Quasi Riemann Surfaces

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Introduction

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Quasi Riemann surfaces, in preparation

rough definition

a *quasi Riemann surface* is a space with the properties of the space of integral currents in a Riemann surface.

as speculative physics:

the geometric setting for a new kind of quantum field theory — for every 2d qft there is to be a qft of $(n-1)$-dimensional extended objects in a space-time of dimension $d = 2n$

as speculative mathematics:

possibly a new application of analysis to the theory of manifolds

1-forms on $\mathbb{R}^2$ are integrated over 1-d objects

coordinates: $x = (x^1, x^2)$ $\quad (\mathbb{R}^2 = \mathbb{C}, \quad z = x^1 + ix^2)$

a 1-form: $\omega = \omega_\mu(x) dx^\mu = \omega_1(x^1, x^2) dx^1 + \omega_2(x^1, x^2) dx^2$

a 1-simplex (all simplices are oriented):

$$\Delta_1 = [0, 1] \qquad \sigma : \Delta_1 \to \mathbb{R}^2 \qquad \sigma(t) = (\sigma^1(t), \sigma^2(t))$$

pull the 1-form ω on $\mathbb{R}^2$ back to a 1-form $\sigma^*\omega$ on Δ_1

$$\sigma^*\omega = \sigma^*\omega(t) dt = \omega_\mu(\sigma(t)) d\sigma^\mu(t) = \omega_\mu(\sigma(t)) \frac{d\sigma^\mu(t)}{dt} dt$$

integrate over Δ_1

$$\int_{\Delta_1} \sigma^*\omega = \int_0^1 \sigma^*\omega(t) dt = \int_0^1 \omega_\mu(\sigma(t)) \frac{d\sigma^\mu(t)}{dt} dt$$

independent of changes of coordinate in x or t if

$$x \mapsto y(x) \qquad \omega_\mu(x) dx^\mu = \omega_\mu(y) dy^\mu \qquad dy^\mu = \frac{\partial y^\mu}{\partial x^\nu} dx^\nu$$

Distributions on $\mathbb{R}^2$

distributions = linear functionals on the smooth functions

$$f \in C^\infty(\mathbb{R}^2) \qquad \eta\colon f \mapsto (\eta, f) \in \mathbb{R}$$

aka generalized functions

$$(\eta, f) = \int_{\mathbb{R}^2} f(x)\, \eta(x) d^2x \qquad \eta = \eta(x) d^2x$$

examples: the Dirac delta-function

$$(\delta_{x_1}, f) = f(x_1) \qquad \delta_{x_1} = \delta^2(x - x_1) d^2x$$

derivatives of delta-functions

$$\partial_\mu \delta_{x_1} = \partial_\mu \delta^2(x - x_1) d^2x \qquad (\partial_\mu \delta_{x_1}, f) = -\partial_\mu f(x_1)$$

$$(\partial_\mu \delta_{x_1}, f) = \int_{\mathbb{R}^2} f(x) \partial_\mu \delta^2(x - x_1) d^2x = \int_{\mathbb{R}^2} (-\partial_\mu f(x)) \delta^2(x - x_1) d^2x$$

1-currents are distributions on 1-forms

$$\int_{\Delta_1} \sigma^* \omega = \int_0^1 \sigma^* \omega(t) dt = \int_0^1 \omega_\mu(\sigma(t)) \frac{d\sigma^\mu(t)}{dt} dt$$

$$= \int_{\mathbb{R}^2} \omega_\mu(x)\, \sigma^\mu(x) d^2x$$

$$\sigma^\mu(x) d^2x = \left(\int_0^1 \delta^2(x - \sigma(t)) \frac{d\sigma^\mu(t)}{dt} dt \right) d^2x$$

$\sigma^\mu(x) d^2x$ is a 1-current = a distribution on 1-forms

write

$$\int_\sigma \omega = \int_{\mathbb{R}^2} \omega_\mu(x)\, \sigma^\mu(x) d^2x = \int_{\Delta_1} \sigma^* \omega$$

