_4.

Personal motivation

Hypothetical application in a speculative physics theory [DF, 2003].

The *lambda model* is a modified 2-d nonlinear model whose target space is the space of space-time fields, e.g., $\mathcal{A}/\mathcal{G}$.

The modification consists of interspersing the ordinary dilation operator of the nonlinear model with the gradient flow of the space-time action, e.g., S_{YM}.

The dilation operator of the lambda model generates a measure on the target manifold — a space-time quantum field theory.

A stable loop for the $SU(2)$ Yang-Mills flow is a classical winding mode for the lambda model.

If the stable loop can be quantized in the lambda model, it might give low energy states that are not in lagrangian $SU(2)$ quantum gauge field theory, and that might be observable.

The result

A plausible outcome of the Y-M flow would be for one point on the loop to flow to a saddle point with a one-dimensional unstable manifold. The outgoing flows in both directions along the unstable manifold would end at the flat connection. The unstable manifold would thus form a stable loop.

Here, I find such a saddle point, with $S_{YM} = 2$, and show that its unstable manifold is one-dimensional, so forms a stable loop.

Actually, the saddle point is not a *point*, but rather a loop of fixed points.

This is *naive* mathematics: entirely explicit, elementary calculations.

Previous work

Sibner, Sibner and Uhlenbeck (1989) constructed nontrivial loops of connections in the trivial bundle over S^4, with certain given $U(1)$ symmetry.

Then they minimized over loops with the given $U(1)$ symmetry the maximum of S_{YM} on the loop, and showed that this min-max was realized by a solution of the Yang-Mills equations.

As far as I can tell, these solutions have $S_{YM} > 2$. Each of these solutions should have a one-dimensional unstable manifold within the space of connections of the given $U(1)$ symmetry, but the full unstable manifold, within the space of all connections, presumably has dimension > 1.

They conjectured the existence of an additional solution, not given by their construction. Presumably, their missing solution is the saddle point described here.

$SU(3)/SU(2) = S^5 \subset \mathbb{C}^3$

Lacking intuition, I looked for an explicit nontrivial loop of $SU(2)$ connections, planning to run the Y-M flow numerically to see what would happen.

The $SU(2)$ bundles over S^5 are classified by $\pi_4 SU(2)$, because they are made by gluing together trivial bundles on the north and south hemispheres using a map from the equator, S^4, to $SU(2)$. So there is a unique nontrivial $SU(2)$ bundle over S^5.

$SU(3) \to SU(3)/SU(2) = S^5 \subset \mathbb{C}^3$ is a homogeneous model of the nontrivial bundle, with a canonical connection invariant under $SU(3)_L \times U(1)_R$.

Pull back along a suitable map $S^1 \times S^4 \to S^5$ to get a nontrivial loop of $SU(2)$ connections on S^4,

$$\sigma \in [0, 2\pi] \mapsto D_\sigma$$

with D_0 and $D_{2\pi}$ nontrivially gauge equivalent flat connections.

Parametrizations

$$S^3 \subset \mathbb{C}^2 : \qquad z = (z_1, z_2) \qquad |z_1|^2 + |z_2|^2 = 1$$

$$S^4 \subset \mathbb{R} \oplus \mathbb{C}^2 : \qquad (\cos\theta, z_1 \sin\theta, z_2 \sin\theta) \qquad 0 \le \theta \le \pi$$

$$r = e^x = \tan\frac{\theta}{2}$$

$U(2)$ symmetry

The map $S^1 \times S^4 \to S^5$ can be chosen so that each connection D_σ is invariant under the action of $U(2)$ on $S^3 \subset \mathbb{C}^2$.

For example, take

$$(\sigma, \theta, z_1, z_2) \mapsto \left(\cos\theta + i \sin\theta \cos\frac{\sigma}{2}, z_1 \sin\theta \sin\frac{\sigma}{2}, z_2 \sin\theta \sin\frac{\sigma}{2}\right)$$

An additional $\mathbb{Z}_2$ symmetry exchanges $D_\sigma \leftrightarrow D_{2\pi-\sigma}$.

So the midpoint D_π has an enhanced $U(2) \rtimes \mathbb{Z}_2$ symmetry.

