A new kind of quantum field theory of $(n-1)$-dimensional defects in $2n$ dimensions

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Abstract

I describe a project to develop a new kind of constructible conformal field theory in $2n$ dimensions. For each ordinary 2d cft, there is to be a corresponding new cft of $(n-1)$-dimensional defects in any $2n$-dimensional conformal space-time manifold M.

The quantum fields live on "quasi Riemann surfaces", which are certain complete metric spaces of integral $(n-1)$-currents in M. These metric spaces have analytic structure analogous to ordinary Riemann surfaces. The new cfts are to be constructed on the quasi Riemann surfaces by analogy with the construction of ordinary 2d cfts on ordinary Riemann surfaces.

The global symmetry group of the ordinary 2d cft will become the gauge group of a local gauge symmetry in the new cft.

I envision a wide expanse of new quantum field theory to explore.

See `http://www.physics.rutgers.edu/~friedan/#res` for the slides of the talk and other material.

QFT is a still young subject.

The current ideas of *what is a QFT* might not be the last word.

I'm proposing a new territory of QFT.

I'm recruiting explorers.

This is best done among the naive and foolish so please pretend.

Put aside the sophisticated technologies and problems of QFT that you know.

Keep only the elementary 2d CFT of the 1970's.

The space-time manifold M

The euclidean space-time M is

- an oriented manifold of dimension $2n$
- compact and without boundary (for simplicity)
- with a conformal structure

The basic examples are $\quad M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$.

When $n = 1$, M is a Riemann surface.

The Hodge $*$-operator acting on n-forms is conformally invariant

$$(*\omega)_{\nu_1 \cdots \nu_n}(x) = \omega_{\mu_1 \cdots \mu_n}(x) \, \frac{1}{n!} \, \epsilon^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n}(x) \qquad\qquad *^2 = (-1)^n$$

Nothing else of the conformal structure will be used.

(Better "*space with Hodge-$*$ in the middle dimension*" instead of "*manifold with conformal structure*".)

Limitation: $3 \neq 2n$. For physics, maybe only $2n = 4$, $n = 2$.

Recall the 2d gaussian model (the free 1-form in 2d)

j is a 1-form on a Riemann surface with equations of motion

$$dj = 0 \qquad d(*j) = 0$$

Its integrals are 0-forms ϕ, ϕ^* defined up to global symmetries

$$d\phi = j \qquad d\phi^* = *j$$

$$\phi(x) \to \phi(x) + a \qquad \phi^*(x) \to \phi^*(x) + a^*$$

The vertex operators describe 0-dimensional (point) defects

$$V_{p,p^*}(x) = e^{ip\phi(x)+ip^*\phi^*(x)} \qquad V_{p,p^*}(x) \to V_{p,p^*}(x)\, e^{ipa+ip^*a^*}$$

$U(1) \times U(1)$ conditions (neeed for IR sanity in 2d)

$$\phi(x) \in S^1_R = \mathbb{R}/2\pi R\mathbb{Z} \qquad \phi^*(x) \in S^1_{R^*} = \mathbb{R}/2\pi R^*\mathbb{Z} \qquad RR^* = 1$$

symmetry group: $a, a^* \in U(1)\times U(1)$ charges: $p, p^* \in \frac{1}{R}\mathbb{Z} \times \frac{1}{R^*}\mathbb{Z}$

Express quantization of the free field theory by the Schwinger-Dyson equation on the 2-pt functions

$$\langle\phi(x)\,j(x')\rangle,\quad \langle\phi^*(x)\,j(x')\rangle,\quad \langle\phi(x)*j(x')\rangle,\quad \langle\phi^*(x)*j(x')\rangle$$

Using complex coordinate z and the chiral field basis

$$j_\pm = \frac{1}{2}(j \pm i^{-1}*j), \qquad \phi_\pm = \frac{1}{2}(\phi \pm i^{-1}\phi^*)$$

the S-D equation is exactly the Cauchy-Riemann equation

$$\frac{\partial}{\partial\bar z}\,\frac{1}{z-z'} = \pi\delta^2(z-z')$$

which is the foundation for complex analysis on Riemann surfaces.

