Quasi Riemann Surfaces

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Introduction

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Quasi Riemann surfaces, in preparation

rough definition

a *quasi Riemann surface* is a space with the properties of the space of integral currents in a Riemann surface.

as speculative physics:

the geometric setting for a new kind of quantum field theory — for every 2d qft there is to be a qft of $(n-1)$-dimensional extended objects in a space-time of dimension $d = 2n$

as speculative mathematics:

possibly a new application of analysis to the theory of manifolds

Vocabulary

M = a manifold of dimension $d = 2n$
compact and without boundary
endowed with an orientation and a conformal structure
example: $S^d = \mathbb{R}^d \cup \{\infty\}$

k-forms $\omega = \omega_{\mu_1 \cdots \mu_k}(x) dx^{\mu_1} \cdots dx^{\mu_k}$ $\quad k = 0, 1, \ldots, d$

$\mathcal{D}_k^{distr}(M)$ = the k-currents $\xi = \xi^{\mu_1 \cdots \mu_k}(x) d^d x$ $\quad k = 0, 1, \ldots, d$
the distributions (linear functionals) on k-forms

$$\int_\xi \omega = \int_M \omega_{\mu_1 \cdots \mu_k}(x) \xi^{\mu_1 \cdots \mu_k}(x) d^d x$$

∂ = the boundary operator $\mathcal{D}_k^{distr}(M) \to \mathcal{D}_{k-1}^{distr}(M)$

$$(\partial \xi)^{\mu_2 \cdots \mu_k} = -\partial_{\mu_1} \xi^{\mu_1 \mu_2 \cdots \mu_k}$$

$$\int_\xi d\omega = \int_{\partial \xi} \omega \qquad \partial\partial = 0$$

Vocabulary (2)

$\mathcal{D}_k^{sing}(M)$ = the singular k-currents, representing singular k-chains

$\sigma = \sum_i n_i \sigma_i \qquad \sigma_i \colon \Delta_i \to M$

$$\int_\sigma \omega = \sum_i n_i \int_{\Delta_i} \sigma_i^* \omega$$

examples: k-submanifolds

$\mathcal{D}_k^{int}(M)$ = the integral k-currents (TBD)

a complete metric space

the completion of $\mathcal{D}_k^{sing}(M)$ in the *flat* metric

currents are close if they differ by a small deformation

Vocabulary (3)

$I_M(\xi_1, \xi_2)$ = the intersection form on currents
non-zero only if $k_1 + k_2 = d$

$$I_M(\xi_1, \xi_2) = \int_M \frac{1}{k_1! k_2!} \epsilon_{\mu_1 \cdots \nu_1 \cdots}(x) \xi_1^{\mu_1 \cdots}(x) \xi_2^{\nu_1 \cdots}(x) d^d x$$

defined on almost all currents
integer values on singular currents

$*$ = the Hodge $*$-operator on n-forms and n-currents
conformally invariant (expresses conformal structure)
$*^2 = (-1)^n$

$$(*\xi)^{\mu_1 \cdots \mu_n}(x) = \tfrac{1}{n!} \epsilon^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n}(x) \xi^{\nu_1 \cdots \nu_n}(x)$$

The basic example

Σ = a compact Riemann surface without boundary
a conformal manifold of dimension $d = 2$
example: the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$

$I_\Sigma(\eta_1, \eta_2)$ = the intersection form on currents in Σ
non-zero only if $k_1 + k_2 = 2$

J = Hodge-$*$ on 1-forms and on 1-currents
$J^2 = -1 \qquad Jdz = idz \qquad Jd\bar{z} = -id\bar{z}$

$P_\pm$ = projections $\frac{1}{2}(1 \mp iJ)$

The basic example (2)

The bi-augmented chain complex of Σ

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, equal] & & & & \mathbb{Z} \arrow[d, equal] & \\
0 & \mathcal{D}_{-1}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{0}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{1}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{2}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{3}^{sing} \arrow[l, "\partial"'] & 0 \arrow[l, "\partial"']
\end{tikzcd}$$

$$\mathcal{D}_k^{sing} = \mathcal{D}_k^{sing}(\Sigma)$$

On the left, the boundary operator on 0-currents is

$$\partial \eta_0 = \int_{\eta_0} 1 \qquad\qquad \partial \sum_i n_i \delta_{x_i} = \sum_i n_i$$