D_π should flow to the saddle point.

The $\mathbb{Z}_2$ symmetry would exchange the two branches of the unstable manifold.

Instead of running the Y-M flow numerically on the loop, it is considerably easier just to minimize S_{YM} within the space of $U(2) \rtimes \mathbb{Z}_2$-invariant connections.

$U(2)$ acts transitively on S^3, so the $U(2)$-invariant connection forms A_σ are functions only of θ, or x.

The additional $\mathbb{Z}_2$ symmetry reflects $\theta \leftrightarrow \pi - \theta$, $x \leftrightarrow -x$, so the invariant connection forms satisfy reflection symmetry conditions.

Numerical investigations

The connection 1-form $A(\theta)$ has two independent components, satisfying certain symmetry conditions under $\theta \to \pi - \theta$.

Write them as suitable polynomials in $\cos\theta$. S_{YM} is a quartic polynomial in the coefficients of the polynomials.

Minimize S_{YM} numerically on this $2N$ dimensional submanifold of the space of invariant connections, using Sage, Mathematica, and Maple. The programs misbehaved for $N > 15$.

The numerics suggest an absolute minimum at $S_{YM} = 2$.

S_{YM} a small integer suggests topology.

$SU(3)$: $S_{YM} = 2.4$

N	min(S_{YM})
1	2.15627
2	2.06011
3	2.03019
4	2.01735
5	2.01086
6	2.00723
7	2.00504
8	2.00368
9	2.00346
10	2.00313
11	2.00286
12	2.00251
13	2.00202
14	2.00186
15	2.00147

(anti-)self-duality

$$F_{\pm} = \frac{1}{2}(F \pm *F) \qquad * \text{ the Hodge operator}$$

$$S_{\pm} = \frac{1}{8\pi^2}\int_{S^4}(-F_{\pm}*F_{\pm}) = \int_{-\infty}^{\infty} dx\, L_{\pm}(x)$$

$$S_{YM} = S_{+} + S_{-}$$

$$S_{+} - S_{-} \in \mathbb{Z} \text{ is the } \textit{instanton number}$$

The *instanton*: $\qquad F = *F \qquad S_{+} = 1$

$$F_{-} = 0 \qquad L_{-}(x) = 0 \qquad S_{-} = 0$$

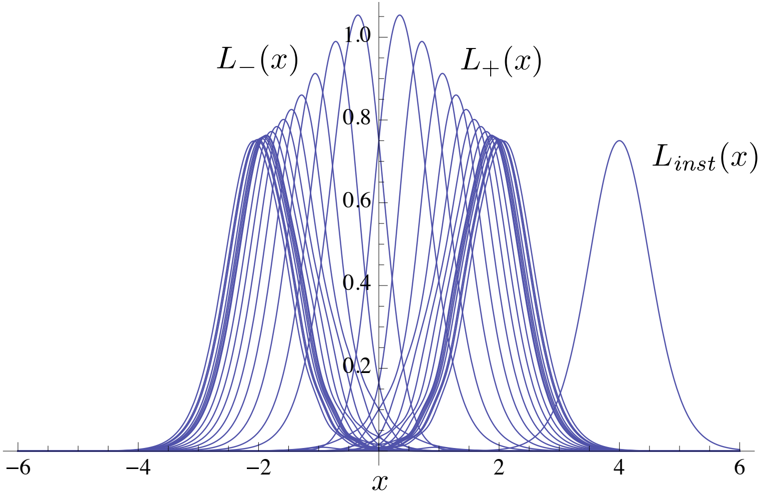
The $U(2)$-invariant instanton centered at the south pole ($x = \infty$) will be written explicitly later. For now, its action density is

$$L_{+}(x) = L_{inst}(x) = .75\cosh^{-4}(x - x_{+})$$

where

$$r_{+} = e^{-x_{+}}$$

is the instanton size.



The numerics suggest that $S_{YM} = 2$ is attained as a zero-size instanton at the south pole (at $x{=}\infty$) combined with a zero-size anti-instanton at the north pole (at $x{=}-\infty$).