Singular 1-currents represent singular 1-chains

a singular 1-chain = a formal sum of 1-simplices

$$\eta = \sum_i n_i \sigma_i \qquad \sigma_i\colon \Delta_1 \to \mathbb{R}^2 \qquad n_i \in \mathbb{Z}$$

represented by the *singular* 1-current

$$\eta^\mu(x) d^2x = \sum_i n_i\, \sigma_i^\mu(x) d^2x$$

$$\int_\eta \omega = \int_{\mathbb{R}^2} \omega_\mu(x)\eta^\mu(x) d^2x = \sum_i n_i \int_{\sigma_i} \omega$$

$\mathcal{D}_1^{sing}(\mathbb{R}^2)$ = the space of singular 1-currents in $\mathbb{R}^2$

an abelian group — closed under addition and subtraction

Many different 1-chains are represented by the same 1-current.
The singular 1-current *is* the 1-d object.

Singular 0-currents represent singular 0-chains

0-forms $\omega = \omega(x) = \omega(x^1, x^2)$ are just functions

a 0-simplex $\Delta_0 = \{0\}$ $\sigma_0 : \Delta_0 \to \mathbb{R}^2$ is a point $\sigma_0(0) \in \mathbb{R}^2$

a 0-form ω on $\mathbb{R}^2$ pulls back to a 0-form $\sigma_1^* \omega$ on Δ_0, a number

integrate $\sigma_0^* \omega$ over Δ_0

$$\int_{\Delta_0} \sigma^* \omega = \omega(\sigma_0(0)) = \int_{\mathbb{R}^2} \omega(x) \sigma_0(x)\, d^2 x = (\delta_{\sigma_0(0)}, \omega) = \int_{\sigma_0} \omega$$

$\sigma_0(x) d^2 x = \delta_{\sigma_0(0)}$ is a 0-current, a distribution on 0-forms

singular 0-chains $\eta_0 = \sum_i n_i \sigma_{0,i}$ $\sigma_{0,i} : \Delta_0 \to \mathbb{R}^2$ $n_i \in \mathbb{Z}$

are represented by singular 0-currents $\eta(x) d^2 x = \sum_i n_i \delta_{\sigma_{0,i}(0)}$

Singular 2-currents represent singular 2-chains

2-forms $\quad \omega = \frac{1}{2}\omega_{\mu_1\mu_2}(x) dx^{\mu_1} dx^{\mu_2} \quad dx^{\mu_1} dx^{\mu_2} = -dx^{\mu_2} dx^{\mu_1}$

a 2-simplex $\quad \Delta_2 =$ a triangle $\quad \sigma_2 : \Delta_2 \to \mathbb{R}^2 \quad \sigma_2(t^1, t^2) \in \mathbb{R}^2$

a 2-form ω on $\mathbb{R}^2$ pulls back to a 2-form $\sigma_2^* \omega(t_1, t_2) dt^1 dt^2$ on Δ_2,

$$dx^\mu = \frac{\partial x^\mu}{\partial t^a} dt^a$$

which we can integrate over Δ_2

$$\int_{\Delta_2} \sigma_2^* \omega(t_1, t_2) dt^1 dt^2 = \int_{\mathbb{R}^2} \frac{1}{2} \omega_{\mu_1\mu_2}(x) \sigma_2^{\mu_1\mu_2}(x)\, d^2x = \int_{\sigma_2} \omega$$

$\sigma_2^{\mu_1\mu_2}(x)\, d^2x$ is a 2-current, a distribution on 2-forms

singular 2-chains $\quad \eta_2 = \sum_i n_i \sigma_{2,i} \quad \sigma_{2,i} \colon \Delta_2 \to \mathbb{R}^2 \quad n_i \in \mathbb{Z}$

are represented by singular 2-currents $\eta_2 = \eta_2^{\mu_1\mu_2}(x) d^2x$

k-currents

Now we have the vector space of distributional k-currents on $\mathbb{R}^2$

$$\mathcal{D}_k^{distr}(\mathbb{R}^2) = \left\{\eta^{\mu_1 \cdots \mu_k}(x) d^2x\right\} \qquad k = 0, 1, 2$$

containing the abelian group of singular k-currents

$$\mathcal{D}_k^{sing}(\mathbb{R}^2) \subset \mathcal{D}_k^{distr}(\mathbb{R}^2) \qquad k = 0, 1, 2$$

There are also some useful vector subspaces of $\mathcal{D}_k^{distr}(\mathbb{R}^2)$.