On the right,

$$\partial \colon \mathbb{Z} \to \mathcal{D}_2^{sing}(\Sigma) \qquad\qquad \partial 1 = \Sigma$$

$\partial\partial = 0$ on the right because $\partial \Sigma = 0$,
on the left because $\partial \sigma_1 = \delta_{x_1} - \delta_{x_0}$

Motivation: generalized Maxwell electromagnetism

F is an n-form on M (e.g., a 2-form in $d = 4$ space-time)
Maxwell's equations

$$dF = 0 \qquad d(*F) = 0$$

$(n{-}1)$-form gauge potentials

$$dA = F \qquad dA^* = *F$$

$(n{-}2)$-form gauge symmetries

$$A \to A + df \qquad A^* \to A^* + df^*$$

A, A^* as scalar fields (functions, 0-forms) on $\mathcal{D}^{sing}_{n-1}(M)$

$$\phi(\xi) = \int_\xi A \qquad \phi^*(\xi) = \int_\xi A^*$$

Motivation (2)

gauge transformations of the scalar fields

$$\phi(\xi) \to \phi(\xi) + \int_{\xi} df = \phi(\xi) + \int_{\partial\xi} f = \phi(\xi) + f(\partial\xi)$$

$$\phi(\xi) \to \phi(\xi) + f(\partial\xi) \qquad \phi^*(\xi) \to \phi^*(\xi) + f^*(\partial\xi)$$

so, if we fix some $(n{-}2)$-boundary $\partial\xi$ and restrict to the ξ' with $\partial\xi' = \partial\xi$, then ϕ and ϕ^* are transformed by the two numbers $f(\partial\xi)$ and $f^*(\partial\xi)$.

The gauge symmetry of the $(n{-}1)$-form gauge potentials becomes a $U(1) \times U(1)$ symmetry of the scalar fields ϕ and ϕ^*.

Want to do field theory on the spaces of relative $(n{-}1)$-cycles in M

$$\mathcal{D}^{sing}_{n-1}(M)_{\mathbb{Z}\partial\xi} = \left\{ \xi' \in \mathcal{D}^{sing}_{n-1}(M) \colon \partial\xi' \in \mathbb{Z}\partial\xi \right\}$$

$\mathcal{D}^{sing}_{n-1}(M)$ = the extended objects in space-time M

Motivation (3)

tangent vectors in $\mathcal{D}^{sing}_{n-1}(M)_{\mathbb{Z}\partial\xi}$ = small deformations = small integral n-currents in M

so an n-form on M gives a 1-form on $\mathcal{D}^{sing}_{n-1}(M)_{\mathbb{Z}\partial\xi}$

F and F^* become 1-forms j and j^* on $\mathcal{D}^{sing}_{n-1}(M)_{\mathbb{Z}\partial\xi}$

$$d\phi = j \qquad d\phi^* = j^*$$

looks very much like the basic 2-d conformal field theory, the free 1-form aka the free massless scalar aka the $c = 1$ gaussian model

essentially all 2-d quantum field theories can be constructed from the $c = 1$ gaussian model

want to do "2-d" qft on the spaces $\mathcal{D}^{sing}_{n-1}(M)_{\mathbb{Z}\partial\xi}$ to get new qfts in space-time dimensions $d = 2n$

want function theory on the spaces $\mathcal{D}^{sing}_{n-1}(M)_{\mathbb{Z}\partial\xi}$ analogous to function theory on Riemann surfaces

Observations of an analogy between Σ and M

1. The intersection form on currents in M pairs k_1-currents and k_2-currents when $(k_1 - n + 1) + (k_2 - n + 1) = 2$.

   $$\begin{array}{ccc} \mathcal{D}_0^{sing}(\Sigma) & \longleftrightarrow & \mathcal{D}_{n-1}^{sing}(M) \\ \mathcal{D}_1^{sing}(\Sigma) & \longleftrightarrow & \mathcal{D}_{n}^{sing}(M) \\ \mathcal{D}_2^{sing}(\Sigma) & \longleftrightarrow & \mathcal{D}_{n+1}^{sing}(M) \end{array}$$

2. We can define a J-operator

   $$J \colon \mathcal{D}_n^{distr}(M) \to \mathcal{D}_n^{distr}(M) \qquad J^2 = -1$$

   by

   $$J = \epsilon_n * \qquad \epsilon_n^2 = (-1)^{n-1}$$

   When n is even, J is imaginary, so we have to go to complex currents to have $J^2 = -1$ for all n.