The $U(2)$-invariant $su(2)$-valued forms on S^3

$U(2)$ invariance:

$$\omega(hz) = h\omega(z)h^{-1} \qquad h \in U(2)$$

Identify S^3 with $SU(2)$,

$$\hat{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad z = g\hat{e} \qquad g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

There is a single invariant 0-form:

$$\phi(z) = i(P - Q) = g \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g^{-1} \qquad P = zz^\dagger, \;\; Q = 1 - P$$

There are three invariant 1-forms:

$$v_+ = PdP \qquad v_- = (dP)P \qquad v_3 = (z^\dagger dz)(Q - P)$$

The Maurer-Cartan form on $SU(2)$ is

$$\Omega = gd(g^{-1}) = v_+ - v_- + v_3$$

The instanton and the anti-instanton

The basic instanton of size r_+ located at the south pole ($x = \infty$)

$$D_+ = d + f_+(x)\Omega \qquad f_+ = \frac{1}{1 + e^{2(x-x_+)}} \qquad r_+ = e^{-x_+}$$

The basic anti-instanton of size r_- at the north pole ($x = -\infty$)

$$D_- = d + f_-(x)\Omega \qquad f_- = \frac{1}{1 + e^{-2(x-x_-)}} \qquad r_- = e^{x_-}$$

The instanton and anti-instanton twisted by $h_\pm \in SU(2)/\{\pm\mathbf{1}\}$

$$(gh_+g^{-1})D_+(gh_+g^{-1})^{-1} \qquad (gh_-g^{-1})D_-(gh_-g^{-1})^{-1}$$

8 (anti-)instanton moduli:	4	the location in S^4
	1	the size $r_\pm$
	3	the twist $h_\pm$

The twisted-pair

Send $r_\pm \to 0$. Then, away from the poles, $f_{\pm 1} = 1$, so

$$D_\pm = gdg^{-1}, \quad \text{so} \quad (gh_\pm g^{-1})D_\pm(gh_\pm g^{-1})^{-1} = gdg^{-1}$$

so it makes sense to patch the twisted pair at the equator, defining

$$D(h_-, h_+) = \begin{cases} (gh_-g^{-1})D_-(gh_-g^{-1})^{-1} & x \leq 0 \\ (gh_+g^{-1})D_+(gh_+g^{-1})^{-1} & x \geq 0 \end{cases}$$

The 15 parameter conformal group of S^4 absorbs 14 of the 16 moduli. Use 8 to put the instanton and the anti-instanton at the poles, and use dilation to scale r_+ and r_-^{-1}, to make $r_+ = r_-$.

This leaves the 6 parameters of $O(4) = SU(2)_L \times SU(2)_R / \{\pm \mathbf{1}\}$.

$SU(2)_L \times SU(2)_R/\{\pm\mathbf{1}\}$ acts on the twists by

$$(h_+, h_-) \mapsto (g_R h_+ g_L^{-1}, g_R h_- g_L^{-1})$$

Absorb h_+ by taking $g_L = g_R h_+$, leaving $SU(2)_R/\{\pm\mathbf{1}\}$ acting by conjugation on h_-

$$(\mathbf{1}, h_-) \mapsto (\mathbf{1}, g_R h_- g_R^{-1})$$

Thus the zero-size twisted-pair has 14 conformal moduli, plus a 1 parameter symmetry, plus 1 additional parameter

$$\cos\frac{\sigma}{2} = \frac{1}{2}\mathrm{tr}(h_- h_+^{-1}) \qquad 0 \le \sigma \le 2\pi$$

with $\sigma = 0 \sim \sigma = 2\pi$ because the twists act by conjugation.

The loop of zero-size twisted pairs

The relative twist $h_- h_+^{-1}/\{\pm\mathbf{1}\}$ lies in $SU(2)/\{\pm\mathbf{1}\} = SO(3)$.

$\pi_1 SO(3) = \mathbb{Z}_2$. We want to show that the nontrivial loops of zero-size twisted pairs are nontrivial loops in $\mathcal{A}/\mathcal{G}$.

