Global structure of euclidean quantum gravity

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Euclidean quantum gravity (EQG) separates into a local theory and a global theory. The local theory operates in every compact d-manifold with boundary to produce a state on the boundary. The global theory then sums these boundary states over the diffeomorphism classes of d-manifolds with boundary to make the Hartle-Hawking state. Global EQG is formulated here as classical statistical physics. The Hartle-Hawking state is the probability measure of a mathematically natural classical statistical system, analogous to the functional measure of euclidean quantum field theory. General principles of global EQG determine the numerical weights $w(M)$ in the sum over diffeomorphism classes M.

Advertisement of another project

A first principles theory of cosmology. A simple initial condition prior to the EW transition determines all cosmology after.

The only assumptions:

1. the Standard Model + classical General Relativity (with Λ)
2. a certain $\mathrm{Spin}(4)$ symmetry of the SM+GR fields
3. a high initial energy

What follows from these assumptions:

1. a cosmology with no free parameters
2. an expanding universe dominated by dark matter
3. a specific explanation of dark matter within the SM
4. initial thermal fluctuations
5. flatness + homogeneity
6. a mechanism for the EW transition

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The dark matter is a cosmological SU(2)-weak gauge field:

- a rapidly oscillating classical solution of the SU(2)-weak Yang-Mills-Higgs equations completely determined by the $\mathrm{Spin}(4)$ symmetry
- behaves macroscopically like a dark perfect fluid with calculable equation of state $P(\rho)$

at low density $P(\rho) = 0.5\, c^2 \dfrac{\rho^2}{\rho_b}$ cold! $\rho_b = \dfrac{c^3}{\hbar^3} m_{\mathrm{Higgs}}^4$ = weak-scale density

- this perfect fluid collapses gravitationally (Oppenheimer-Volkoff equations for $P(\rho)$) to:

dark matter stars	mass $< 10^{-5}\mathrm{M}_\odot$	Chandrasekhar limit for $P(\rho)$ is
black holes	mass $> 10^{-5}\mathrm{M}_\odot$	$0.3 \left(\frac{c^6}{4\pi G^3}\right)^{1/2} \frac{1}{\sqrt{\rho_b}} = 10^{-5}\mathrm{M}_\odot$

- all the low mass dark matter stars have the same diameter

star mass $\ll 10^{-5}\mathrm{M}_\odot$ $\Longrightarrow$ star diameter $= \sqrt{\dfrac{\pi}{G\rho_b}} = 27\,\mathrm{cm}$

- microlensing puts upper bound $10^{-11}\mathrm{M}_\odot$ on masses of compact objects in halos
  $\Longrightarrow$ halo dark matter consists of 27 cm diameter dark matter stars

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A real possibility of a complete first principles cosmology.

A cosmology to match the SM — assume only symmetry, nothing more.

No speculation about beyond the SM physics (no inflaton).

“Just one of many, many conjectural theories of dark matter.”

NO! This theory of dark matter comes out of a simple, complete theory of cosmology with all the other attributes listed earlier. There are no comparable theories of dark matter.

Advertisement (4)

The theory should be put to the test:

- Do the initial thermal fluctuations give the correct spectrum of density fluctuations?
- Is the equation of state consistent with the evolution of density fluctuations?
- Detect the cosmological gauge field by a resonance phenomenon?
  - 21 cm glow from the halo? (is 27 cm star diameter close enough to 21 cm?)
  - sitting on the W mass in a collider to detect when dark matter stars pass through?
- . . .

The theory has no free parameters. There is no wiggle room. It's eminently killable.

The payoff is big if it turns out to be right — cosmology on the same footing as the SM + GR.

No dark matter particles to find!

The only mystery left in SM physics & cosmology would be neutrino mixing.

I would be glad if people more expert than me in SM physics and cosmology would look into testing the theory.

The Hartle-Hawking state

The Hartle-Hawking state

Fix a closed 3-manifold. For every compact 4-manifold with that 3-manifold as boundary, do a generally covariant functional integral in the bulk 4-manifold, producing a state on the boundary 3-manifold. Sum over all 4-manifolds with given boundary. This is the Hartle-Hawking state of the 3-manifold. The collection of all the Hartle-Hawking states over all 3-manifolds is *the* Hartle-Hawking state.

