Free k-form cft in $2n$ dimensions

Daniel Friedan

New High Energy Theory Center, Rutgers University
and Natural Science Institute, The University of Iceland
dfriedan@gmail.com

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Abstract

Some elementary observations are made about the non-unitary conformal field theory of the free k-form in $2n$ space-time dimensions. The prototype is the free 1-form, aka the free conformal scalar field. The Ward identities are written in terms of the intersection form on currents. The charge-carrying and dual-charge-carrying vertex operators are described. Some peculiarities are remarked in the construction of the theories on a general conformal space-time manifold.

Contents

1 The free scalar cft

The free conformal scalar field in d=$2n$ dimensional euclidean space-time has classical action

$$S[\phi] = \int d^d x \; \frac{1}{2} \phi(x) \Box^n \phi(x) \qquad \Box = -g^{\mu\nu}\partial_\mu \partial_\nu \tag{1.1}$$

$\phi(x)$ has scaling dimension 0. The shift by a constant,

$$\phi(x) \rightarrow \phi(x) + a \tag{1.2}$$

is a global internal symmetry. The classical action can be written on an arbitrary $2n$-manifold such that it is

(1) covariant in the space-time metric $g_{\mu\nu}(x)$,
(2) conformally invariant, i.e., invariant under $g_{\mu\nu}(x) \rightarrow e^{f(x)} g_{\mu\nu}(x)$, and
(3) shift invariant.

The conformally invariant version of the operator $\Box^n$ was constructed in [1]. Its shift invariance was established in [2] (or possibly earlier). The conformal field theory was discussed in [3].

The Schwinger-Dyson equation

$$\langle \Box^n \phi(x) \, \phi(x_0) \rangle = \delta^d(x - x_0) \tag{1.3}$$

gives the for the euclidean 2-point function

$$\langle \phi(x) \, \phi(x_0) \rangle = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-x_0)}}{(p^2)^n} \tag{1.4}$$

which is log-divergent at $p = 0$. The integral must be cut off near $p = 0$ and renormalized at some infrared scale μ. The resulting 2-point function grows logarithmically at large distance

$$\langle \phi(x) \, \phi(x_0) \rangle \longrightarrow \mathrm{const}(-\ln \mu |x - x_0|) \tag{1.5}$$

violating cluster decomposition.

We can attempt to cure this IR pathology by the strategy that works for the 2d theory [?]. First, the shift symmetry is made into a gauge symmetry. That is, only shift-invariant quantities are observable. These are generated by the 1-form $j(x) = d\phi(x)$. The scalar field $\phi(x)$ is no longer a local field in the quantum field theory. Rather, $\phi(x)$ is the multi-valued integral of $j(x) = d\phi(x)$. At this point it seems appropriate to re-name the theory *the free conformal* 1*-form*.

As in 2d, this is not a complete cure. The problem remains that the ground state is not normalizable because the zero-mode of $\phi(x)$ is unbounded. The observable

$$e^{ik[\phi(x)-\phi(x_0)]} = e^{ik\int_{x_0}^{x} j(x)} \tag{1.6}$$

has expectation value

$$\langle e^{ik[\phi(x)-\phi(x_0)]} \rangle = \delta(k) \; |x - x_0|^{-k^2} \tag{1.7}$$

where the $\delta(k)$ factor comes from the integral over the unbounded zero-mode. Taking $k \to 0$ gives $\langle 1 \rangle = \infty$ where $\langle 1 \rangle$ is the square of the norm of the ground state.

The second step of the cure is to modify the theory so that the zero-mode becomes bounded. The scalar field $\phi(x)$ is made to take values in a circle of radius R. The identification $\phi(x) \sim \phi(x) + 2\pi R$ is made at each point x in space-time. The parameter k in the observable (1.6) now takes discrete values $k = m/R$, $m \in \mathbb{Z}$, and $\delta(k)$ is the discrete delta-function, so $\delta(0)$ is finite. The resulting theory remains conformally invariant because the shift symmetry commutes with the conformal group.

Such a construction can be carried out as a straightforward generalization of the familiar case $d = 2$. The fundamental field is the 1-form $j(x)$ satisfying $dj = 0$. Conjugate to $j(x)$ is the divergenceless 1-current

$$J^\mu(x) = g^{\mu\nu}\Box^{n-1} j_\nu(x) \qquad \partial_\mu J^\mu(x) = 0 \tag{1.8}$$

which can equally well be regarded as a closed $(d-1)$-form

$$j^*_{\mu_1\cdots\mu_{d-1}}(x) = \epsilon_{\mu_1\cdots\mu_{d-1}\mu}\, J^\mu(x) \qquad dj^* = 0 \tag{1.9}$$

Charges are associated to integral $(d-1)$-cycles ξ_{d-1} and dual charges are associated to integral 1-cycles ξ_1

$$Q^*(\xi_{d-1}) = i^{-1}\int_{\xi_{d-1}} j^* \qquad Q(\xi_1) = i^{-1}\int_{\xi_1} j \tag{1.10}$$

$Q^*(\xi_{d-1})$ generates the shift symmetry. $Q(\xi_1)$ measures the winding number around ξ_1. A multi-valued 0-form $\phi(x)$ is constructed by integrating $d\phi = j$. A multi-valued $(d-2)$-form $\phi^*(x)$ is constructed by integrating $d\phi^* = j^*$. The basic charged fields under the shift symmetry are the vertex operators $e^{ik[\phi(x)-\phi(x_0)]}$ or, more generally, $e^{ik\int_{\xi_0}\phi}$ for ξ_0 an integral 0-cycle. The extended objects carrying dual charge (winding number) are the dual vertex operators $e^{ik\int_{\xi_{d-2}}\phi^*}$ which live on integral $(d-2)$-cycles ξ_{d-2} in space-time.

