Geometry of the 2D renormalization group

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Mostly joint work with Anatoly Konechny.

RG flow in quantum field theory

$$\mu \frac{d}{d\mu} \lambda^i = \beta^i(\lambda)$$

Gradient conjecture (Wallace and Zia, 1974)

$$\partial_i S(\lambda) = -G_{ij}(\lambda) \beta^j(\lambda)$$

The gradient property implies

- Critical exponents are real. At $\beta^i = 0$, a fixed point, the matrix $\partial_i \beta^j$ is symmetric if G_{ij} is.
- $\beta^i = 0 \iff \partial_i S = 0$
- S decreases along the flow

$$\mu \frac{dS}{d\mu} = \beta^i \partial_i S = -G_{ij} \beta^i \beta^j \leq 0$$

so there cannot be limit cycles.

2D general nonlinear model (GNLM aka NLSM)

2D fields $\quad X^\alpha(x) \in M,\quad$ the target manifold

2D action $$A(X) = \int d^2x\, g_{\alpha\beta}(X)\partial_\mu X^\alpha \partial^\mu X^\beta$$

DF, 1979 The GNLM is renormalizable, with RG flow

$$\mu \frac{d}{d\mu} g_{\alpha\beta}(X) = -R_{\alpha\beta}(X) + \cdots$$

which drives the metric to a fixed point $R_{\alpha\beta} = 0$.

The 2D RG flow produces classical field theory (M = spacetime).

- Can it produce a realistic field theory?
- Where does *quantum* field theory come from?

A gradient formula looks like a field theory action principle.

DF, 1980 formulated and proved a gradient formula to two loops for particular classes of 2D general nonlinear models (GNLM).

Fradkin, Tseytlin, 1985 introduced the dilaton couplings into the GNLM — a crucial ingredient for a gradient formula.

Callan, DF, Martinec, Perry, 1985 showed that at one loop $\beta^i = 0$ is equivalent to $\partial_i S = 0$ for S the classical field theory action functional that gives the low-energy tree-level string S-matrix.

Tseytlin, 1986 checked a gradient formula for GNLM to two loops.

In string theory, $\beta^i = 0$ is the consistency condition for the classical string background.

What is the *quantum* string background?

Zamolodchikov, 1986 proved the c-theorem

$$\mu \frac{dc}{d\mu} = \beta^i \partial_i c = -g_{ij} \beta^i \beta^j \leq 0$$

and showed to first order around a fixed point that

$$\partial_i c = -g_{ij} \beta^j$$

Osborn, 1990 showed that a (modified) gradient formula exists for the GNLM to all orders in α' assuming there is a certain integration measure for the zero modes.

Osborn, 1991 proved a formula

$$\partial_i c = -g_{ij} \beta^j - b_{ij} \beta^j$$

for 2D QFT's subject to some power counting assumptions. b_{ij} is an antisymmetric tensor on the space of theories.

Boundary RG flow

Consider a fixed 2D conformal field theory — a 1+1D quantum critical wire. The boundary conditions at an end of the wire are parametrized by boundary coupling constants λ^a. The boundary RG flow is

$$\mu \frac{d}{d\mu} \lambda^a = \beta^a(\lambda)$$

DF and Konechny, 2003 proved a gradient formula

$$\partial_a s = -g_{ab} \beta^b$$

where s is the boundary entropy. The proof assumed canonical short distance behavior. In contrast to $c \geq 0$, no lower bound on s is known.

DF, Konechny and Schmidt-Colinet, 2012 proved a lower bound on $s = \ln g$ at fixed points, $\beta^a = 0$, assuming that the most relevant scaling dimension of the bulk CFT satisfies $\Delta_1 > (c-1)/12$.

DF, 2005

- argued that 1+1D quantum critical circuits are ideal physical systems for asymptotically large scale quantum computers.
- defined a conserved entropy current to study the flow of entropy in such circuits
- used real-time response functions for the entropy current to re-prove the boundary gradient formula.