Structures on complex currents in M

Define the sesquilinear form $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ on complex currents

$$I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = \epsilon_{n,k_2-n} I_M(\bar{\xi}_1,\xi_2) \qquad \epsilon_{n,k'} = (-1)^{nk'+k'(k'+1)/2}\,\epsilon_n^{-1}\,.$$

The properties of $J=\epsilon_n *$ and $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ are the same for all d.

- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = -\,\overline{I_M\langle\,\bar{\xi}_2,\xi_1\,\rangle}$ skew-hermitian
- $I_M\langle\,\overline{\partial\xi}_1,\xi_2\,\rangle + I_M\langle\,\bar{\xi}_1,\partial\xi_2\,\rangle = 0$
- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle \neq 0$ only if $(k_1-n+1)+(k_2-n+1)=2$
- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ is densely defined
- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ is nondegenerate
- $I_M\langle\,\bar{\xi}_1,J\xi_2\,\rangle$ is hermitian and positive definite on $k-n+1=1$ forms

n odd

For simplicity, assume n odd, so J is real, so all currents can be taken real, and $I_M\langle\, \xi_1, \xi_2 \,\rangle$ is a skew-symmetric bilinear form.

Suppose $\partial\xi$ is an $(n{-}2)$-boundary, $\quad \partial\xi \in \partial\mathcal{D}^{sing}_{n-1}(M)$

Define the abelian group

$$\mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi} = \left\{ \xi' \in \mathcal{D}^{sing}_{n-1}(M) \colon \partial\xi' \in \mathbb{Z}\partial\xi \right\}$$

so we have

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, equal] & & & & \\
0 & \mathbb{Z}\partial\xi \arrow[l, "\partial"'] & \mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi} \arrow[l, "\partial"'] & \mathcal{D}^{sing}_{n} \arrow[l, "\partial"'] & \mathcal{D}^{sing}_{n+1} \arrow[l, "\partial"'] & \mathcal{D}^{sing}_{n+2} \arrow[l, "\partial"']
\end{tikzcd}$$

Now divide $\mathcal{D}^{sing}_{n+1}$ and $\mathcal{D}^{sing}_{n+2}$ by the null spaces wrt the intersection form with $\mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi}$ and $\mathbb{Z}\partial\xi$ respectively, to get

The quasi Riemann surfaces

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, equal] & & & & \mathbb{Z} \arrow[d, equal] & \\
0 & \mathcal{Q}^{sing}_{-1} \arrow[l] & \mathcal{Q}^{sing}_{0} \arrow[l, "\partial" above] & \mathcal{Q}^{sing}_{1} \arrow[l, "\partial" above] & \mathcal{Q}^{sing}_{2} \arrow[l, "\partial" above] & \mathcal{Q}^{sing}_{3} \arrow[l, "\partial" above] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{aligned}
\mathcal{Q}^{sing}_{-1} &= \mathbb{Z}\partial\xi && \subset \partial\mathcal{D}^{sing}_{n-1}(M) \subset \mathcal{D}^{sing}_{n-2}(M) \\
\mathcal{Q}^{sing}_{0} &= \mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi} && \subset \mathcal{D}^{sing}_{n-1}(M) \\
\mathcal{Q}^{sing}_{1} &= \mathcal{D}^{sing}_{n}(M) && \\
\mathcal{Q}^{sing}_{2} &= \mathcal{D}^{sing}_{n+1}(M)/\mathcal{N}^{sing}_{n+1}(\mathbb{Z}\partial\xi) && \\
\mathcal{Q}^{sing}_{3} &= \mathcal{D}^{sing}_{n+2}(M)/\mathcal{N}^{sing}_{n+2}(\mathbb{Z}\partial\xi) &&
\end{aligned}$$

with a non-denegerate skew form $I_{\mathcal{Q}}\langle\, \eta_1, \eta_2 \,\rangle$, and a J-operator on $\mathcal{Q}^{distr}_1 = \mathcal{D}^{distr}_n(M)$, with exactly the same properties as those of a Riemann surface Σ

The bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$ of quasi Riemann surfaces

Call this quasi Riemann surface $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$

Let

$$\mathcal{PB}(M) = \left\{ \mathbb{Z}\partial\xi \subset \partial\mathcal{D}_{n-1}^{sing}(M) \right\} = P_{\mathbb{Z}}\partial\mathcal{D}_{n-1}^{sing}(M)$$

be the space of "integer lines" in the abelian group $\partial\mathcal{D}_{n-1}^{sing}(M)$

The quasi Riemann surfaces $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$ are the fibers of a bundle

$$\mathcal{Q}(M) \to \mathcal{PB}(M)$$

of quasi Riemann surfaces, naturally associated to M.

Want function theory on a quasi Riemann surface

Riemann surface Σ:

$$\begin{tikzcd}
& \mathbb{Z} \arrow[d, equal] & & & & \mathbb{Z} \arrow[d, equal] & \\
0 & \mathcal{D}_{-1}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{0}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{1}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{2}^{sing} \arrow[l, "\partial"'] & \mathcal{D}_{3}^{sing} \arrow[l, "\partial"'] & 0 \arrow[l, "\partial"']
\end{tikzcd}$$

manifold M:

$$\begin{tikzcd}
& \mathbb{Z} \arrow[d, equal] & & & & \mathbb{Z} \arrow[d, equal] & \\
0 & \mathcal{Q}_{-1}^{sing} \arrow[l] & \mathcal{Q}_{0}^{sing} \arrow[l, "\partial"'] & \mathcal{Q}_{1}^{sing} \arrow[l, "\partial"'] & \mathcal{Q}_{2}^{sing} \arrow[l, "\partial"'] & \mathcal{Q}_{3}^{sing} \arrow[l, "\partial"'] & 0 \arrow[l]
\end{tikzcd}$$

goal: construct quantum field theories on $\mathcal{Q}_0^{sing} = \mathcal{D}_{n-1}^{sing}(M)_{\mathbb{Z}\partial\xi}$

function theory on $\Sigma \longrightarrow$ function theory on $\mathcal{D}_0^{sing}(\Sigma)$

$\longrightarrow$ function theory on $\mathcal{Q}_0^{sing}$ recall $G(\eta_0, \partial\eta_2) = I_\Sigma(\eta_0, \eta_2)$

The Cauchy kernel

The Cauchy kernel

$$G(w;z)dz = \frac{1}{2\pi i}\,\frac{1}{z-w}\,dz$$

(used as a basic 2-point function in 2d conformal field theory)
If R is a nice region in $\mathbb{R}^2$, say a disk,

$$\oint_{\partial R} G(w;z)dz = \ 1 \text{ if } w \in R, \quad 0 \text{ if } w \notin R$$

For fixed w, think of the 1-form $G(w;z)dz$ as a function on 1-currents

$$G(\delta_w, \eta_1) = \int_{\eta_1} G(w;z)dz$$

The Cauchy kernel can be considered as a function $G(\eta_0, \eta_1)$ of a 0-current and a 1-current characterized by

$$G(\eta_0, \partial\eta_2) = I_{\mathbb{R}^2}(\eta_0, \eta_2)$$

All of function theory of 1-complex variable can be expressed in terms of currents, the intersection forma and the J-operator.

Integral currents

want calculus on $\mathcal{D}_0^{sing}(\Sigma)$ and on $\mathcal{Q}_0^{sing} = \mathcal{D}_{n-1}^{sing}(M)_{\mathbb{Z}\partial\xi}$