Section 2 summarizes the mathematical language of forms and currents and some of the basic results from [2] on conformally invariant differential operators. Section 3 describes the charge structure of the free conformal 1-form cft. Section 4 explicitly constructs the free 1-form cft on euclidean space-time and its conformal compactification $S^d = \mathbb{R}^d \cup \{\infty\}$. Section 5 generalizes to the free conformal k-form and sketches a formulation that does not assume an orientation of space-time. Section 6 mentions some peculiar features of the theories on a general conformal space-time.

2 Mathematical preliminaries

Space-time is a manifold M of even dimension $d = 2n$. It has a conformal riemannian structure, i.e., a riemannian metric $g_{\mu\nu}(x)$ defined up to Weyl transforms $g_{\mu\nu}(x) \to e^{f(x)} g_{\mu\nu}(x)$. For simplicitly, M is assumed to be compact. The basic example is the d-sphere $S^d = \mathbb{R}^d \cup \{\infty\}$ which is the conformal compactification of euclidean space-time. For convenience, we assume M to be oriented. The assumption of an orientation can be avoided at the cost of some minor complications (see section 5.1).

2.1 Forms and currents

The basic fields of the theory will be differential forms on M. Correlation functions in quantum field theory are distributions, so the differential forms in the correlation functions are to be integrated against smooth "smearing functions". The natural "smearing functions" for k-forms are the *k-currents* (k-vector-valued densities)

$$\xi_k(x) = \xi_k^{\mu_1\cdots\mu_k}(x)\, d^d x \tag{2.1}$$

The "smearing" — the pairing between a k-form $\omega^k(x)$ and a k-current $\xi_k(x)$ — will be written variously

$$(\omega^k, \xi_k) = \omega^k(\xi_k) = \int_{\xi_k} \omega^k = \int \omega^k_{\mu_1\cdots\mu_k}(x)\, \frac{1}{k!}\, \xi_k^{\mu_1\cdots\mu_k}(x)\, d^d x \tag{2.2}$$

The smearing currents $\xi_k^{\mu_1\cdots\mu_k}(x)\, d^d x$ should be smooth in x. However, correlation functions are nonsingular except at coincident points, more general distributional smearing currents can be used as long as account is taken of the singularities at coincident points. A special kind of distributional k-current is the generalized Dirac delta-function $\xi_k(x)$ localized on an oriented k-submanifold N_k in space-time, defined by

$$\int_{\xi_k} \omega^k = \int_{N_k} \omega^k \tag{2.3}$$

where the rhs is the integral of the k-form $\omega^k(x)$ over the oriented k-submanifold N_k. The current $\xi_k(x)$ is said to *represent* the oriented submanifold N_k. More generally, there is a distributional k-current representing any integral k-chain N_k in space-time, defined by the same formula (2.3).

The space of k-forms and the space of k-currents

$$\Omega^k = \text{the space of } k\text{-forms} \qquad \mathcal{D}_k = \text{the space of } k\text{-currents} \tag{2.4}$$

are formally dual to each other under the pairing (2.2)

$$\mathcal{D}_k = (\Omega^k)^* \qquad \Omega^k = (\mathcal{D}_k)^* \tag{2.5}$$

The *boundary operator* ∂ on currents is dual to the exterior derivative d on forms,

$$d\colon \Omega^k \to \Omega^{k+1} \qquad \partial\colon \mathcal{D}_{k+1} \to \mathcal{D}_k \qquad \partial = d^* \qquad d = \partial^* \qquad \int_{\partial\xi_{k+1}} \omega^k = \int_{\xi_{k+1}} d\omega^k \tag{2.6}$$

$$(d\omega^k)_{\mu_0\cdots\mu_k} = \partial_{\mu_0}\omega^k_{\mu_1\cdots\mu_k} - \partial_{\mu_1}\omega^k_{\mu_0\mu_2\cdots\mu_k} + \cdots \qquad (\partial\xi_{k+1})^{\mu_1\cdots\mu_k}(x) = -\partial_{\mu_0}\xi_{k+1}^{\mu_0\mu_1\cdots\mu_k}(x) \tag{2.7}$$

The boundary operator on currents agrees with the usual boundary operator on oriented submanifolds and integral chains.

2.2 The intersection form on currents

The orientation of M is expressed by two invariants (which are inverses of each other)

$$\epsilon^{\mu_1\cdots\mu_d}\, d^d x \qquad (d^d x)^{-1}\epsilon_{\mu_1\cdots\mu_d} \tag{2.8}$$

The latter gives the bilinear *intersection form* on currents

$$I(\xi_k, \xi_{d-k}) = \int_M \frac{1}{k!(d-k)!}\, \xi_k^{\mu_1\cdots\mu_k}(x)\, \xi_{d-k}^{\mu_{k+1}\cdots\mu_d}(x)\, \epsilon_{\mu_1\cdots\cdots\mu_d}\, d^d x \tag{2.9}$$