For example, there are the *smooth* currents, which have the $\eta^{\mu_1 \cdots \mu_k}(x)$ infinitely differentiable.

The boundary operator ∂

The boundary of the k-simplex is a sum of $(k-1)$-simplices, which gives a boundary operator taking a k-simplex to a $(k-1)$-chain

$$\sigma_k \colon \Delta_k \to \mathbb{R}^2 \qquad \partial \sigma_k = (\sigma_k)_{/\partial \Delta_k}$$

which extends by additivity to a boundary operator on k-chains, which gives a boundary operator on the singular k-currents

$$\mathcal{D}_0^{sing}(\mathbb{R}^2) \overset{\partial}{\leftarrow} \mathcal{D}_1^{sing}(\mathbb{R}^2) \overset{\partial}{\leftarrow} \mathcal{D}_2^{sing}(\mathbb{R}^2) \qquad \partial\partial = 0$$

For example,

$$\partial \Delta_1 = \partial([0,1]) = \{1\} - \{0\}$$

so the boundary of a 1-simplex is a 0-chain

$$\sigma_1 \colon [0,1] \to \mathbb{R}^2 \qquad \partial \sigma_1 = \delta_{\sigma_1(1)} - \delta_{\sigma_1(0)}$$

Stokes' theorem

The exterior derivative takes 0-forms to 1-forms, 1-forms to 2-forms

$$(d\omega)_\mu = \partial_\mu \omega \qquad (d\omega)_{\mu_1\mu_2} = \partial_{\mu_1}\omega_{\mu_2} - \partial_{\mu_2}\omega_{\mu_1}$$

Integration by parts on each k-simplex gives Stokes' theorem

$$\int_{\partial\eta} \omega = \int_\eta d\omega$$

Writing η as a distribution, and integrating by parts,

$$(\partial\eta)(x)d^2x = -\partial_{\mu_1}\eta^{\mu_1}(x)d^2x \qquad (\partial\eta)^{\mu_2}(x)d^2x = -\partial_{\mu_1}\eta^{\mu_1\mu_2}(x)d^2x$$

so ∂ makes sense as an operator on distributional currents

$$\mathcal{D}_0^{distr}(\mathbb{R}^2) \overset{\partial}{\leftarrow} \mathcal{D}_1^{distr}(\mathbb{R}^2) \overset{\partial}{\leftarrow} \mathcal{D}_2^{distr}(\mathbb{R}^2) \qquad \partial\partial = 0$$

Examples

a singular 1-current (a line segment)

$$\sigma(t) = (t, 0) \qquad \sigma^\mu(x) d^2x = \theta_{[0,1]}(x^1)\delta(x^2)\delta^\mu_1 d^2x$$

$$\begin{aligned}(\partial\sigma)(x)d^2x &= -\partial_\mu \theta_{[0,1]}(x^1)\delta(x^2)\delta^\mu_1 d^2x = -\partial_1 \theta_{[0,1]}(x^1)\delta(x^2) d^2x \\ &= [\delta(x^1 - 1) - \delta(x^1)]\delta(x^2) d^2x \\ &= \delta_{(1,0)} - \delta_{(0,0)}\end{aligned}$$

a singular 2-current (a disk of radius R)

$$\eta^{\mu_1\mu_2}(x) d^2x = \theta(R - |x|)\epsilon^{\mu_1\mu_2} d^2x \qquad \epsilon^{12} = 1 = -\epsilon^{21}$$

$$(\partial\eta)^{\mu_2}(x) d^2x = -\partial_{\mu_1}\eta^{\mu_1\mu_2}(x) d^2x = \delta(|x| - R)\hat{x}_{\mu_1}\epsilon^{\mu_1\mu_2} d^2x$$

which is the singular 1-current representing the circle of radius R, and which also represents the 1-simplex

$$\sigma(t) = (R\cos 2\pi t, R\sin 2\pi t)$$

Intersection number

two singular currents — η_1 a k_1-current and η_2 a k_2-current — can have an intersect number if $k_1 + k_2 = 2$

$$I_{\mathbb{R}^2}(\eta_1, \eta_2) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} \frac{1}{2}\epsilon_{\mu_1\mu_2}\eta_1(x)\eta_2^{\mu_1\mu_2}(x)d^2x & k_1 = 0,\ k_2 = 2 \\ \displaystyle\int_{\mathbb{R}^2} \epsilon_{\mu_1\mu_2}\eta_1^{\mu_1}(x)\eta_2^{\mu_1}(x)d^2x & k_1 = 1,\ k_2 = 1 \end{cases}$$