The 2d Gaussian model might have led to complex analysis on Riemann surfaces had mathematicians not already provided it.

Heuristic

The 2d Gaussian model was one of the *ur-cfts*. It is only a slight exaggeration to say that all of 2d cft (and non-conformal 2d qft) emerged from the 2d Gaussian model:

- the Virasoro algebra
- 2d nonabelian current algebra (at $R = R^* = 1$)
- torus models (several free 1-forms)
- orbifolds of the above models
- perturbation theory (sigma models, general nonlinear models)
- conformal perturbation theory
- $\cdots$
- axiomatic formulations

The 2d Gaussian model can serve as a starting point from which to explore the whole galaxy of 2d quantum field theory.

Recall the free n-form in $2n$ dimensions

$F(x)$ is an n-form on M with equations of motion

$$dF = 0 \qquad d(*F) = 0$$

Its integrals are $(n-1)$-forms A, A^* defined up to local gauge symmetries given by $(n-2)$-forms f, f^*

$$dA = F \qquad dA^* = *F$$

$$A \to A + df \qquad A^* \to A^* + df^*$$

$(n-1)$-dimensional defects are described by fields that live on $(n-1)$-dimensional objects ξ with boundary $\partial\xi$

$$V_{p,p^*}(\xi) = e^{ip\int_\xi A + ip^*\int_\xi A^*} \qquad V_{p,p^*}(\xi) \to V_{p,p^*}(\xi) e^{ip\int_{\partial\xi} f + ip^*\int_{\partial\xi} f^*}$$

$$\int_\xi A \in S^1_R = \mathbb{R}/2\pi R\mathbb{Z} \qquad \int_\xi A^* \in S^1_{R^*} = \mathbb{R}/2\pi R^*\mathbb{Z} \qquad RR^* = 1$$

Singular currents in M

A k-current (in the math sense) is a distribution on k-forms

$$\omega \mapsto \int_\xi \omega = \int_M \omega_{\mu_1\cdots\mu_k}(x)\, \xi^{\mu_1\cdots\mu_k}(x)\, d^{2n}x$$

The oriented k-simplex Δ^k is the basic k-dimensional object:

$\Delta^0 =$ a point, $\quad \Delta^1 =$ a line interval, $\quad \Delta^2 =$ a triangle, $\quad \ldots$

An oriented k-simplex σ in M gives a k-current $[\sigma]$

$$\sigma\colon \Delta^k \to M \qquad \int_{[\sigma]} \omega = \int_{\Delta^k} \sigma^*\omega$$

$[\sigma]$ is the characteristic δ-function localized on $\sigma(\Delta^k) \subset M$.

The *singular* k-currents are the integer linear combinations

$$\mathcal{D}_k^{sing}(M) = \{\, \textstyle\sum_i m_i[\sigma_i], m_i \in \mathbb{Z}\}$$

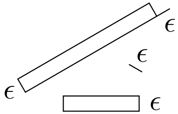
They include the k-submanifolds of M. The singular k-current is the physical object in M, the linear functional on k-forms, independent of how it is made out of k-simplices as $\sum_i m_i[\sigma_i]$.

The flat metric on $\mathcal{D}_k^{int}(M)$

Geometric Measure Theory introduces on $\mathcal{D}_k^{sing}(M)$ the *flat norm* $\|\xi\|_{\mathit{flat}}$ which measures the physical ease of deforming $\xi \to 0$.

$$\|\xi\|_{\mathit{flat}} = \inf_{\xi_{k+1} \in \mathcal{D}_{k+1}^{sing}(M)} \Big[(k+1)\text{-volume}(\xi_{k+1}) + k\text{-volume}(\xi - \partial \xi_{k+1}) \Big]$$

e.g., a small 1-current $\xi =$



$\|\xi\|_{flat} = O(\epsilon)$

$\mathcal{D}_k^{sing}(M)$ is a metric space with the *flat metric*

$$\operatorname{dist}(\xi_1, \xi_2)_{\mathit{flat}} = \|\xi_1 - \xi_2\|_{\mathit{flat}}$$

Integral currents

The metric completion $\mathcal{D}_k^{int}(M)$ is the space of *integral* k-currents

$$\mathcal{D}_k^{sing}(M) \subset \mathcal{D}_k^{int}(M) \subset \mathcal{D}_k^{distr}(M)$$

The additional currents, the limits of Cauchy sequences, are fractal.