When ξ_k and ξ_{d-k} are distributional currents representing oriented submanifolds or integral chains in general position, i.e., such that the integral (2.9) makes sense, then $I(\xi_k, \xi_{d-k})$ is the integer intersection number. The intersection form satisfies an integration by parts formula

$$I(\partial\xi_k, \xi_{d-k+1}) = (-1)^k I(\xi_k, \partial\xi_{d-k+1}) \tag{2.10}$$

and a symmetry formula

$$I(\xi_{d-k}, \xi_k) = (-1)^{k(d-k)} I(\xi_k, \xi_{d-k}) \tag{2.11}$$

2.3 Currents as axial forms, forms as axial currents

We are using the mathematical language in which k-forms are integrated over oriented k-submanifolds which are represented by distributional k-currents. In physics usage, a 1-current $\xi_1^\mu(x)\, d^d x$ is integrated over a hypersurface (with a choice of normal direction) to give a flux. More generally, a k-current is integrated over a co-oriented $(d{-}k)$-submanifold N_{d-k} to give a generalized flux. The co-orientation is an orientation of the normal bundle of the submanifold. Co-oriented $(d{-}k)$-submanifolds are then represented by distributional k-forms.

Given an orientation of the space-time M, these two languages are exactly equivalent. The mathematical language seems simpler to apply in a general setting, so we assume an orientation and translate physics k-currents to $(d{-}k)$-forms.

Given an orientation of the space-time M, an co-orientation of a submanifold is equivalent to an orientation. So there is an equivalence between k-forms and $(d{-}k)$-currents, where the equivalence changes sign with a change of the space-time orientation. A $(d{-}k)$-form is an *axial* k-current.

Given an orientation of M, the integral of a k-current $\xi_k(x)$ over a (co-)oriented $(d{-}k)$-submanifold N_{d-k} is

$$\int_{\xi_{d-k}} \xi_k(x) = I(\xi_k, \xi_{d-k}) \tag{2.12}$$

where ξ_{d-k} is the distributional $(d{-}k)$-current that represents the submanifold N_{d-k}. On the rhs, the distributional current $\xi_k(x)$ is integrated against the smooth current $\xi_{d-k}(x)$. For example, suppose $M = \mathbb{R}^d$ and N_{d-k} is the hyperplane spanned by $x^{k+1}, \ldots, x^d$. Then

$$I(\xi_k, \xi_{d-k}) = \int \xi_k^{1,\ldots,k}(x)\, dx^{k+1}\cdots dx^d \tag{2.13}$$

which is indeed the integral of the $(d-k)$-current over the hyperplane.

Stokes theorem for currents has the form

$$\int_{\xi_{d-k+1}} \mathbf{div}\,\xi_k = \int_{\partial\xi_{d-k+1}} \xi_k \qquad \mathbf{div}\colon \mathcal{D}_k \to \mathcal{D}_{k-1} \tag{2.14}$$

which is

$$I(\mathbf{div}\,\xi_k, \xi_{d-k+1}) = I(\xi_k, \partial\xi_{d-k+1}) = (-1)^k I(\partial\xi_k, \xi_{d-k+1}) \tag{2.15}$$

therefore the divergence operator is

$$\mathbf{div}\,\xi_k = (-1)^k \partial\xi_k \qquad (\mathbf{div}\,\xi_k)^{\mu_1\cdots\mu_{k-1}}(x) = \partial_{\mu_k}\xi_k^{\mu_1\cdots\mu_k}(x) \tag{2.16}$$

The orientation-dependent equivalence between k-currents and $(d-k)$-forms is

$$\varepsilon\colon \mathcal{D}_k \to \Omega^{d-k} \qquad \int_{\xi_{d-k}} \varepsilon\xi_k = \int_{\xi_{d-k}} \xi_k \qquad (\varepsilon\xi_k, \xi_{d-k}) = I(\xi_k, \xi_{d-k})$$

$$\varepsilon^{-1}\colon \Omega^{d-k} \to \mathcal{D}_k \qquad \int_{\xi_{d-k}} \omega^{d-k} = \int_{\xi_{d-k}} \varepsilon^{-1}\omega^{d-k} \qquad (\omega^{d-k}, \xi_{d-k}) = I(\varepsilon^{-1}\omega^{d-k}, \xi_{d-k},)$$

$$(\varepsilon\xi_k)_{\mu_{k+1}\cdots\mu_d}(x) = \frac{1}{k!}\,\xi_k^{\mu_1\cdots\mu_k}(x)\,\epsilon_{\mu_1\cdots\mu_d} \qquad (\varepsilon^{-1}\omega^{d-k})^{\mu_1\cdots\mu_k}(x)\,d^dx = \frac{1}{k!}\omega^{d-k}_{\mu_{k+1}\cdots\mu_d}(x)\,\epsilon^{\mu_1\cdots\mu_d}\,d^dx \tag{2.17}$$

The divergence on currents is equivalent to the exterior derivative on forms

$$\mathbf{div} = \epsilon^{-1}d\epsilon \tag{2.18}$$

The symmetry formula becomes

$$(\varepsilon\xi_k, \varepsilon^{-1}\omega^k) = (-1)^{k(d-k)}(\xi_k, \omega^k) \tag{2.19}$$

2.4 Conformally invariant differential operators

So far only the orientation of M has been used to construct the intersection form on currents and the equivalence between k-currents and $(d-k)$-forms.