Example: $\eta_1 = \delta_{x_1} \quad \eta_2 = D_1$ the unit disk

$$I_{\mathbb{R}^2}(\delta_{x_1}, D_1) = \int_{\mathbb{R}^2} \frac{1}{2}\epsilon_{\mu_1\mu_2}\delta^2(x - x_1)\theta(1 - |x|)\epsilon^{\mu_1\mu_2}d^2x$$

$$= \quad 1 \text{ if } |x| < 1, \quad 0 \text{ if } |x| > 1, \quad ? \text{ if } |x| = 1$$

Intersection number (2)

Example: two integral 1-currents

$\eta_1 = [a_1, a_2]$ on x-axis $\qquad \eta_2 = [b_1, b_2]$ on y-axis

$\eta_1 = \theta_{[a_1,a_2]}(x^1)\delta(x^2)\delta_1^\mu d^2x \qquad \eta_2 = \delta(x^1)\theta_{[b_1,b_2]}(x^2)\delta_2^\mu d^2x$

$$I_{\mathbb{R}^2}(\eta_1, \eta_2) = \int_{\mathbb{R}^2} \epsilon_{\mu_1\mu_2}\eta_1^{\mu_1}(x)\eta_2^{\mu_1}(x)d^2x = \theta_{[a_1,a_2]}(0)\theta_{[b_1,b_2]}(0)$$
$$= 1 \text{ if intersect,} \quad = 0 \text{ if disjoint,} \quad =? \text{ if they only touch}$$

$I_{\mathbb{R}^2}(\eta_1, \eta_2)$ is a bilinear form on currents

- $\neq 0$ only if $k_1 + k_2 = 2$
- defined almost everywhere ("in general position")
- takes integer values on singular currents

Complex structure

The J-operator on 1-forms

$$Jdx^1 = -dx^2 \quad Jdx^2 = dx^1$$

$J^2 = -1$ so J has eigenspaces $J = \pm i$

$$dz = dx^1 + idx^2 \qquad d\bar{z} = dx^1 - idx^2$$

$$Jdz = J(dx^1 + idx^2) = idz \qquad Jd\bar{z} = J(dx^1 - idx^2) = -id\bar{z}$$

projection operators on 1-forms

$$P_+ = P_{(1,0)} = \frac{1}{2}(1 - iJ) \qquad P_- = P_{(0,1)} = \frac{1}{2}(1 + iJ)$$

$$P_+ dz = dz \qquad P_+ d\bar{z} = 0$$

$$P_+\omega = \omega \quad \Longleftrightarrow \quad \omega = \omega(z, \bar{z})dz$$

$$d\omega = \frac{\partial \omega}{\partial \bar{z}} d\bar{z} dz$$

$P_+\omega = \omega$ and $d\omega = 0 \quad \Longleftrightarrow \quad \omega = \omega(z)dz$

The Cauchy kernel

$$G(w;z)dz = \frac{1}{2\pi i}\,\frac{1}{z-w}\,dz$$

(used as a basic 2-point function in 2d conformal field theory)
If R is a nice region in $\mathbb{R}^2$, say a disk,

$$\oint_{\partial R} G(w;z)dz = \ 1 \text{ if } w \in R, \quad 0 \text{ if } w \notin R$$

For fixed w, think of the 1-form $G(w;z)dz$ as a function on 1-currents

$$G(\delta_w, \eta_1) = \int_{\eta_1} G(w;z)dz$$

The Cauchy kernel can be considered as a function $G(\eta_0, \eta_1)$ of a 0-current and a 1-current characterized by

$$G(\eta_0, \partial\eta_2) = I_{\mathbb{R}^2}(\eta_0, \eta_2)$$

All of function theory of 1-complex variable can be expressed in terms of currents, the intersection forma and the J-operator.