A convenient nontrivial loop of zero-size twisted pairs is

$$D_\sigma = \begin{cases} D_\sigma^- &= e^{\frac{1}{4}\sigma\phi} D_- e^{-\frac{1}{4}\sigma\phi} \qquad x \leq 0 \\ D_\sigma^+ &= e^{-\frac{1}{4}\sigma\phi} D_+ e^{\frac{1}{4}\sigma\phi} \qquad x \geq 0 \end{cases}$$

$$g h_- g^{-1} = e^{\frac{1}{4}\sigma\phi} = g \begin{pmatrix} e^{\frac{i\sigma}{4}} & 0 \\ 0 & e^{-\frac{i\sigma}{4}} \end{pmatrix} g^{-1}$$

$$h_- h_+^{-1} = \begin{pmatrix} e^{\frac{i\sigma}{2}} & 0 \\ 0 & e^{-\frac{i\sigma}{2}} \end{pmatrix}$$

Non-triviality of the loop of twisted pairs in $\mathcal{A}/\mathcal{G}$

The twisted pairs, as written, are singular at the poles (before the limit $r_\pm \to 0$). The connections can be made regular everywhere on S^4 by a gauge transformation, giving

$$D_\sigma = \begin{cases} D_\sigma^- & = \Phi_\sigma^- D_- (\Phi_\sigma^-)^{-1} \qquad x \leq 0 \\ D_\sigma^+ & = \Phi_\sigma^+ D_+ (\Phi_\sigma^+)^{-1} \qquad x \geq 0 \end{cases}$$

$$\Phi_\sigma^-(x,g) = e^{\frac{1}{2}\sigma k_-(x)\phi} \qquad \Phi_\sigma^+(x,g) = e^{\left(\pi - \frac{1}{2}\sigma k_+(x)\right)\phi}$$

$$k_+(-\infty) = 0 \quad k_+(-\infty) = 1 \qquad k_- + k_+ = 1\,.$$

$$D_{2\pi} = \Phi D_0 \Phi^{-1}$$

$$\Phi = e^{-\pi k_+(x)\phi} = g \begin{pmatrix} e^{-i\pi k_+(x)} & 0 \\ 0 & e^{i\pi k_+(x)} \end{pmatrix} g^{-1}$$

$\Phi = \Sigma H : S^4 \to S^3$, the suspension of the Hopf map $H : S^3 \to S^2$, representing the nontrivial element in $\pi_4 S^3$.

Stability

The zero-size twisted-pairs are all critical points of the Y-M action, so the loop of zero-size twisted pairs is pointwise fixed under the Y-M flow.

Is the loop of zero-size twisted-pairs stable under the flow?

The instanton and the anti-instanton are individually stable, so we need only calculate the flow in the two zero-modes r_+ and σ, in the limit where r_+ is asymptotically small.

To determine the topology of the flow, it should be enough to calculate $\dot{r}_+$ and $\dot{\sigma}$ as functions of r_+ and σ, at least to leading order in r_+.

Calculation of $\dot{r}_+$ and $\dot{\sigma}$

Write the general asymptotically small $U(2)$-invariant perturbation

$$D_\sigma = \begin{cases} e^{\frac{1}{4}\sigma\phi}\left(D_- + \delta A_-\right) e^{-\frac{1}{4}\sigma\phi} & x \leq 0 \\ e^{-\frac{1}{4}\sigma\phi}\left(D_+ + \delta A_+\right) e^{\frac{1}{4}\sigma\phi} & x \geq 0 \end{cases}$$

(1) Enforce the flow equation to leading order

$$\dot{r}_- \partial_{r_-} D_- + \frac{1}{4}\dot{\sigma}[\phi,\, D_-] = *D_- * D_- \delta A_-$$

$$\dot{r}_+ \partial_{r_+} D_+ - \frac{1}{4}\dot{\sigma}[\phi,\, D_+] = *D_+ * D_+ \delta A_+$$

(2) Require D_σ to be regular at $x = 0$,

$$e^{\frac{1}{4}\sigma\phi}\left(D_- + \delta A_-\right) e^{-\frac{1}{4}\sigma\phi} - e^{-\frac{1}{4}\sigma\phi}\left(D_+ + \delta A_+\right) e^{\frac{1}{4}\sigma\phi} = 0 + O(x^2)$$

Together, (1) and (2) ensure that D_σ satisfies the flow equation.