Now the conformal structure The conformal structure has played no role to this point

Branson, Thomas and Gover, A. Rod, *Conformally invariant operators, differential forms, cohomology and a generalisation of q-curvature*, arXiv:math/0309085

$\mathcal{E}^k$ = the space of k-forms

$\mathcal{E}_k$ = the space of k-currents

$\mathcal{C}^k$ = the space of closed k-forms

$\mathcal{C}_k = (\mathcal{C}^k)^* = \mathcal{E}_k/\partial\mathcal{E}_{k+1}$

For $k \le n$, there is a conformally invariant differential operator, formally self-adjoint,

$$L_k\colon \mathcal{E}^k \to \mathcal{E}_k \qquad \mathrm{order}(L_k) = d - 2k \tag{2.20}$$

$$L_k = \delta M_k d \tag{2.21}$$

where

$$M_k\colon \mathcal{C}^k \to \mathcal{C}_k \tag{2.22}$$

is conformally invariant.

There is an elliptic complex (the *kth de Rham detour complex*)

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{L_k} \mathcal{E}_k \xrightarrow{d} \mathcal{E}_{k-1} \cdots \xrightarrow{d} \mathcal{E}_0 \tag{2.23}$$

$H^k_L(M)$ is the cohomology in degree k. There is an injection $H^k(M) \to H^k_L(M)$.

3 The 1-form cft

1. Start with a 1-form field $j(x)$ with canonical scaling dimension 1 and satisfying the equation of motion

$$dj = 0 \tag{3.1}$$

There is an associated dimensionless generalized charge (the winding number)

$$Q^*(\xi_1) = \int_{\xi_1} j \qquad \dim(Q^*) = -1 + \dim(j) = 0 \tag{3.2}$$

for every 1-cycle ξ_1 in space-time (e.g., the 1-submanifolds without boundary). The charge $Q^*(\xi_1)$ depends only on the homology class of ξ_1 because $dj = 0$.

2. Construct (locally) the 0-form $\phi(x)$ as the solution of

$$d\phi = j \qquad \partial_\mu \phi(x) = j_\mu(x) \qquad \dim(\phi) = \dim(j) - 1 = 0 \tag{3.3}$$

$\phi(x)$ is determined by (3.3) up to a constant shift

$$\phi(x) \to \phi(x) + a \tag{3.4}$$

3. Impose the identification at each point x

$$\phi(x) \sim \phi(x) + 2\pi R \tag{3.5}$$

4. Construct the dual $(d-1)$-form $j^*(x)$

$$j^*(x) = \Box^{n-1} i^{-1} {*j} \qquad j^*_{\mu_1\cdots\mu_{d-1}}(x) = \Box^{n-1}\, i^{-1} \epsilon_{\mu_1\cdots\mu_{d-1}}{}^{\nu}\, j_\nu(x) \qquad \dim(j^*) = \dim(j) + 2(n-1) = d-1 \tag{3.6}$$

5. Impose the second equation of motion

$$dj^* = 0 \tag{3.7}$$

This is equivalent to $\Box^n \phi = 0$ because

$$dj^* = 0 \quad\Longleftrightarrow\quad 0 = i{*d}j^* = -\Box^{n-1}\partial^\mu j_\mu = \Box^n \phi \tag{3.8}$$

There is an associated dimensionless charge

$$Q(\xi_{d-1}) = \int\limits_{\xi_{d-1}} j^* \qquad \dim(Q) = -(d-1) + \dim(j^*) = 0 \tag{3.9}$$

for every $(d-1)$–cycle ξ_{d-1} in space-time (e.g., the $(d-1)$-submanifolds without boundary). The charge $Q(\xi_{d-1})$ depends only on the homology class of ξ_{d-1} because $dj^* = 0$.

6. Construct a $(d-2)$-form $\phi^*(x)$ locally by integrating j^*

$$d\phi^* = j^* \qquad \dim(\phi^*) = \dim(j^*) - 1 = d-2 \tag{3.10}$$

where $\phi^*(x)$ is defined up to $(d-3)$-form gauge transformations $a^*(x)$

$$\phi^* \to \phi^* + da^* \tag{3.11}$$

Note that all of the forms j, ϕ, j^*, ϕ^* have canonical scaling dimensions.

7. Construct the free quantum theory by writing the Schwinger-Dyson equations for the 2-point correlation functions $\langle\, j^*(x)\, \phi(x')\,\rangle$ and $\langle\, \phi^*(x)\, j(x')\,\rangle$. This is done in the following sections. The S-D equation for $\langle\, j^*(x)\, \phi(x')\,\rangle$ is

$$\langle\, iQ(\partial\xi_d) \int\limits_{\xi_0} \phi \,\rangle = I(\xi_d, \xi_0) \tag{3.12}$$

The S-D equation for $\langle\, \phi^*(x)\, j(x')\,\rangle$ is

$$\langle\, iQ^*(\partial\xi_2) \int\limits_{\xi_{d-2}} \phi^* \,\rangle = I(\xi_{d-2}, \xi_2) \tag{3.13}$$

In (3.12), ξ_d is an arbitrary d-chain (e.g. a d-submanifold) whose boundary is the $(d-1)$-cycle $\partial\xi_d$, and ξ_0 is a 0-cycle (a sum of points with integer coefficients). In (3.13), ξ_2 is a 2-chain (e.g. a 2-submanifold) with boundary the 1-cycle $\partial\xi_2$, and ξ_{d-2} is a $(d-2)$-cycle. In both (3.12) and (3.13), $I(\xi_{d-k}, \xi_k)$ is the intersection number.