Σ is a compact Riemann surface without boundary

(= a space covered by coordinate neighborhoods that look like $\mathbb{C}$)

For example, $\Sigma =$ the Riemann sphere $\mathbb{C} \cup \{\infty\} = S^2$

have k-forms on Σ, k-chains in Σ

singular k-currents in Σ: $\mathcal{D}_k^{sing}(\Sigma)$

intersection form $I_\Sigma(\eta_1, \eta_2)$ on currents

J-operator on 1-forms

Augmented chain complex

The bi-augmented chain complex is

$$\begin{array}{ccccccccccccc} & & \mathbb{Z} & & & & & & & & \mathbb{Z} & & \\ & & \| & & & & & & & & \| & & \\ 0 & \xleftarrow{\partial} & \mathcal{D}^{sing}_{-1} & \xleftarrow{\partial} & \mathcal{D}^{sing}_{0} & \xleftarrow{\partial} & \mathcal{D}^{sing}_{1} & \xleftarrow{\partial} & \mathcal{D}^{sing}_{2} & \xleftarrow{\partial} & \mathcal{D}^{sing}_{3} & \xleftarrow{\partial} & 0 \end{array}$$

writing $\mathcal{D}^{sing}_k$ for $\mathcal{D}^{sing}_k(\Sigma)$

On the left, the boundary operator on 0-currents is

$$\partial\eta_0 = \int_{\eta_0} 1 \qquad\qquad \partial \sum_i n_i \delta_{x_i} = \sum_i n_i$$

On the right,

$$\partial\colon \mathbb{Z} \to \mathcal{D}^{sing}_2(\Sigma) \qquad \partial 1 = \Sigma$$

$\partial\partial = 0$ on the right because $\partial\Sigma = 0$,
on the left because $\partial\sigma_1 = \delta_{x_1} - \delta_{x_0}$

$\mathbb{R}^d$, $d = 2n$

k-forms, k-chains, for $k = 0, 1, 2, \ldots, d$

k-currents: $\xi = \xi^{\mu_1 \cdots \mu_k}(x) d^d x$

an integral 0-current: a point $\delta_{x_1} = \delta^d(x - x_1) d^d x$

an integral d-current: the unit ball $B_1 = \theta(1 - |x|)\epsilon^{\mu_1 \cdots \mu_d} d^d x$

intersection form $I_{\mathbb{R}^d}(\xi_1, \xi_2) \neq 0$ only if $k_1 + k_2 = d$

$$I_{\mathbb{R}^d}(\xi_1, \xi_2) = \int_{\mathbb{R}^d} \frac{1}{k_1! k_2!} \epsilon_{\mu_1 \cdots \mu_{k_1} \nu_1 \cdots \nu_{k_2}} \xi_1^{\mu_1 \cdots \mu_{k_1}}(x) \xi_2^{\nu_1 \cdots \nu_{k_2}}(x) d^d x$$

Hodge $*$-operator on n-forms and n-currents

$$(*\omega)_{\mu_1 \cdots \mu_n} = \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n}{}^{\nu_1 \cdots \nu_n} \omega_{\nu_1 \cdots \nu_n} \qquad *^2 = (-1)^n$$

e.g., for $d = 4$, $n = 2$: $(*\omega)_{12} = \omega_{34}$, $(*\omega)_{34} = \omega_{12}$,

M a conformal manifold of dimension $d = 2n$

M is a compact, oriented manifold without boundary, of dimension $d = 2n$, with a conformal structure

Think $M = \mathbb{R}^d \cup \{\infty\} = S^d$, $d = 2n$

k-forms on M, k-chains in M, $\quad k = 0, 1, \ldots, d$

singular k-currents in M: $\quad \mathcal{D}_k^{sing}(M)$

intersection form $I_M(\eta_1, \eta_2) \neq 0$ only if $k_1 + k_2 = d$

$$I_M(\xi_1, \xi_2) = \int_M \frac{1}{k_1! k_2!} \epsilon_{\mu_1 \cdots \mu_{k_1} \nu_1 \cdots \nu_{k_2}} \xi_1^{\mu_1 \cdots \mu_{k_1}}(x) \xi_2^{\nu_1 \cdots \nu_{k_2}}(x) d^d x$$

conformal structure $\Longrightarrow$ Hodge $*$-operator on n-forms

$$(*\omega)_{\mu_1 \cdots \mu_n}(x) = \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n}{}^{\nu_1 \cdots \nu_n}(x)\, \omega_{\nu_1 \cdots \nu_n}(x) \qquad *^2 = (-1)^n$$