$\mathcal{D}_k^{int}(M)$ is a metric abelian group — an abelian group that is a complete metric space, the group operations respecting the metric.

The boundary of an integral current is an integral current

$$\mathcal{D}_k^{int}(M) \xrightarrow{\partial} \mathcal{D}_{k-1}^{int}(M)$$

where the boundary operator on currents is the dual of d

$$\int_{\partial\xi} \omega = \int_{\xi} d\omega \qquad (\partial\xi)^{\mu_2\cdots\mu_k}(x) = -\partial_{\mu_1}\xi^{\mu_1\mu_2\cdots\mu_k}(x) \qquad \partial^2 = 0$$

What are the $(n-1)$-dimensional objects ξ ?

We take $\mathcal{D}^{int}_{n-1}(M)$ as the space of $(n-1)$-dimensional objects ξ.

Recall that A, A^* are $(n-1)$-forms and

$$V_{p,p^*}(\xi) = e^{ip\int_\xi A + ip^*\int_\xi A^*} \qquad \int_\xi A \in \mathbb{R}/2\pi R\mathbb{Z} \qquad \int_\xi A^* \in \mathbb{R}/2\pi R^*\mathbb{Z}$$

The rationale is

1. ξ is manifestly an $(n-1)$-current — a distribution on forms.
2. The $(n-1)$-simplices $[\sigma]$ are basic objects.
3. The $U(1)$ conditions are closed under integer linear combinations. So $\mathcal{D}^{sing}_{n-1}(M)$ is the minimal space of objects.
4. Take the metric completion $\mathcal{D}^{int}_{n-1}(M)$ to do calculus.

Note that $\mathcal{D}^{int}_{n-1}(M)$ is the space of extended objects for the free n-form, not necessarily for any other theory.

The free n-form as 2d gaussian model

Define scalar fields on $\mathcal{D}^{int}_{n-1}(M)$

$$\phi(\xi) = \int_\xi A \qquad \phi^*(\xi) = \int_\xi A^* \quad \text{so} \quad V_{p,p^*}(\xi) = e^{ip\phi(\xi)+ip^*\phi^*(\xi)}$$

Under a gauge transformation $A \to A + df$, $A^* \to A^* + df^*$

$$\phi(\xi) \to \phi(\xi) + a(\partial\xi) \qquad \phi^*(\xi) \to \phi^*(\xi) + a^*(\partial\xi)$$

$$a(\partial\xi) = \int_\xi df = \int_{\partial\xi} f \qquad a^*(\partial\xi) = \int_\xi df^* = \int_{\partial\xi} f^*$$

$$V_{p,p^*}(\xi) \longrightarrow V_{p,p^*}(\xi)\; e^{ipa(\partial\xi)+ip^*a^*(\partial\xi)}$$

Fix an $(n-2)$-boundary $\partial\xi_0$ and consider only ξ with $\partial\xi = \partial\xi_0$

$$\mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0} = \left\{\xi \in \mathcal{D}^{int}_{n-1}(M) \colon\; \partial\xi = \partial\xi_0\right\}$$

On $\mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0}$ the symmetries are just two numbers, $a(\partial\xi_0)$ and $a^*(\partial\xi_0)$, independent of ξ. The gauge symmetry acts on $\mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0}$ as a global $U(1)\times U(1)$.