8. The S-D equation (3.12) says that the charge $Q(\partial\xi_d)$ generates the shift of $\int_{\xi_0}\phi$ when ξ_d intersects ξ_0. For example, take ξ_d to be a ball in space-time, so $\partial\xi_{d-1}$ is the boundary $(d-1)$-sphere. Take ξ_0 to be the Dirac delta-function $\delta_{x'}$ concentrated at a point x' inside the ball ξ_d. Then $I(\xi_d, \xi_0) = 1$ and (3.12) becomes

$$\langle\, iQ(\partial\xi_d)\ \phi(x')\,\rangle = 1 \tag{3.14}$$

In the radial quantization around the center of the ball, this becomes the operator equation

$$[iQ(\partial\xi_d),\ \phi(x')] = 1 \tag{3.15}$$

which is to say that the charge operator $Q(\partial\xi_d)$ generates the shift of $\phi(x')$

$$e^{iaQ(\partial\xi_d)}\phi(x')e^{-iaQ(\partial\xi_d)} = \phi(x') + a \tag{3.16}$$

9. The second S-D equation (3.13) says that the generalized charge $Q^*(\partial\xi_2)$ generates the shift of $\int_{\xi_{d-2}} \phi^*$ when ξ_{d-2} intersects ξ_2, which is to say when the 1-cycle $\partial\xi_2$ links ξ_{d-2}. For example, take ξ_2 to be a disk so $\partial\xi_2$ is the boundary circle. The extended charge $Q^*(\partial\xi_2)$ measures the winding of $\phi(x)$ around the circle $\partial\xi_2$.

10. $\phi(x)$ is dimensionless, so we can exponentiate it to construct the vertex operators

$$V_p(x) = e^{ip\phi(x)} \tag{3.17}$$

The S-D equation (3.12) implies the operator product formula

$$Q(\partial\xi_d)\, V_p(x) = pV_p(x) \tag{3.18}$$

when x is inside $\partial\xi_d$. So $V_p(x)$ carries charge $Q = p$.

11. $\int_{\xi_{d-2}} \phi^*$ is dimensionless, so we can exponentiate it to construct the dual generalized vertex operators

$$V_{p^*}(\xi_{d-2}) = e^{ip^* \int_{\xi_{d-2}} \phi^*} \tag{3.19}$$

which live on $(d-2)$-cycles ξ_{d-2}. The S-D equation (3.13) implies the operator product formula

$$Q^*(\partial\xi_2)\, V_{p^*}(\xi_{d-2}) = p^* V_{p^*}(\xi_{d-2}) \tag{3.20}$$

when $\partial\xi_2$ has linking number 1 with ξ_{d-2}. So $V_{p^*}(\xi_{d-2})$ carries generalized charge $Q^* = p^*$.

We construct $\phi(x')$ in the presence of a dual vertex operator $V_{p^*}(\xi_{d-2})$ by integrating the 1-form j along a path to x' from some base-point x

$$\phi(x)\, V_{p^*}(\xi_{d-2}) = \int_x^{x'} j \;\; V_{p^*}(\xi_{d-2}) \tag{3.21}$$

so $\phi(x)$ is multi-valued around any loop $\partial\xi_2$ that links with ξ_{d-2}, shifting around the loop by

$$\int_{\partial\xi_2} j\, V_{p^*}(\xi_{d-2}) = Q^*(\partial\xi_2)\, V_{p^*}(\xi_{d-2}) = p^* V_{p^*}(\xi_{d-2}) \tag{3.22}$$

The integral of $j = d\phi$ around a 1-cycle must be $2\pi R$ times an integer winding number, so

$$p^* = 2\pi R m^* \qquad m^* \in \mathbb{Z} \quad \text{(the winding number)} \tag{3.23}$$

12. The product $V_{p^*}(\xi_{d-2})\, V_p(x')$ should be single-valued. If we move x' around a loop $\partial\xi_2$ that links ξ_{d-2}, we pick up a factor which can be evaluated using the S-D equation (3.12)

$$e^{\langle ip^* \int_{\partial\xi_2} j\; ip\phi(x') \rangle} = e^{ip^* p I(\xi_2, \delta_{x'})} = e^{ip^* p} \tag{3.24}$$

so we get the Dirac quantization condition

$$p^* p \in 2\pi\mathbb{Z} \tag{3.25}$$

Since $p^* = 2\pi R m^*$, the charges of the $V_p(x)$ must lie in the dual lattice

$$p = mR^{-1} \qquad m \in \mathbb{Z} \tag{3.26}$$

13. The general class of gauge-invariant observables is formed by operator products of the vertex operators and dual vertex operators.

14. It remains to write the stress-energy tensor $T_{\mu\nu}(x)$ in terms of $j(x)$ and $j^*(x)$ and derive the conformal properties of the vertex operators and the dual vertex operators

15. The partition function on a general space-time manifold M must include a sum over charge sectors $Q(\xi_{d-1}) = m(\xi_{d-1})/R$ associated to each $(d-1)$-homology class $[\xi_{d-1}]$ in M, and a sum over dual charge sectors $Q^*(\xi_1) = m^*(\xi_1)R$ associated to each 1-homology class $[\xi_1]$ in M.

16. Everything about the construction of the theory from j is manifestly conformal invariance except equation (3.6) for $j^*(x)$. Counting powers of the metric $g_{\mu\nu}$, $\Box \sim g^{-1}$ and $*$ acting on 1-forms goes as g^{n-1}, so $j^* \sim g^0$. Thus Weyl invariance of the theory is given by the existence of a Weyl-invariant covariant operator $(\Box^{n-1}i^{-1}*)_{\mathrm{cov}}$ from 1-forms to $(d-1)$-forms which is equal to $\Box^{n-1}i^{-1}*$ in flat space-time.