Observations of an analogy between Σ and M

1. The intersection form on currents in M pairs k_1-currents and k_2-currents when $(k_1 - n + 1) + (k_2 - n + 1) = 2$.

$$\begin{array}{ccc} \mathcal{D}_0^{sing}(\Sigma) & \longleftrightarrow & \mathcal{D}_{n-1}^{sing}(M) \\ \mathcal{D}_1^{sing}(\Sigma) & \longleftrightarrow & \mathcal{D}_{n}^{sing}(M) \\ \mathcal{D}_2^{sing}(\Sigma) & \longleftrightarrow & \mathcal{D}_{n+1}^{sing}(M) \end{array}$$

2. We can define a J-operator

$$J : \mathcal{D}_n^{distr}(M) \to \mathcal{D}_n^{distr}(M) \qquad J^2 = -1$$

by

$$J = \epsilon_n * \qquad \epsilon_n^2 = (-1)^{n-1}$$

When n is even, J is imaginary, so we have to go to complex currents to have $J^2 = -1$ for all n.

Structures on complex currents in M

Define the sesquilinear form $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ on complex currents

$$I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = \epsilon_{n,k_2-n} I_M(\bar{\xi}_1,\xi_2) \qquad \epsilon_{n,k'} = (-1)^{nk'+k'(k'+1)/2}\,\epsilon_n^{-1}\,.$$

The properties of $J=\epsilon_n *$ and $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ are the same for all d.

- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = -\,\overline{I_M\langle\,\bar{\xi}_2,\xi_1\,\rangle}$ skew-hermitian
- $I_M\langle\,\overline{\partial\xi}_1,\xi_2\,\rangle + I_M\langle\,\bar{\xi}_1,\partial\xi_2\,\rangle = 0$
- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle \neq 0$ only if $(k_1-n+1)+(k_2-n+1)=2$
- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ is densely defined
- $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ is nondegenerate
- $I_M\langle\,\bar{\xi}_1,J\xi_2\,\rangle$ is hermitian and positive definite on $k-n+1=1$ forms

n odd

For simplicity, assume n odd, so J is real, so all currents can be taken real, and $I_M\langle\, \xi_1, \xi_2\,\rangle$ is a skew-symmetric bilinear form.

Suppose $\partial\xi$ is an $(n-2)$-boundary, $\quad \partial\xi \in \partial\mathcal{D}^{sing}_{n-1}(M)$

Define the abelian group

$$\mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi} = \left\{\xi' \in \mathcal{D}^{sing}_{n-1}(M)\colon \partial\xi' \in \mathbb{Z}\partial\xi\right\}$$

so we have

$$\begin{array}{c}\mathbb{Z} \\ || \\ 0 \xleftarrow{\ \partial\ } \mathbb{Z}\partial\xi \xleftarrow{\ \partial\ } \mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi} \xleftarrow{\ \partial\ } \mathcal{D}^{sing}_{n} \xleftarrow{\ \partial\ } \mathcal{D}^{sing}_{n+1} \xleftarrow{\ \partial\ } \mathcal{D}^{sing}_{n+2}\end{array}$$

Now divide $\mathcal{D}^{sing}_{n+1}$ and $\mathcal{D}^{sing}_{n+2}$ by the null spaces wrt the intersection form with $\mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi}$ and $\mathbb{Z}\partial\xi$ respectively, to get

The quasi Riemann surfaces

$$
\begin{array}{ccccccccccc}
 & & \mathbb{Z} & & & & & & \mathbb{Z} & & \\
 & & \| & & & & & & \| & & \\
0 & \longleftarrow & \mathcal{Q}_{-1}^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_{0}^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_{1}^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_{2}^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_{3}^{sing} & \longleftarrow & 0
\end{array}
$$