The bundle of quasi Riemann surfaces

Consider as a fiber bundle

$$\mathcal{D}^{int}_{n-1}(M) \xrightarrow{\partial} \partial\mathcal{D}^{int}_{n-1}(M) \quad \text{with fibers} \quad \mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0} \xrightarrow{\partial} \{\partial\xi_0\}$$

All the $\mathcal{D}^{int}_{n-1}(M)_{m\partial\xi_0}$, $m \in \mathbb{Z}$ see the same $U(1)\times U(1)$ because $a(m\partial\xi_0) = ma(\partial\xi_0)$. So combine them to form the abelian group

$$\mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} = \bigoplus_{m\in\mathbb{Z}} \mathcal{D}^{int}_{n-1}(M)_{m\partial\xi_0}$$

These are the "quasi Riemann surfaces" (modulo some technical niceties). They form a fiber bundle of quasi Riemann surfaces

$$\mathcal{Q}(M) \to \mathcal{B}(M) \quad \text{with fibers} \quad \mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \xrightarrow{\partial} \mathbb{Z}\partial\xi_0$$

For brevity, write $\mathcal{Q} \xrightarrow{\partial} \mathbb{Z}$ for any one of the fibers.

There is a 2d cft – the gaussian model – on each $\mathcal{Q}$. A copy of the global symmetry 2d group $G = U(1) \times U(1)$ acts on each fiber, comprising a local gauge symmetry over the base space $\mathcal{B}(M)$.

The prototype quasi Riemann surface is the metric abelian group of integral 0-currents in a Riemann surface Σ (the case $n = 1$)

$$\mathcal{Q}(\Sigma) = \mathcal{D}_0^{int}(\Sigma) \qquad \mathcal{Q}(\Sigma) \xrightarrow{\partial} \mathbb{Z} \qquad \partial\xi = \int_\xi 1$$

GMT provides a construction of integral currents in any complete metric space. So we are given $\mathcal{D}_j^{int}(\mathcal{Q})$, the metric abelian group of integral j-currents in the metric space $\mathcal{Q}$.

Define the j-forms on $\mathcal{Q}$ by

$$\Omega_j(\mathcal{Q}) = \mathrm{Hom}(\mathcal{D}_j^{int}(\mathcal{Q}), \mathbb{R}) \qquad \Omega_j(\mathcal{Q}) \xrightarrow{d} \Omega_{j+1}(\mathcal{Q}) \quad d\omega(\xi) = \omega(\partial\xi)$$

A j-form is determined by its values on the infinitesimal j-simplices, which generate $\mathcal{D}_j^{int}(\mathcal{Q})$. So the tangent bundle can be defined as the set of infinitesimal 1-simplices in $\mathcal{Q}$

$$T\mathcal{Q} = \left\{\epsilon^{-1}[\sigma_\epsilon]\colon \quad \sigma_\epsilon\colon [0,\epsilon] \to \mathcal{Q}\right\}$$

This version of tensor analysis, if the metric space happens to be a manifold M, is equivalent to the usual tensor analysis on manifolds.

The maps $\Pi_{j,n-1} \colon \mathcal{D}_j^{int}(\mathcal{Q}) \to \mathcal{D}_{j+n-1}^{int}(M)$

There are natural maps $\quad \Pi_{j,k} \colon \mathcal{D}_j^{int}(\mathcal{D}_k^{int}(M)) \to \mathcal{D}_{j+k}^{int}(M)$

which are derived from the equivalence

$$\Delta^j \to \left\{\Delta^k \to M\right\} \;=\; \Delta^j \times \Delta^k \to M \;=\; \Delta^{j+k} \to M$$

In particular, taking $k = n-1$ and restricting to $\mathcal{Q}$

$$\Pi_{j,n-1} \colon \mathcal{D}_j^{int}(\mathcal{Q}) \to \mathcal{D}_{j+n-1}^{int}(M) \qquad \Pi_{j,n-1}\partial = \partial\Pi_{j,n-1}$$

$\Pi_{0,n-1}\colon$ 0-currents in $\mathcal{Q}$ $\to$ $(n-1)$-currents in M

$\Pi_{1,n-1}\colon$ 1-currents in $\mathcal{Q}$ $\to$ n-currents in M

$\Pi_{2,n-1}\colon$ 2-currents in $\mathcal{Q}$ $\to$ $(n+1)$-currents in M

A crucial technical point (requiring the flat metric completion)

> $\Pi_{1,n-1}$ *identifies each tangent space* $T_\xi\mathcal{Q}$ *with a certain subspace of* $\mathcal{D}_n^{distr}(M)$ *which is closed under Hodge-$*$.*
> *Therefore $*$ acts on the tangent spaces* $T_\xi\mathcal{Q}$.