Bulk 2D QFT — the stress energy tensor

Response to varying the 2D metric, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}(x)$

$$\delta \ln Z = \frac{1}{2} \int d^2x \, \delta g_{\mu\nu}(x) \, \langle T^{\mu\nu}(x) \rangle$$

For conformally flat metrics, $g_{\mu\nu}(x) = \mu(x)^2 \delta_{\mu\nu}$,

$$\mu \frac{\delta \ln Z}{\delta \mu(x)} = \langle \Theta(x) \rangle \,, \qquad \Theta(x) = g^{\mu\nu} T_{\mu\nu}(x)$$

Global scale variation of correlation functions

$$\mu \frac{\partial}{\partial \mu} \langle \cdots \rangle_c = \int d^2x \, \langle \Theta(x) \cdots \rangle_c$$

Action principle (Schwinger)

The λ^i couple to a complete set of local spin-0 fields $\phi_i(x)$

$$\frac{\partial}{\partial \lambda^i} \langle \cdots \rangle_c = \int d^2x \, \langle \phi_i(x) \cdots \rangle_c$$

Let the coupling constants λ^i become local sources $\lambda^i(x)$

$$\frac{\delta \ln Z}{\delta \lambda^i(x)} = \langle \phi_i(x) \rangle$$

Correlation functions are distributions

$$\langle \phi_{i_1}(x_1) \phi_{i_2}(x_2) \ldots \Theta(y_1) \Theta(y_2) \ldots \rangle_c =$$

$$\left(\frac{\delta}{\delta \lambda^{i_1}(x_1)} \, \frac{\delta}{\delta \lambda^{i_2}(x_2)} \, \cdots \, \mu \frac{\delta}{\delta \mu(y_1)} \, \mu \frac{\delta}{\delta \mu(y_2)} \, \cdots \right) \ln Z$$

Since the $\phi_i(x)$ form a complete set of fields,

$$\Theta(x) = \beta^i \phi_i(x) \quad \text{up to contact terms.}$$

Using power counting (dimensional analysis),

$$\Theta(x) = \beta^i \phi_i(x) + \frac{1}{2}\mu^2 R_2\, C(x) + \partial_\mu \lambda^i J_i^\mu(x) + \partial^\mu \left[W_i(x) \partial_\mu \lambda^i \right] + \frac{1}{2} \partial_\mu \lambda^i \partial^\mu \lambda^j G_{ij}(x)$$

where $C(x)$, $W_i(x)$, $G_{ij}(x)$ are spin-0 fields, $J_i^\mu(x)$ are spin-1 fields, and $R_2(x)$ is the 2D curvature.

strict power counting (as near unitary fixed points):
dimension 0 fields must be proportional to the identity

$$C(x) = c\mathbf{1},\ \ W_i(x) = w_i \mathbf{1},\ \ G_{ij}(x) = g_{ij}\mathbf{1}$$

loose power counting (as in GNLM):
$C(x)$, $W_i(x)$, $G_{ij}(x)$ can have dimensions near zero — slightly irrelevant.

Osborn, 1991 assumed strict power counting (and neglected the $\partial_\mu \lambda^i J_i^\mu$ term) to derive

$$\partial_i c = -g_{ij}\beta^j - b_{ij}\beta^j \qquad b_{ij} = \partial_i w_j - \partial_j w_i$$

The derivation was essentially nonperturbative. The main ingredient was the Wess-Zumino consistency conditions on local Weyl variations in the presence of curved metric and sources.

Freedman, Headrick and Lawrence, 2005 showed that the naive gradient formula $\partial_i c = -g_{ij}\beta^j$ cannot hold because $\partial_k(g_{ij}\beta^j) - (i \leftrightarrow k) \neq 0$ at 2nd order in conformal perturbation theory.

Resolution: w_i and b_{ij} are nonzero at this order.

DF and Konechny 2009 proved a generalized version of the Osborn formula without assuming any power counting restrictions at all.

The result can be specialized to situations where power counting applies.

We prove a general gradient formula

$$\partial_i c = -(g_{ij} + \Delta g_{ij})\beta^j - b_{ij}\beta^j$$

where c and g_{ij} are as in the c-theorem.

$$G_\Lambda(x) = 3\pi x^2 \theta(1 - \Lambda|x|)$$

$$c = -\int d^2x \;\; G_\Lambda(x) \, \langle \, \Theta(x) \, \Theta(0) \, \rangle_c$$

$$g_{ij} = -\Lambda \frac{\partial}{\partial \Lambda} \int d^2x \;\; G_\Lambda(x) \, \langle \, \phi_i(x) \, \phi_j(0) \, \rangle_c$$

$$w_i = \int d^2x \; G_\Lambda(x) \langle \phi_i(x) \Theta(0) \rangle_c$$

$$b_{ij} = \partial_i w_j - \partial_j w_i = \int d^2y \int d^2x \; G_\Lambda(x) \langle \phi_i(y) \phi_j(x) \Theta(0) \rangle_c - (i \leftrightarrow j)$$

The correction term Δg_{ij} is described below.