3.1 Forms and charges

3.2 Ward identities

3.3 Vertex operators

4 The 1-form cft on S^d

In the following sections we derive the smeared Schwinger-Dyson equation (3.12) for $\langle\, j^*(x)\, \phi(x')\, \rangle$,

$$\langle\, i \int_{\partial\xi_d} j^* \int_{\xi_0} \phi \,\rangle = I(\xi_d, \xi_0) \tag{4.1}$$

and S-D equation (3.13) for $\langle\, j(x)\, \phi^*(x')\, \rangle$,

$$\langle\, i \int_{\partial\xi_2} j \int_{\xi_{d-2}} \phi^* \,\rangle = I(\xi_{d-2}, \xi_2) \qquad \partial\xi_{d-2} = 0 \tag{4.2}$$

These equations are to hold for arbitrary smooth smearing currents, so they are equivalent to differential equations on the unsmeared 2-point functions, which is the usual form for the Schwinger-Dyson equations. Actually, we will derive the differential equations, then smear with currents to get (4.1) and (4.2). These differential equations determine the 2-point functions completely on euclidean space-time, and therefore determine the free quantum theory completely. The S-D equations are independent of the space-time metric, so they define the quantum field theory on any space-time manifold.

4.1 2-point functions

The Schwinger-Dyson equation for $\langle\, j^*(x)\phi(x')\,\rangle$

Euclidean symmetry determines $\langle\, j^*(x)\phi(x')\,\rangle$ up to normalization,

$$\langle\, j^*_{\mu_1\cdots\mu_{d-1}}(x)\,\phi(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,\epsilon_{\mu_1\cdots\mu_{d-1}\sigma}\,p^\sigma \tag{4.3}$$

Calculate

$$\langle\, \partial_1 j^*_{2\cdots d}(x)\,\phi(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,ip_1\epsilon_{2\cdots d1}\,p^1 = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,(-i\epsilon_{12\cdots d})\,p_1^2 \tag{4.4}$$

so

$$\langle\, dj^*_{\mu_1\cdots\mu_d}(x)\,\phi(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,(-i\epsilon_{\mu_1\cdots\mu_d})p^2 \tag{4.5}$$

giving the S-D equation

$$\langle\, idj^*_{\mu_1\cdots\mu_d}(x)\,\phi(x')\,\rangle = \delta^d(x-x')\epsilon_{\mu_1\cdots\mu_d} \tag{4.6}$$

We can write (4.6)

$$\langle\, i{*}dj^*(x)\,\phi(x')\,\rangle = \delta^d(x-x')$$

which agrees with (1.3) since $i{*}dj^* = \Box^n\phi$. Thus the normalization of (4.3) is correct.

Smear the S-D equation (4.6) against a d-current $\xi_d(x)$ and a 0-current $\xi_0(x')$ to get

$$\langle\, i\int_{\xi_d} dj^* \int_{\xi_0}\phi\,\rangle = I(\xi_d,\xi_0) \tag{4.7}$$

or, equivalently,

$$\langle\, i\int_{\partial\xi_d} j^* \int_{\xi_0}\phi\,\rangle = I(\xi_d,\xi_0) \tag{4.8}$$

which is (4.1), the first of the two smeared S-D equations to be derived.

The 2-point function $\langle\,\phi^*(x)j(x')\,\rangle$

Euclidean symmetry implies, up to normalization,

$$\langle\, \phi^*_{\mu_1\cdots\mu_{d-2}}(x)j_\nu(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,\epsilon_{\mu_1\cdots\mu_{d-2}\nu\sigma}\,(-p^\sigma) \tag{4.9}$$

To check the normalization, evaluate

$$\begin{aligned}\langle\, d\phi^*_{\mu_1\cdots\mu_{d-1}}(x)j_\nu(x')\,\rangle &= \langle\, \partial_{\mu_1}\phi^*_{\mu_2\cdots\mu_{d-1}}(x)j_\nu(x')\,\rangle - \langle\, \partial_{\mu_2}\phi^*_{\mu_1\mu_3\cdots\mu_{d-1}}(x)j_\nu(x')\,\rangle + \cdots + \langle\, \partial_{\mu_{d-1}}\phi^*_{\mu_1\cdots\mu_{d-2}}(x)j_\nu(x') \\ &= \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,i\left[p_{\mu_1}\epsilon_{\mu_2\cdots\mu_{d-1}\nu\sigma} - p_{\mu_2}\epsilon_{\mu_1\mu_3\cdots\mu_{d-1}\nu\sigma} + \cdots + p_{\mu_{d-1}}\epsilon_{\mu_1\cdots\mu_{d-2}\nu\sigma}\right](-p^\sigma)\end{aligned}$$

Now derive the identity

$$p_{\mu_1}\epsilon_{\mu_2\cdots\mu_{d-1}\nu\sigma} - p_{\mu_2}\epsilon_{\mu_1\mu_3\cdots\mu_{d-1}\nu\sigma} + \cdots + p_{\mu_{d-1}}\epsilon_{\mu_1\cdots\mu_{d-2}\nu\sigma} = \epsilon_{\mu_1\cdots\mu_{d-1}\sigma}p_\nu - \epsilon_{\mu_1\cdots\mu_{d-1}\nu}p_\sigma \tag{4.10}$$

by contracting both sides with $\frac{1}{(d-1)!}\epsilon^{\mu_1\cdots\mu_{d-1}\sigma'}$ because both sides are completely antisymmetric in $\mu_1\cdots\mu_{d-1}$

$$(\delta^{\mu_1}_{\nu}\delta^{\sigma'}_{\sigma} - \delta^{\mu_1}_{\sigma}\delta^{\sigma'}_{\nu})p_{\mu_1} = \delta^{\sigma'}_{\sigma}p_\nu - \delta^{\sigma'}_{\nu}p_\sigma \tag{4.11}$$