The dual maps on forms are

$$\Pi^*_{j,n-1}\colon\ \Omega_{j+n-1}(M) \to \Omega_j(\mathcal{Q}) \qquad d\Pi^*_{j,n-1} = \Pi^*_{j,n-1} d$$

$$\Pi^*_{0,n-1}\colon\ (n{-}1)\text{-forms on } M \ \to\ 0\text{-forms on } \mathcal{Q} \qquad A(x) \mapsto \phi(\xi)$$

$$A^*(x) \mapsto \phi^*(\xi)$$

$$\Pi^*_{1,n-1}\colon\ n\text{-forms on } M \ \to\ 1\text{-forms on } \mathcal{Q} \qquad F(x) \mapsto j(\xi)$$

$$*F(x) \mapsto *j(\xi)$$

$$\Pi^*_{2,n-1}\colon\ (n{+}1)\text{-forms on } M \ \to\ 2\text{-forms on } \mathcal{Q}$$

Now we have the classical 2d gaussian model on each fiber $\mathcal{Q}$

$$d\phi(\xi) = j(\xi) \qquad d\phi^*(\xi) = *j(\xi) \qquad V_{p,p^*}(\xi) = e^{ip\phi(\xi)+ip^*\phi^*(\xi)}$$

The intersection form on currents in M

There is a natural bilinear intersection form on currents in M

$$I_M(\xi_1, \xi_2) = \int_M \xi_1^{\mu_1 \cdots \mu_{k_1}}(x)\, \xi_2^{\nu_1 \cdots \nu_{k_2}}(x)\, \frac{\epsilon_{\mu_1 \cdots \mu_{k_1} \nu_1 \cdots \nu_{k_2}}(x)}{k_1!\, k_2!}\, d^{2n}x$$

- depending only on the orientation of M
- well-defined on all smooth ξ_1, ξ_2 and almost all integral ξ_1, ξ_2 (then giving the integer intersection number)
- nonzero only if $\dim(\xi_1) + \dim(\xi_2) = \dim(M)$

$$I_M(\xi_1, \xi_2) \neq 0 \quad \text{only if} \quad k_1 + k_2 = 2n$$

Pull back to $\mathcal{Q}$: $\qquad I_{\mathcal{Q}}(\eta_1, \eta_2) = I_M(\Pi_{j_1, n-1}\eta_1, \Pi_{j_2, n-1}\eta_2)$

$$I_{\mathcal{Q}}(\eta_1, \eta_2) \neq 0 \quad \text{only if} \quad j_1 + j_2 = 2$$

just as the intersection form of currents in a 2-manifold.

Write the Schwinger-Dyson equation for $\langle A(x)\, F(x')\rangle$ smeared against two currents ξ_1, ξ_2. Roughly (detail to follow) the S-D equation for the free n-form on M is

$$\langle \int_{\xi_1} A_\pm \int_{\xi_2} dF_\pm \rangle = 2\pi i I_M(\xi_1, \xi_2) \qquad k_1 = n-1, \quad k_2 = n+1$$

Pulled back to $\mathcal{Q}$, the S-D equation is

$$\langle \int_{\eta_1} \phi_\pm \int_{\eta_2} dj_\pm \rangle = 2\pi i I_{\mathcal{Q}}(\eta_1, \eta_2) \qquad j_1 = 0, \quad j_2 = 2$$

which has exactly the same form as the S-D eqn of the 2d gaussian model on a Riemann surface Σ

$$\langle \int_{\eta_1} \phi_\pm \int_{\eta_2} dj_\pm \rangle = 2\pi i I_{\Sigma}(\eta_1, \eta_2) \qquad j_1 = 0, \quad j_2 = 2$$

which (unsmeared) is the Cauchy-Riemann equation

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z - z'} = \pi \delta^2(z - z')$$

The S-D equations (details to be skipped)

We want the S-D equations to look the same for all n, so that when they are pulled back to $\mathcal{Q}$ they will all be the same.