GMT: put a metric on $\mathcal{D}_k^{sing}$ and complete $\longrightarrow \quad \mathcal{D}_k^{int}$

define normed vector space of flat currents

$$\mathcal{D}_k^{sing} \subset \mathcal{D}_k^{flat} \subset \mathcal{D}_k^{distr}$$

flat currents = measure-like distributions, that take no derivatives

completion of $\mathcal{D}_k^{sing}$ in the flat metric is $\mathcal{D}_k^{int}$

$$\mathcal{D}_k^{sing} \subset \mathcal{D}_k^{int} \subset \mathcal{D}_k^{flat} \subset \mathcal{D}_k^{distr}$$

the flat metric:

$$M(\xi) = k\text{-volume of } k\text{-current } \xi$$

$$\|\xi\|_{flat} = \inf_{\xi'} \left[M(\xi - \partial\xi') + M(\xi') \right]$$

Currents in spaces of integral currents

currents can be defined in any complete metric space, also singular, flat, and integral currents: $\mathcal{D}_j^{int}(\mathcal{D}_k^{int})$

$\exists$ natural maps $\Pi_*^{j,k} \colon \mathcal{D}_j^{int}(\mathcal{D}_k^{int}) \to \mathcal{D}_{j+k}^{int}$

because

$$\Delta_j \to (\Delta_k \to M) \quad = \quad \Delta_j \times \Delta_k \to M$$

and $\Delta_j \times \Delta_k$ is a singular $(j+k)$-chain

so every quasi Riemann surface $\mathcal{Q}$ comes with natural maps

$$\Pi_*^{j,k} \colon \mathcal{D}_j^{int}(\mathcal{Q}_k^{int}) \to \mathcal{Q}_{j+k}^{int}$$

in particular,

$$\Pi_*^{j,0} \colon \mathcal{D}_j^{int}(\mathcal{Q}_0^{int}) \to \mathcal{Q}_j^{int}$$

can be used to pull back the intersection form $I_{\mathcal{Q}}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$ and J to give a skew-hermitian form and a J-operator on j-currents in $\mathcal{Q}_0^{int}$

The J-operator on $\mathcal{D}_1^{distr}(\mathcal{Q}_0^{int})$

To talk of $(1,0)$-forms and holomorphic 1-forms on $\mathcal{Q}_0^{int}$, we need a J-operator that acts on $\mathcal{D}_1^{distr}(\mathcal{Q}_0^{int})$

A tangent vector in $\mathcal{Q}_0^{int}$ is an infinitesimal 1-simplex = tiny arrow

$\Pi_*^{1,0}$ maps that 1-simplex to a tiny element of $\mathcal{Q}_1^{int}$.

For a Riemann surface Σ: $\mathcal{Q}_1^{int} = \mathcal{D}_1^{int}(\Sigma)$, so this is a tiny integral 1-current in Σ. and J takes it to another such.

For a manifold M: $\mathcal{Q}_1^{int} = \mathcal{D}_n^{int}(\Sigma)$, so this tangent vector is a tiny integral n-current in M.

It is not obvious that $J = \epsilon_n *$ takes this to another such.

I have "proved" this. The argument made crucial use of the metric closure in the flat metric. This is my main motivation for adopting the integral currents.

Speculation on the classification

I speculate that the quasi Riemann surfaces can be classified by data analogous to the Jacobian of an ordinary Riemann surface — the integer homology group H_1 as a lattice in a finite dimensional complex Hilbert space.

This would mean that the $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$ would all be isomorphic to the $\mathcal{Q}(\Sigma)$ for some 2-dimensional space Σ (something more general than a Riemann surface).

Such isomorphisms would

- give the possibility of directly transfering 2-d quantum field theories on Σ to $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$.
- lead to pictures in which the each bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$ of quasi Riemann surfaces is naturally embedded in a universal homogeneous bundle of quasi Riemann surfaces

Quantum field theory

All this started with consideration of the free quantum field theory of an n-form F, satisfying the field equations

$$dF = 0 \qquad d(*F) = 0$$

generalizing $d = 4$ Maxwell electromagnetism.

The map

$$\Pi_*^{1,n-1} \colon \mathcal{D}_1^{int}(\mathcal{D}_{n-1}^{int}(M)) \to \mathcal{D}_n^{int}(M)$$

interprets the n-form F on M as a 1-form on $\mathcal{D}_{n-1}^{int}(M)$.

The qft becomes formally identical to the 2-d cft of the free 1-form — the $c = 1$ gaussian model.

The $(n{-}1)$-dimensional extended objects of the n-form theory are just the vertex operators of the 2-d cft.

Essentially all of 2-d qft can be built on the free 1-form theory.