Assumptions

We need two assumptions about infrared behavior:

- The one-point and two-point functions of the ϕ_i and $T_{\mu\nu}$ are at least once differentiable w.r.t. the coupling constants λ^i.
- Spontaneously broken global conformal symmetry does not occur:

$$\lim_{|x|\to\infty} |x|^3 \langle J_\mu(x)\, T_{\alpha\beta}(0)\rangle_c = 0$$

for all spin-1 fields $J_\mu(x)$

The proof

We want to calculate

$$-r_i = \partial_i c + g_{ij}\beta^j + b_{ij}\beta^j \, .$$

Each term on the RHS is an integrated 3 point function.

$$\partial_i c = -\int d^2y \int d^2x \;\; G_\Lambda(x) \, \langle \, \phi_i(y) \, \Theta(x) \, \Theta(0) \, \rangle_c$$

$$g_{ij}\beta^j = \int d^2y \int d^2x \; G_\Lambda(x) \, \langle \, \Theta(y) \, \phi_i(x) \, \Theta(0) \, \rangle_c$$

$$b_{ij}\beta^j = \int d^2y \int d^2x \; G_\Lambda(x) \langle \phi_i(y) \beta^j \phi_j(x) \Theta(0) \rangle_c \\ - \int d^2y \int d^2x \; G_\Lambda(x) \langle \phi_i(y) \beta^j \phi_j(x) \Theta(0) \rangle_c$$

Combining, we get

$$r_i = \int d^2y \int d^2x \, G_\Lambda(x) \left[\langle \phi_i(y) \, D(x) \, \Theta(0) \rangle - \langle D(y) \, \phi_i(x) \, \Theta(0) \rangle \right]$$

where

$$D(x) = \Theta(x) - \beta^i \phi_i(x)$$

is a pure contact field.

We do not want to make any assumptions about the form of $D(x)$. In particular, we do not want to use power counting. We could expand $D(x)$ in all possible local fields, with coefficients all possible products of derivatives of the sources and the metric, constrained only by covariance. This would be unwieldy. Instead, we introduce a formalism that allows a general analysis of contact fields such as $D(x)$.

Operations

Let us add sources for all the higher spin fields, so we now have a source for every local field in the theory. Then the generating functional satisfies a differential equation

$$\left[\mu\frac{\delta}{\delta\mu(x)} - \beta^i(\lambda(x))\frac{\delta}{\delta\lambda^i(x)} - \mathcal{D}(x)\right]\ln Z = 0$$

where $\mathcal{D}(x)$ is a first order differential operator in the sources whose coefficients vanish when all the sources and the metric are made constant and all the higher spin sources are set to zero.

We study correlation functions of the contact field $D(x)$ by expressing them in terms of the differential operator $\mathcal{D}(x)$.

An operation $\mathcal{O}(x)$ is a local first order differential operator on functionals of the sources and the metric, a sum of terms each of which is a product of sources and the metric and their derivatives at x times a variation wrt a source or the metric at x. We write

$$\Theta(x) = \mu \frac{\delta}{\delta \mu(x)}\,, \qquad \phi_i(x) = \frac{\delta}{\delta \lambda^i(x)}$$

depending on context to avoid confusion.

Example of an operation:

$$\partial_\mu \lambda^i \partial^\mu \lambda^j \, g^k_{ij}(\lambda)\, \phi_k(x)$$

For every operation $\mathcal{O}(x)$ there is an ordinary field $\underline{\mathcal{O}}(x)$ obtained by setting all the sources and the metric to constants (and the higher-spin sources to zero). A pure contact operation is one for which $\underline{\mathcal{O}}(x)$ vanishes.

The above renormalization identity is now written

$$[\Theta(x) - \beta(x) - \mathcal{D}(x)] \ln Z = 0\,, \qquad \beta(x) = \beta^i(\lambda(x))\phi_i(x)$$

We use the operation $\mathcal{D}(x)$ to construct the correlators involving the pure contact field $D(x) = \Theta(x) - \beta(x)$. For example,

$$\langle\, D(y)\, \phi_i(x)\, \Theta(0)\, \rangle_c = \langle\langle \phi_i(x)\, \Theta(0)\, \mathcal{D}(y) \rangle\rangle$$

where on the right are a series of operations, '$\rangle\rangle$' stands for $\ln Z$, and '$\langle\langle$' stands for setting the sources and metric to constants (and the higher spin sources to zero). Then we commute $\mathcal{D}(y)$ to the left, and finally use $\langle\langle \mathcal{D}(y) = 0$.