$$
\begin{aligned}
\mathcal{Q}_{-1}^{sing} &= \mathbb{Z}\partial\xi & &\subset \partial\mathcal{D}_{n-1}^{sing}(M) \subset \mathcal{D}_{n-2}^{sing}(M) \\
\mathcal{Q}_{0}^{sing} &= \mathcal{D}^{sing}(M)_{\mathbb{Z}\partial\xi} & &\subset \mathcal{D}_{n-1}^{sing}(M) \\
\mathcal{Q}_{1}^{sing} &= \mathcal{D}_{n}^{sing}(M) & & \\
\mathcal{Q}_{2}^{sing} &= \mathcal{D}_{n+1}^{sing}(M)/\mathcal{N}_{n+1}^{sing}(\mathbb{Z}\partial\xi) & & \\
\mathcal{Q}_{3}^{sing} &= \mathcal{D}_{n+2}^{sing}(M)/\mathcal{N}_{n+2}^{sing}(\mathbb{Z}\partial\xi) & &
\end{aligned}
$$

with a non-denegerate skew form $I_{\mathcal{Q}}\langle\,\eta_1, \eta_2\,\rangle$, and a J-operator on $\mathcal{Q}_1^{distr} = \mathcal{D}_n^{distr}(M)$, with exactly the same properties as those of a Riemann surface Σ

The bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$ of quasi Riemann surfaces

Call this quasi Riemann surface $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$

Let

$$\mathcal{PB}(M) = \left\{ \mathbb{Z}\partial\xi \subset \partial\mathcal{D}_{n-1}^{sing}(M) \right\} = P_{\mathbb{Z}}\partial\mathcal{D}_{n-1}^{sing}(M)$$

be the space of "integer lines" in the abelian group $\partial\mathcal{D}_{n-1}^{sing}(M)$

The quasi Riemann surfaces $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$ are the fibers of a bundle

$$\mathcal{Q}(M) \to \mathcal{PB}(M)$$

of quasi Riemann surfaces, naturally associated to M.

Want function theory on a quasi Riemann surface

Riemann surface Σ:

$$\begin{array}{ccccccccccccc} & & \mathbb{Z} & & & & & & & & \mathbb{Z} & & \\ & & \| & & & & & & & & \| & & \\ 0 & \overset{\partial}{\longleftarrow} & \mathcal{D}_{-1}^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{D}_0^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{D}_1^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{D}_2^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{D}_3^{sing} & \overset{\partial}{\longleftarrow} & 0 \end{array}$$

manifold M:

$$\begin{array}{ccccccccccccc} & & \mathbb{Z} & & & & & & & & \mathbb{Z} & & \\ & & \| & & & & & & & & \| & & \\ 0 & \longleftarrow & \mathcal{Q}_{-1}^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_0^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_1^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_2^{sing} & \overset{\partial}{\longleftarrow} & \mathcal{Q}_3^{sing} & \longleftarrow & 0 \end{array}$$

goal: construct quantum field theories on $\mathcal{Q}_0^{sing} = \mathcal{D}_{n-1}^{sing}(M)_{\mathbb{Z}\partial\xi}$

$\mathcal{D}_{n-1}^{sing}(M)$ = the extended objects in space-time M

function theory on $\Sigma \longrightarrow$ function theory on $\mathcal{D}_0^{sing}(\Sigma)$

$\longrightarrow$ function theory on $\mathcal{Q}_0^{sing}$ recall $G(\eta_0, \partial\eta_2) = I_\Sigma(\eta_0, \eta_2)$

want calculus on $\mathcal{D}_0^{sing}(\Sigma)$ and on $\mathcal{Q}_0^{sing} = \mathcal{D}_{n-1}^{sing}(M)_{\mathbb{Z}\partial\xi}$

GMT: put a metric on $\mathcal{D}_k^{sing}$ and complete $\longrightarrow$ $\mathcal{D}_k^{\text{int}}$

define normed vector space of flat currents

$$\mathcal{D}_k^{sing} \subset \mathcal{D}_k^{flat} \subset \mathcal{D}_k^{distr}$$

flat currents = measure-like distributions, that take no derivatives

completion of $\mathcal{D}_k^{sing}$ in the flat metric is $\mathcal{D}_k^{\text{int}}$

$$\mathcal{D}_k^{sing} \subset \mathcal{D}_k^{\text{int}} \subset \mathcal{D}_k^{flat} \subset \mathcal{D}_k^{distr}$$

the flat metric:

$$M(\xi) = k\text{-volume of } k\text{-current } \xi$$

$$\|\xi\|_{flat} = \inf_{\xi'} \left[M(\xi - \partial\xi') + M(\xi') \right]$$

Currents in spaces of integral currents

currents can be defined in any complete metric space, also singular, flat, and integral currents: $\mathcal{D}_j^{\text{int}}(\mathcal{D}_k^{\text{int}})$