Hartle and Hawking proposed this state to be "the state of the universe", "the initial quantum state of the universe".

A different proposal: the Hartle-Hawking state is the probability measure of a classical statistical universe.

Analogy: the normalized functional measure of euclidean quantum field theory.

$$\langle \phi(x_1)\,\phi(x_2)\,\cdots\rangle = \int_{\text{fields}} \phi(x_1)\,\phi(x_2)\,\cdots\;Z^{-1}e^{-S(\phi)}d\phi$$

Local and global

Work in arbitrary space-time dimension d.

local EQG $=$ a generally covariant functional integral in each d-manifold,
producing a state on the boundary $(d-1)$-manifold

global EQG $=$ sum over d-manifolds to make the Hartle-Hawking state

We assume the local theory is given.

All we need are some general properties of the boundary states, expressed in a few axioms of local EQG.

Otherwise the local theory is unspecified.

Anindya Bannerjee & Greg Moore (2022) (and references therein) calculate the Hartle-Hawking state in $d = 2$ examples where the local theory is a topological quantum field theory.

Diffeomorphism invariant states of $(d{-}1)$-manifolds

Principle 1: The Hartle-Hawking state on a $(d{-}1)$-manifold is invariant under diffeomorphisms of the $(d{-}1)$-manifold.

Physics is diffeomorphism invariant (general covariance). It lives on the diffeomorphism equivalence classes of manifolds (*diff-classes*).

$$Y = \{N[y]\} = \{\text{diff-classes of } \textit{connected} \text{ closed } (d{-}1)\text{-manifolds}\}$$

$$\mathcal{N} = \{N^{\mathbf{n}}\} = \{\text{diff-classes of closed } (d{-}1)\text{-manifolds}\}$$

$$N^{\mathbf{n}} = \bigsqcup_{y} \sqcup^{\mathbf{n}[y]} N[y] \qquad \mathbf{n}[y] \in \mathbb{N} = \{0, 1, 2, \ldots\} \qquad N^0 = \emptyset$$

Every diff-class is a disjoint union of connected diff-classes (with multiplicities).

Diffeomorphism invariant states of $(d{-}1)$-manifolds (2)

$$\mathcal{S}[y] \;=\; \text{the diffeomorphism invariant states on } N[y]$$

The invariant states on the general $N^{\mathbf{n}}$ are tensor products of states on the individual components — symmetric tensor products because permutations of identical components are diffeomorphisms.

$$\mathcal{S}(N^{\mathbf{n}}) \;=\; \bigotimes_{y} \otimes_s^{\mathbf{n}[y]} \mathcal{S}[y]$$

The vector space of all invariant states:

$$\mathcal{S} = \bigoplus_{\mathbf{n}} \mathcal{S}(N^{\mathbf{n}}) = \bigoplus_{n=0}^{\infty} \otimes_s^{n} \Big(\oplus_y \, \mathcal{S}[y] \Big) = \mathrm{Sym}\Big(\oplus_y \, \mathcal{S}[y] \Big) = \mathrm{Sym}(\Gamma_Y)$$

Γ_Y is the vector space of invariant states on the diff-classes of connected $(d{-}1)$-manifolds.

$$\Gamma_Y = \bigoplus_{y} \mathcal{S}[y]$$

Not Fock space

Tom Banks (1989) constructed $\mathrm{Sym}(\Gamma_Y)$ as the Fock space of quantum states of “baby universes” .

The symmetric algebra $\mathrm{Sym}(V)$ on a vector space V is familiar in quantum mechanics as the Fock space on the “single particle” states V. One supposes a Hilbert space norm on V, which in turn gives the Fock space a Hilbert space norm. The Fock space is the Hilbert space of “multi-particle” quantum states.

But Bannerji & Moore calculated the norm of the Hartle-Hawking state in their $d = 2$ examples. They got ∞. The Hartle-Hawking state is not a normalizable quantum state.