But $*^2 = (-1)^n$ and the intersection form $I_M(\xi_1, \xi_2)$ has symmetry properties and various other properties that depend explicitly on n.

Solution: use complex currents and set

$$\mathcal{Q} = \mathcal{D}_{n-1}^{int}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}_n^{int}(M)$$

The tangent spaces $T_\xi\mathcal{Q}$ are now complex vector spaces on which $*$ acts. Define

$$J = \epsilon_n * \qquad \epsilon_n^2 = (-1)^{n-1}$$

so

$$J^2 = -1$$

For n odd, ϵ_n is real. Then all currents can be taken real.

Modify the intersection form so its properties are independent of n.

$$I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle = \epsilon_{n,k_2-n} I_M(\bar{\xi}_1, \xi_2) \qquad \epsilon_{n,m} = (-1)^{nm+m(m+1)/2}\, \epsilon_n^{-1}$$

- $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle = -\,\overline{I_M\langle\, \bar{\xi}_2, \xi_1 \,\rangle}$ (skew-hermitian)
- $I_M\langle\, \overline{\partial\xi}_1, \xi_2 \,\rangle + I_M\langle\, \bar{\xi}_1, \partial\xi_2 \,\rangle = 0$ (integration by parts)
- $I_M\langle\, \bar{\xi}_1, J\xi_2 \,\rangle$ is hermitian and positive definite on n-currents, i.e., on $(j{+}n{-}1)$-currents with $j = 1$

The Schwinger-Dyson equation is now the same for all n

$$\langle \int_{\bar{\xi}_1} A^\dagger_\alpha \int_{\xi_2} dF_\beta \rangle = -2\pi i \gamma_{\alpha\beta} I_M \langle\, \bar{\xi}_1, \xi_2 \,\rangle$$

$$\gamma_{++} = 1\,, \quad \gamma_{--} = -1\,, \quad \gamma_{+-} = \gamma_{-+} = 0$$

written in terms of the chiral fields and their euclidean adjoints (F and A are now complex fields):

$$P_\pm = \frac{1}{2}\left(1 \pm i^{-1} J\right) \qquad F_\pm = P_\pm F \qquad dA_\pm = F_\pm$$

$$F^\dagger_\pm = \overline{F_\mp} \qquad A^\dagger_\pm = \overline{A_\mp}$$

Pulled back to $\mathcal{Q}$, the S-D equation is the same as on a Riemann surface Σ

$$\langle \int_{\bar{\eta}_1} \phi^\dagger_\alpha \int_{\eta_2} dj_\beta \rangle = -2\pi i \gamma_{\alpha\beta} I_{\mathcal{Q}} \langle\, \bar{\eta}_1, \eta_2 \,\rangle$$

where $I_{\mathcal{Q}}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$ has the same properties as $I_\Sigma\langle\, \bar{\eta}_1, \eta_2 \,\rangle$.

Definition of quasi Riemann surface (details to be skipped)

Now we can define a quasi Riemann surface as a metric space whose currents have the structures needed to write the Cauchy-Riemann equation in the same form as the C-R equation for an ordinary Riemann surface.

Here is the definition for the real case, which covers the examples with n odd.

A real quasi Riemann surface is

- a metric abelian group $\mathcal{Q}$ with a morphism $\mathcal{Q} \xrightarrow{\partial} \mathbb{Z}$
- a translation-invariant linear operator J on the tangent spaces $T_\xi \mathcal{Q} = T_0 \mathcal{Q}$
- a densely defined translation-invariant integral bilinear form $I_\mathcal{Q}\langle\, \eta_1, \eta_2 \,\rangle$ on $\bigoplus_j \mathcal{D}_j^{int}(\mathcal{Q})$