We make use of the WZ consistency constraints on $\mathcal{D}(x)$

$$[\Theta(x) - \beta(x) - \mathcal{D}(x),\, \Theta(y) - \beta(y) - \mathcal{D}(y)] \ln Z = 0$$

Callan-Symanzik equations

We can derive general Callan-Symanzik equations for correlators at finite separation

$$\mu\frac{\partial}{\partial\mu}\langle\,\phi_i(x)\cdots\rangle_c = \beta^i\frac{\partial}{\partial\lambda^i}\langle\,\phi_i(x)\cdots\rangle_c + \langle\,\Gamma\phi_i(x)\cdots\rangle_c + \cdots$$

by writing

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\langle\,\phi_i(x)\cdots\rangle_c &= \langle\langle\phi_i(x)\cdots\int d^2y\;\Theta(y)\rangle\rangle & (1)\\ &= \langle\langle\phi_i(x)\cdots\int d^2y\;[\beta(y)+\mathcal{D}(y)]\rangle\rangle & (2)\end{aligned}$$

and commuting to the left to get the C-Z equations with

$$\Gamma\phi_i(x) = \partial_i\beta^j\phi_j(x) - \partial_\mu J_i^\mu(x)\,.$$

where

$$\mathcal{D}\phi_i(x) = \int d^2y\;[\phi_i(x),\,\mathcal{D}(y)]\,,\qquad \underline{\mathcal{D}\phi_i}(x) = -\partial_\mu J_i^\mu(x)$$

which is a total derivative because $\int d^2x\;\underline{\mathcal{D}\phi_i}(x) = 0$.

The analysis of the remainder term r_i is long and painful. We have to introduce an infrared cutoff distance L and carefully control infrared errors. We end with an expression

$$r_i^L = \Delta g_{ij}^L \beta^j$$

$$\Delta g_{ij}^L = \int_{|x|<L} dx^2 \, [G_\Lambda(x) - G_0(x)] \langle \, \partial_\mu J_i^\mu(x) \, \phi_j(0) \, \rangle_c + i \leftrightarrow j$$

where $\partial_\mu J_i^\mu(x)$ is the term we just saw in the C-Z equations. On the way, we do an intgration by parts which produces an error of the form

$$L^3 \langle \, J_i^\mu(L) \, T_{\alpha\beta}(0) \, \rangle_c \, .$$

The assumption of no spontaneously broken conformal invariance is needed in order to be able to discard this error.

We were able to show that $r_i^L = \Delta g_{ij}^L \beta^j$ has a finite limit as $L \to \infty$, but that does not imply the same for Δg_{ij}^L. We only know that any infrared divergence in Δg_{ij}^L vanishes when contracted with β^j. Therefore it does no harm to subtract the infrared divergence, defining

$$\Delta g_{ij} = \Delta g_{ij}^L - \text{IR subtractions}$$

to get

$$r_i = \Delta g_{ij} \beta^j$$

which gives the gradient formula.

Comments

• Konechny has a nice example showing the necessity of the IR condition. Consider a free compact boson $X(x)$ with background charge Q at ∞. The action is

$$\frac{1}{8\pi}\int d^2x\,(\lambda\partial_\mu X\partial^\mu X + QX\mu^2 R_2)$$

on the plane, where $\int d^2x\,\sqrt{g}R_2 = 0$ so the zero mode integral is well defined. Take λ to be the coupling constant. The beta function vanishes. The central charge is $c = 1 + \frac{3Q^2}{\lambda}$. The gradient formula fails because c is certainly not stationary in λ. Global conformal invariance is broken by the background charge

$$J^\mu_\lambda = \frac{Q}{\lambda}\partial^\mu X, \qquad \langle\, J^\mu_\lambda(x)\, T_{\alpha\beta}(0)\,\rangle_c = O(|x|^{-3})$$

• Konechny has calculated that $\Delta g_{ij} \neq 0$ for dilaton couplings in the GNLM. So there exist theories for which the correction term Δg_{ij} is needed in the gradient formula.

Question

- The IR condition is distasteful, as is the non-local character of the formula for Δg_{ij}. Can $c(\lambda)$ in the gradient formula be modified so that Δg_{ij} and the IR condition are not needed?

A recent geometric result

DF and Konechny, 2012 derive a formula for the curvature tensor on the space of 2D conformal field theories, based on Kutasov, 1989.

$$R_{ijkl} = \mathrm{RV} \int \frac{d^2\eta}{2\pi} \, (-\ln|\eta|) \, \langle \, \phi_i(1)\phi_j(\eta)\phi_k(\infty)\phi_l(0) \, \rangle_c$$

The letters 'RV' denote a particular prescription for regularizing and subtracting the divergences — a hard-sphere (point-splitting) cutoff followed by minimal subtraction. A similar curvature formula is derived for the space of boundary conformal field theories.

One possible motivation is a conjecture of DF, 2003 that the supermanifold of susy string backgrounds satisfies an Einstein equation $R_{ij} = \frac{1}{4} g_{ij}$.