Substitute the identity (4.10) in (**??**) to get

$$\langle\, d\phi^*_{\mu_1\cdots\mu_{d-1}}(x) j_\nu(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\, i^{-1}\left(\epsilon_{\mu_1\cdots\mu_{d-1}\sigma}p^\sigma p_\nu - \epsilon_{\mu_1\cdots\mu_{d-1}\nu}p^2\right) \tag{4.12}$$

Now use (4.3) to calculate

$$\langle\, j^*_{\mu_1\cdots\mu_{d-1}}(x)\, d\phi_\nu(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\, i^{-1}\epsilon_{\mu_1\cdots\mu_{d-1}\sigma}\, p^\sigma p_\nu \tag{4.13}$$

which differs from (4.12) only by a contact term

$$\langle\, j^*_{\mu_1\cdots\mu_{d-1}}(x)\, d\phi_\nu(x')\,\rangle - \langle\, d\phi^*_{\mu_1\cdots\mu_{d-1}}(x) j_\nu(x')\,\rangle = i^{-1}\epsilon_{\mu_1\cdots\mu_{d-1}\nu}\delta^d(x-x') \tag{4.14}$$

so the normalization of (4.9) is correct. It gives the correct 2-point function away from coincident points.

We see from (4.14) that we cannot have both of

$$\langle\, j^*(x)\; j(x')\,\rangle = \langle\, d\phi^*(x)\; j(x')\,\rangle \qquad \langle\, j^*(x)\; j(x')\,\rangle = \langle\, j^*(x)\; d\phi(x')\,\rangle \tag{4.15}$$

There is an unavoidable contact term ambiguity in $\langle\, j^*(x)\; j(x')\,\rangle$. Smearing (4.14) against currents gives

$$\langle\, i\int_{\xi_{d-1}} j^* \int_{\xi_1} d\phi\,\rangle - \langle\, i\int_{\xi_{d-1}} d\phi^* \int_{\xi_1} j\,\rangle = I(\xi_{d-1},\xi_1) \tag{4.16}$$

which should have an interpretation as the chiral anomaly.

The Schwinger-Dyson equation for $\langle\, \phi^*(x) j(x')\,\rangle$

The equation of motion $dj = 0$ implies that $\langle\, \phi^*(x) dj(x')\,\rangle$ vanishes up to a contact term, which is the Schwinger-Dyson equation for $\langle\, \phi^*(x) j(x')\,\rangle$. However, equation (4.9) for $\langle\, \phi^*(x) j(x')\,\rangle$ gives

$$\langle\, \phi^*_{\mu_1\cdots\mu_{d-2}}(x) dj_{\nu_1\nu_2}(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\, i^{-1}\left[\epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\sigma}p_{\nu_2} \quad - \;(\nu_1\leftrightarrow\nu_2)\right]\, p^\sigma \tag{4.17}$$

which is not a pure contact term. However, because of the gauge symmetry $\phi^*\to\phi^*+da^*$, we should only expect a contact term up to a gauge transformation. Re-write the identity (4.10)

$$\begin{aligned}\epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\sigma}p_{\nu_2} = \epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\nu_2}p_\sigma + &\\ + \big(p_{\mu_1}\epsilon_{\mu_2\cdots\mu_{d-2}\nu_1\nu_2\sigma} - p_{\mu_2}\epsilon_{\mu_1\mu_3\cdots\mu_{d-2}\nu_1\nu_2\sigma} + \cdots - p_{\mu_{d-2}}\epsilon_{\mu_1\cdots\mu_{d-3}\nu_1\nu_2\sigma} + p_{\nu_1}\epsilon_{\mu_1\cdots\mu_{d-2}\nu_2\sigma}\big)&\end{aligned}$$

or

$$\epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\sigma}p_{\nu_2} - (\nu_1\leftrightarrow\nu_2) = \epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\nu_2}p_\sigma + \big(p_{\mu_1}\epsilon_{\mu_2\cdots\mu_{d-2}\nu_1\nu_2\sigma} - p_{\mu_2}\epsilon_{\mu_1\mu_3\cdots\mu_{d-2}\nu_1\nu_2\sigma} + \cdots - p_{\mu_{d-2}}\epsilon_{\mu_1\cdots\mu_{d-3}\nu_1\nu_2\sigma}\big) \tag{4.18}$$

Substitute in the Schwinger-Dyson equation (4.17) to get

$$
\begin{aligned}
&\langle\, \phi^*_{\mu_1\cdots\mu_{d-2}}(x) dj_{\nu_1\nu_2}(x')\,\rangle = i^{-1}\epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\nu_2}\delta^d(x-x') \\
&+\int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\; i^{-1}\left(p_{\mu_1}\epsilon_{\mu_2\cdots\mu_{d-2}\nu_1\nu_2\sigma} - p_{\mu_2}\epsilon_{\mu_1\mu_3\cdots\mu_{d-2}\nu_1\nu_2\sigma} + \cdots - p_{\mu_{d-2}}\epsilon_{\mu_1\cdots\mu_{d-3}\nu_1\nu_2\sigma}\right)\, p^\sigma
\end{aligned}
\tag{4.19}
$$

which we can write as a contact term plus a gauge term,

$$
\langle\, \phi^*_{\mu_1\cdots\mu_{d-2}}(x) dj_{\nu_1\nu_2}(x')\,\rangle = i^{-1}\epsilon_{\mu_1\cdots\mu_{d-2}\nu_1\nu_2}\delta^d(x-x') + \langle\, da^*_{\mu_1\cdots\mu_{d-2}}(x) dj_{\nu_1\nu_2}(x')\,\rangle \tag{4.20}
$$

with

$$
\langle\, a^*_{\mu_2\cdots\mu_{d-2}}(x) dj_{\nu_1\nu_2}(x')\,\rangle = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip(x-x')}}{p^2}\,\epsilon_{\mu_2\cdots\mu_{d-2}\nu_1\nu_2\sigma}\,(-p^\sigma) \tag{4.21}
$$