Working in $A_x = 0$ gauge, expand in the invariant 1-forms on S^3

$$\delta A_\pm = \delta A_\pm^+(x) v_+ + \delta A_\pm^-(x) v_- + \delta A_\pm^3(x) v_3$$

The flow equations are thus second order inhomogeneous linear ordinary differential equations in x, with non-constant 3×3 matrix coefficients. They can be diagonalized and integrated exactly.

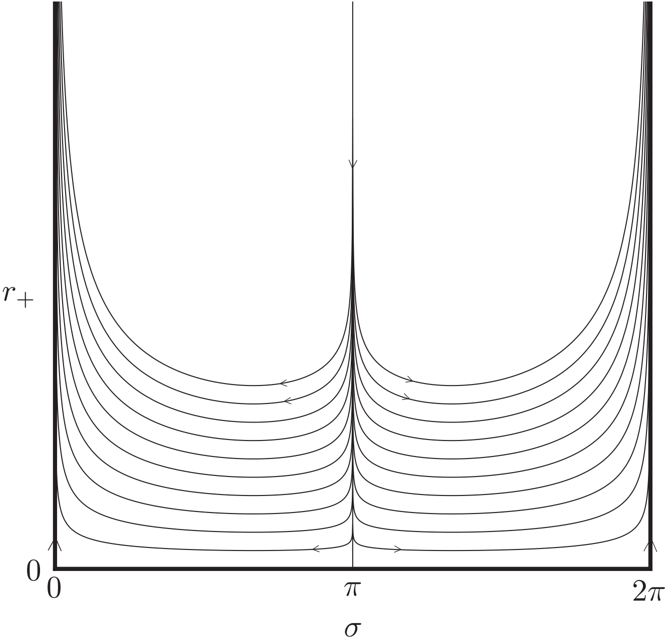
Then there are 6 patching equations: on the values of the coefficients of v_+, v_-, v_3 at $x = 0$, and the first derivatives.

After elementary but laborious calculations, I get

$$\dot{r}_+ = r_+^3(1 + 2\cos\sigma) + O(r_+^5)$$

$$\dot{\sigma} = -8 r_+^2 \sin\sigma + O(r_+^4)$$

The topology of the flow is easiest understood by looking at the flow lines.


r_+
0
0
π
2π
σ

The stable loop

There is a stable loop consisting of two branches.

One branch consists of the line of fixed points at $r_+ = 0$ from $\sigma = \pi$ to $\sigma = 0$, then the outgoing trajectory along the vertical axis at $\sigma = 0$.

The second branch consists of the line of fixed points at $r_+ = 0$ from $\sigma = \pi$ to $\sigma = 2\pi$, then out along the vertical axis at $\sigma = 2\pi$.

Recall that the two vertical axes, $\sigma = 0$ and $\sigma = 2\pi$ are gauge equivalent, under a non-trivial gauge transformation.

I strongly suspect that the outgoing flows at $\sigma = 0$ and $\sigma = 2\pi$ end at the flat connection.

Geometry

The metric inherited from the space of connections $\mathcal{A}$ is, to leading order in r_+,

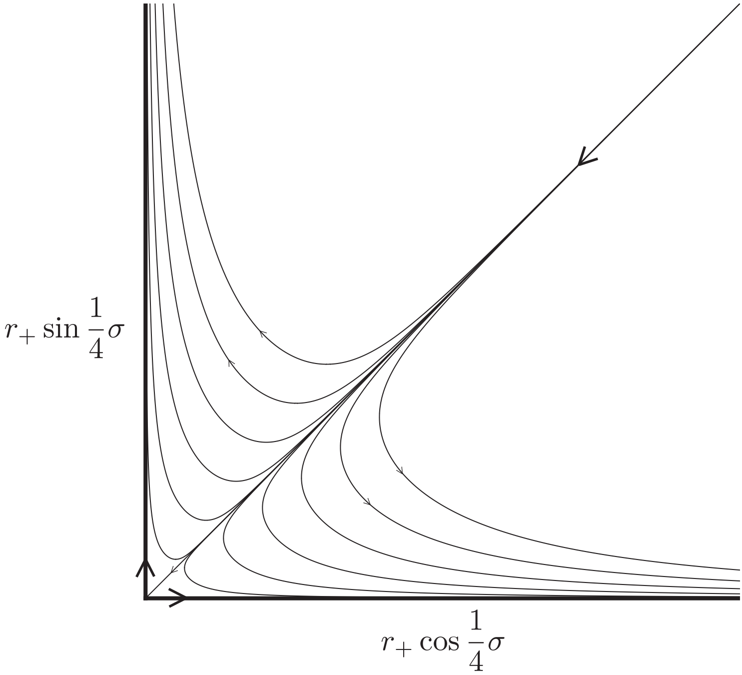
$$(ds)^2 = 64\left[(dr_+)^2 + r_+^2\left(d\sigma'\right)^2\right] \qquad \sigma' = \frac{1}{4}\sigma \qquad 0 \le \sigma' \le \frac{\pi}{2}$$