satisfying

1. $J^2 = -1$ (J is an almost-complex structure)
2. $I_\mathcal{Q}\langle\, \eta_1, \eta_2 \,\rangle \neq 0$ only if $j_1 + j_2 = 2$
3. $I_\mathcal{Q}\langle\, \eta_1, \eta_2 \,\rangle = -I_\mathcal{Q}\langle\, \eta_2, \eta_1 \,\rangle$ (skew symmetric)
4. $I_\mathcal{Q}\langle\, \partial\eta_1, \eta_2 \,\rangle + I_\mathcal{Q}\langle\, \eta_1, \partial\eta_2 \,\rangle = 0$ (integration by parts)
5. $I_\mathcal{Q}\langle\, \eta_1, J\eta_2 \,\rangle$ on 1-currents is symmetric and non-negative

Prospects

We have the 2d gaussian model on the quasi Riemann surfaces $\mathcal{Q}$ and, what is equivalent, the Cauchy-Riemann equation.

So we have the foundation for all the constructions of 2d cft on the quasi Riemann surfaces $\mathcal{Q}$.

Performing these constructions on the quasi Riemann surfaces $\mathcal{Q}(M)$ will produce new cfts of $(n-1)$-dimensional defects in conformal $2n$-manifolds M.

We can further envision extending all the constructions of non-conformal 2d qft to obtain, for each non-conformal 2d qft, a non-conformal qft of defects in M.

A cornucopeia of questions present themselves — foundational technical problems — opportunities to leverage 2d qft to develop a new technology for doing qft in $2n$ dimensions — a new territory of qft to explore. A sampling of possible questions follow.

Conjecture

- A quasi Riemann surface $\mathcal{Q}$ is classified up to isomorphism by its Jacobian — the integral homology in the middle dimension as a lattice in a complex vector space.

- Each $\mathcal{Q}$ is isomorphic to $\mathcal{Q}(\Sigma) = \mathcal{D}_0^{int}(\Sigma)$ where Σ is the 2-dimensional space with the same Jacobian as $\mathcal{Q}$.

$S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$ has trivial Jacobian, so the conjecture implies that all the quasi Riemann surfaces $\mathcal{Q}(S^{2n})$ are isomorphic

$$\mathcal{D}_{n-1}^{int}(S^{2n})_{\mathbb{Z}\partial\xi_0} \cong \mathcal{D}_0^{int}(S^2) \qquad \forall\, n,\ \forall\, \mathbb{Z}\partial\xi_0$$

2. Construct cfts on M via the conjectured isomorphisms $\mathcal{Q} = \mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \cong \mathcal{D}^{int}_0(\Sigma)$

1. Start with an ordinary 2d cft on the 2d space Σ. Call it CFT$_2$.
2. Lift CFT$_2$ to $\mathcal{D}^{int}_0(\Sigma)$. Call this ECFT$_2$ (*extended* CFT$_2$)
   This lifting still remains to be done for the vertex operators of the 2d gaussian model — i.e., renormalize them as fields on the space of *integral* 0-currents $\eta = \sum_i m_i \delta_{x_1}$
   $$V_{p,p^*}(\eta) = e^{ip\int_\eta \phi + ip^* \int_\eta \phi^*} = e^{ip\sum_i m_i\phi(x_i) + ip^*\sum_i m_i \phi^*(x_i)}$$
   In general, such a field $\Phi(\eta)$ might be described by a radial quantization state at each point $x \in \Sigma$.
3. Use an isomorphism $\mathcal{Q} \cong \mathcal{D}^{int}_0(\Sigma)$ to put EQFT$_2$ on each fiber $\mathcal{Q}$.

Note: it also remains to renormalize $V_{p,p^*}(\xi)$ in the n-form theory for *integral* $(n-1)$-currents ξ. These renormalizations for all values of n have to be consistent with the isomorphisms $\mathcal{Q} \cong \mathcal{D}^{int}_0(\Sigma)$.

The choice of isomorphism $\mathcal{Q} \cong \mathcal{D}_0^{int}(\Sigma)$ for each fiber is not unique. Different isomorphisms can produce theories on the fiber that differ by an element of the global symmetry group G of CFT$_2$. The possible theories on the fibers form a principle G-bundle, a gauge bundle, $\mathcal{T}(M) \to \mathcal{B}(M)$.