$\exists$ natural maps $\Pi_*^{j,k}\colon \mathcal{D}_j^{\text{int}}(\mathcal{D}_k^{\text{int}}) \to \mathcal{D}_{j+k}^{\text{int}}$

because

$$\Delta_j \to (\Delta_k \to M) \quad = \quad \Delta_j \times \Delta_k \to M$$

and $\Delta_j \times \Delta_k$ is a singular $(j+k)$-chain

so every quasi Riemann surface $\mathcal{Q}$ comes with natural maps

$$\Pi_*^{j,k}\colon \mathcal{D}_j^{\text{int}}(\mathcal{Q}_k^{\text{int}}) \to \mathcal{Q}_{j+k}^{\text{int}}$$

in particular,

$$\Pi_*^{j,0}\colon \mathcal{D}_j^{\text{int}}(\mathcal{Q}_0^{\text{int}}) \to \mathcal{Q}_j^{\text{int}}$$

can be used to pull back the intersection form $I_{\mathcal{Q}}\langle\,\bar\eta_1, \eta_2\,\rangle$ and J to give a skew-hermitian form and a J-operator on j-currents in $\mathcal{Q}_0^{\text{int}}$

The J-operator on $\mathcal{D}_1^{distr}(\mathcal{Q}_0^{\mathrm{int}})$

To talk of $(1,0)$-forms and holomorphic 1-forms on $\mathcal{Q}_0^{\mathrm{int}}$, we need a J-operator that acts on $\mathcal{D}_1^{distr}(\mathcal{Q}_0^{\mathrm{int}})$

A tangent vector in $\mathcal{Q}_0^{\mathrm{int}}$ is an infinitesimal 1-simplex = tiny arrow

$\Pi_*^{1,0}$ maps that 1-simplex to a tiny element of $\mathcal{Q}_1^{\mathrm{int}}$.

For a Riemann surface Σ: $\mathcal{Q}_1^{\mathrm{int}} = \mathcal{D}_1^{\mathrm{int}}(\Sigma)$, so this is a tiny integral 1-current in Σ. and J takes it to another such.

For a manifold M: $\mathcal{Q}_1^{\mathrm{int}} = \mathcal{D}_n^{\mathrm{int}}(\Sigma)$, so this tangent vector is a tiny integral n-current in M.

It is not obvious that $J = \epsilon_n *$ takes this to another such.

I have "proved" this. The argument made crucial use of the metric closure in the flat metric. This is my main motivation for adopting the integral currents.

Speculation on the classification

I speculate that the quasi Riemann surfaces can be classified by data analogous to the Jacobian of an ordinary Riemann surface — the integer homology group H_1 as a lattice in a finite dimensional complex Hilbert space.

This would mean that the $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$ would all be isomorphic to the $\mathcal{Q}(\Sigma)$ for some 2-dimensional space Σ (something more general than a Riemann surface).

Such isomorphisms would

- give the possibility of directly transfering 2-d quantum field theories on Σ to $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi}$.
- lead to pictures in which the each bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$ of quasi Riemann surfaces is naturally embedded in a universal homogeneous bundle of quasi Riemann surfaces

Quantum field theory

All this started with consideration of the free quantum field theory of an n-form F, satisfying the field equations

$$dF = 0 \qquad d(*F) = 0$$

generalizing $d = 4$ Maxwell electromagnetism.

The map

$$\Pi_*^{1,n-1} \colon \mathcal{D}_1^{\mathrm{int}}(\mathcal{D}_{n-1}^{\mathrm{int}}(M)) \to \mathcal{D}_n^{\mathrm{int}}(M)$$

interprets the n-form F on M as a 1-form on $\mathcal{D}_{n-1}^{\mathrm{int}}(M)$.

The qft becomes formally identical to the 2-d cft of the free 1-form — the $c = 1$ gaussian model.

The $(n-1)$-dimensional extended objects of the n-form theory are just the vertex operators of the 2-d cft.

Essentially all of 2-d qft can be built on the free 1-form theory.