The flow is the gradient flow wrt this metric for

$$S_{YM} = 2 - 16r_+^4(1 + 2\cos\sigma) + O(r_+^6)$$

$\mathcal{A}/\mathcal{G}$ is a cone: the upper right quadrant of the plane, the positive x-axis identified with the positive y-axis.

The space of zero-size twisted pairs lives at the vertex of the cone, at $r_+ = 0$.



$r_+ \sin \frac{1}{4}\sigma$
$r_+ \cos \frac{1}{4}\sigma$

The outgoing trajectory

The outgoing trajectory at $\sigma = 0$ (or $\sigma = 2\pi$) is the flow of connections of the form

$$D = d + f(x)\Omega \qquad f(\pm\infty) = 0, \quad f(x) = f(-x)$$

generated by

$$\frac{df}{dt} = \partial_x^2 f - 4f(1-f)(2f-1)$$

with initial conditions

$$f(x) \to \frac{1}{1 + r_+(t)^2 e^{2|x|}} \qquad t \to -\infty$$

$$r_+(t)^2 = (-6t)^{-1} + O(t^{-2})$$

$$\frac{df}{dt} = \partial_x^2 f - 4f(1-f)(2f-1)$$

is a nonlinear diffusion equation, a special case of the FitzHugh–Nagumo equation

$$\partial_t u = \partial_x^2 u - u(u-1)(u-a) \qquad \text{at } \; a = \frac{1}{2}\,.$$

This is not integrable, but some exact solutions are known, including travelling shock waves which become static when $a = \frac{1}{2}$. These are our instanton and anti-instanton.

The calculation described above guarantees an early time solution that starts as a widely separated shock-anti-shock pair moving slowly towards each other with a speed that goes as $\frac{1}{-2t}$.

Does this solution exist for all t? As $t \to +\infty$, does the shock-anti-shock pair annihilate to $f = 0$ (the flat connection)?

Questions

- What is the outgoing trajectory from the loop of twisted pairs located at separation R in euclidean $\mathbb{R}^4$? How does it approach the flat connection?

  I expect that any low energy states in the lambda model would come from this asymptotic approach to the flat connection on $\mathbb{R}^4$.

- Global stability? Is the attracting basin open and dense? Are there any other locally stable loops?

- Stable 2-spheres (lambda instantons)?
  - $\pi_2(\mathcal{A}/\mathcal{G}) = \pi_5 G$
  - $\pi_5 SU(3) = \mathbb{Z}$. $G_2/SU(3) = S^6$ is a model. Numerics are essentially the same [DF, Pisa Workshop on Geometric Flows, June 2009]. The space of twisted pairs in $SU(3)$ contains a $\mathbf{CP}^1$. The stability calculation has not yet been done.
  - $\pi_5 SU(2) = \mathbb{Z}_2$. I have no idea what a stable 2-sphere might look like. Is there a homogeneous model? The model in the literature,
$$\Sigma H \circ \Sigma^2 H : S^5 \to S^4 \to S^3$$
does not seem useful.
- Ricci flow
  - $\pi_0 Diff(S^4) = 0$? (no exotic 5-spheres?)
  - $\pi_2(Metrics/Diff(S^4)) = 0$?