Consider a CFT$_2$ with nonabelian current algebra $j^a(x)$ in the Lie algebra $\mathfrak{g}$ of the global symmetry group G, e.g. the 2d Gaussian model at $R = R^* = 1$ with $G = SU(2) \times SU(2)$.

The ECFT on each fiber contains 1-forms $j^a(\xi)$ in the local Lie algebra $\mathfrak{g}_{\partial\xi_0}$ of the fiber. These correspond to n-forms $F^a(x)$ on the space-time M, again in $\mathfrak{g}_{\partial\xi_0}$. The $j^a(\xi)$ constitute the curvature tensor of a nonabelian connection in the gauge bundle.

In this way, the CFT$_2$ gives rise to a quantum theory with nonabelian gauge symmetry over the base space $\mathcal{B}(M)$.

In the abelian case — the free n-form — the gauge symmetry over $\mathcal{B}(M)$ reduces to ordinary local gauge symmetry in the space-time M. Is there a local space-time interpretation of the nonabelian gauge symmetry over $\mathcal{B}(M)$?

More generally, the plan is to make a cft of $(n-1)$-dimensional defects in $2n$ dimensions for for each ordinary 2d cft (and a non-conformal qft from each 2d qft). Will these new theories have a local space-time interpretation (assuming the projected construction is successful)? Tiny defects look like points in M, so fields $\Phi(\xi)$ on $\mathcal{Q}(M)$ will when restricted to small ξ give ordinary local quantum fields on M. Will these form a local quantum field theory on M? If so, will they be new local qfts in $2n$-dimensions?

5. Automorphism groups of the quasi Riemann surfaces

If the classification conjecture is correct, then there will be an automorphism group $\mathrm{Aut}(\mathcal{Q}(\Sigma))$ for each Jacobian.

There will be a conjugacy class of group inclusions

$$\mathrm{Conf}(M) \to \mathrm{Aut}(\mathcal{Q}(\Sigma))$$

for every conformal symmetry group of every conformal manifold M with the given Jacobian.

$\mathrm{Aut}(\mathcal{Q}(\Sigma))$ will be the structure group of the fiber bundle of quasi Riemann surfaces $\mathcal{Q}(M) \to \mathcal{B}(M)$. Each fiber will be isomorphic to $\mathcal{Q}(\Sigma)$ up to an element of $\mathrm{Aut}(\mathcal{Q}(\Sigma))$.

There will be a group automorphism

$$\mathrm{Aut}(\mathcal{Q}(\Sigma)) \to \mathrm{Sym}(\mathsf{CFT}_2)$$

for every global symmetry group $\mathrm{Sym}(\mathsf{CFT}_2)$ of a 2d cft.

6. Quasi holomorphic curves in $\mathcal{Q}$

A *quasi holomorphic curve* in $\mathcal{Q}$ is a morphism $\mathcal{Q}(\Sigma) \to \mathcal{Q}$, i.e., a map $C \colon \Sigma \to \mathcal{Q}$ from a Riemann surface Σ to $\mathcal{Q}$ that preserves the J-operator and the skew-hermitian form on currents.

The basic example is $C : z \in \Sigma \mapsto [z] = \delta_z \in \mathcal{D}_0^{int}(\Sigma)$

A *local* qhc is a qhc where $C \colon \mathbb{D}_\epsilon \to \mathcal{Q}$ where $\mathbb{D}_\epsilon$ is a tiny complex disk. A tiny disk in $\mathcal{Q}$ is a tiny $(n+1)$-dimensional object in M.

The local quasi holomorphic curves are local probes. The fields of an ECFT on $\mathcal{Q}$ pull back along C to become *ordinary* 2d conformal fields of the underlying CFT$_2$ on Σ.

For each local qhc, we have the radial quantization, Virasoro algebras, and operator product expansion of the 2d cft on $\mathbb{D}_\epsilon$.

This local 2d cft data on the collection of all local qhc's provides a local description of the ECFT on $\mathcal{Q}$.