Invariant states as measures on Γ_Y

$$\Gamma_Y^* = \{\text{linear functions on } \Gamma_Y\} \qquad \mathrm{Sym}(\Gamma_Y^*) = \{\text{polynomial functions on } \Gamma_Y\}$$

$$\mathrm{Sym}(\Gamma_Y) = \{\text{linear functions on } \mathrm{Sym}(\Gamma_Y^*)\} = \{\text{measures on } \Gamma_Y\}$$

This is more natural than the Fock space construction — no need for an inner product on Γ_Y.

More concretely. Choose a basis for each invariant state space $\mathcal{S}[y]$ and combine to get a basis $\{\phi_\alpha\}$ for Γ_Y. Let $\{\phi^\alpha\}$ be the dual basis for Γ_Y^*. Each ϕ^α is a linear function on Γ_Y with $\phi^\alpha \phi_{\alpha'} = \delta^\alpha_{\alpha'}$. The ϕ^α are coordinate functions on Γ_Y. Polynomial functions on Γ_Y are linear combinations of monomials $\phi^{\alpha_1}\phi^{\alpha_2}\cdots\phi^{\alpha_n}$. Polynomial states are linear combinations of monomials $\phi_{\alpha_1}\phi_{\alpha_2}\cdots\phi_{\alpha_n}$ where multiplication is in the symmetric tensor algebra.
Identify $\mathrm{Sym}(\Gamma_Y)$ with the measures on Γ_Y.

$$1 \mapsto \delta_0 \qquad \phi_\alpha \mapsto -\partial_\alpha \delta_0 \qquad \phi_{\alpha_1}\cdots\phi_{\alpha_n} \mapsto (-\partial_{\alpha_1})\cdots(-\partial_{\alpha_n})\delta_0 \tag{0.1}$$

Multiplication in the symmetric algebra becomes convolution of measures. These polynomials are "generalized" measures.

Hartle-Hawking state as measure on Γ_Y

The Hartle-Hawking state on a diff-class $N^{\mathbf{n}}$ is a polynomial in the ϕ_α that are states of the connected components $N[y]$ with nonzero multiplicity $\mathbf{n}[y] > 0$. The full Hartle-Hawking state is the sum of all these polynomial states.

$$D\phi = \sum_{\mathbf{n}} D\phi(N^{\mathbf{n}})$$

The polynomial states $D\phi(N^{\mathbf{n}})$ are "formal" measures or "generalized" measures on Γ_Y.

It is a familiar idea to sum polynomial measures to get an honest measure.

$$\delta_x = e^{-x^i \partial_i} \delta_0 = \sum_{n=0}^{\infty} \frac{1}{n!} x^{i_1} \cdots x^{i_n} (-\partial_{i_1}) \cdots (-\partial_{i_n}) \delta_0 \qquad \text{Taylor series}$$

$-\partial_i$ generates translation in x^i. We do this in euclidean quantum field theory when we reconstruct the measure on fields from the correlation functions. The correlation functions are the expectation values of polynomial functions of the fields.

We will assume that the sum $D\phi$ is an honest measure on Γ_Y.

The $D\phi(N^{\mathbf{n}})$ are the Taylor series coefficients.

Microscopic statistical system

- microscopic objects $Y = \{N[y]\}$
- microscopic state space $\Gamma_Y = \bigoplus_y \mathcal{S}[y]$ $\qquad$ $\mathcal{S}[y] =$ state space of object y

  Γ_Y is the direct product: $\phi \in \Gamma_Y$ is a state $\phi[y] \in \mathcal{S}[y]$ for each object $y \in Y$

- probability measure $D\phi$ on Γ_Y

$$\text{expectation values} \qquad \langle \phi^{\alpha_1} \cdots \phi^{\alpha_n} \rangle = \int_{\Gamma_Y} \phi^{\alpha_1} \cdots \phi^{\alpha_n} \; D\phi$$

Assumptions:

1. $D\phi$ is an honest measure
2. $D\phi$ is normalized: $D\phi(\emptyset) = \delta_0$ which is 1 in the symmetric algebra $\mathrm{Sym}(\Gamma_Y)$
3. $D\phi$ is non-negative (a positivity condition in the local EQG)