Smeared against currents, this is

$$
\langle\, i\int_{\xi_{d-2}}\phi^*\int_{\partial\xi_2} j\,\rangle = I(\xi_{d-2},\xi_2) + \langle\, i\int_{\partial\xi_{d-2}} a^*\int_{\xi_2} dj\,\rangle \tag{4.22}
$$

Taking ξ_{d-2} to be a $(d-2)$-cycle, we get the gauge-invariant equation

$$
\langle\, i\int_{\xi_{d-2}}\phi^*\int_{\partial\xi_2} j\,\rangle = I(\xi_{d-2},\xi_2) \qquad \partial\xi_{d-2}=0 \tag{4.23}
$$

which is the second Schwinger-Dyson equation (4.2) that was to be derived.

4.2 Conformal invariance

4.3 Ward identities

5 Generalization to the free k-form cft

There is an immediate generalization to a free conformal k-form, $1\le k\le n$:

1. Start with a k-form field $j(x)$ of scaling dimension k satisfying the equation of motion $dj=0$.
2. The generalized charges $Q^*(\xi_k)=\int_{\xi_k} j$ live on k-cycles.
3. Integrate $j=d\phi$ to get a $(k-1)$-form ϕ defined up to $(k-2)$-form gauge transformations $\phi\to\phi+da$.
4. Impose the identifications, for all ξ_{k-1},

$$
\int_{\xi_{k-1}}\phi \sim \int_{\xi_{k-1}}\phi + 2\pi R \tag{5.1}
$$

5. Construct the dual $(d-k)$-form $j^* = \Box^{n-k} i^{-1}{*j}$ (which presumably can be made covariant and Weyl invariant).

6. Impose the second equation of motion $dj^* = 0$.

7. The generalized charges $Q(\xi_{d-k}) = \int_{\xi_{d-k}} j^*$ live on $(d-k)$-cycles.

8. Integrate $j^* = d\phi^*$ to get a $(d-k-1)$-form ϕ^* defined up to $(d-k-2)$-form gauge transformations $\phi^* \to \phi^* + da^*$.

9. Write Schwinger-Dyson equations

$$\langle\, iQ(\partial\xi_{d-k+1}) \int\limits_{\xi_{k-1}} \phi\,\rangle = I(\xi_{d-k+1}, \xi_{k-1}) \qquad \langle\, iQ^*(\partial\xi_{k+1}) \int\limits_{\xi_{d-k-1}} \phi^*\,\rangle = I(\xi_{d-k-1}, \xi_{k+1}) \tag{5.2}$$

   for arbitrary cycles ξ_{k-1} and ξ_{d-k-1} and arbitrary boundaries $\partial\xi_{d-k+1}$ and $\partial\xi_{k+1}$.

10. The vertex operator $V_p(\xi_{k-1}) = e^{ip\int_{\xi_{k-1}}\phi}$ lives on $(k-1)$-cycles ξ_{k-1}. It has generalized charge $Q(\partial\xi_{d-k+1}) = p$ for every $(d-k)$-boundary $\partial\xi_{d-k+1}$ that links ξ_{k-1}.

11. The dual vertex operator $V_{p^*}(\xi_{d-k-1}) = e^{ip^*\int_{\xi_{d-k-1}}\phi^*}$ lives on $(d-k-1)$-cycles ξ_{d-k-1}. It has generalized charge $Q^*(\partial\xi_{k+1}) = p^*$ for every k-boundary $\partial\xi_{k+1}$ that links ξ_{d-k-1}.

12. Note the gauge invariance of the vertex operators and the dual vertex operators. For coherence, the treatment above of the case $k=1$ should be revised because the $V_p(x) = e^{ip(x)}$ are not invariant under the "gauge" symmetry $\phi(x) \to \phi(x) + a$ of the equation $d\phi = j$. Instead, we should use the gauge-invariant vertex operators $V_p(\xi_0) = e^{ip\int_{\xi_0}\phi}$ with ξ_0 a 0-cycle in the sense that its boundary $\int_{\xi_0} 1$ vanishes.

13. The partition function of the qft on a general space-time manifold M is a sum over charge sectors $Q(\xi_{d-k}) = m(\xi_{d-k})R^{-1}$ associated to to the $(d-k)$-homology classes $[\xi_{d-k}]$ of M, and over dual charge sectors $Q^*(\xi_k) = 2\pi R m^*(\xi_k)$associated to the k-homology classes $[\xi_k]$.

5.1 Orientation-independent formulation

6 Peculiarities on general conformal space-times

6.1 Zero and negative eigenvalues

6.2 The partition function on $M = (S^1)^d)$

Consider the free conformal scalar $\phi(x) \sim \phi(x) + 2\pi R$. For simplicity, take $M = (S^1)^d$, the d-torus. Let the lengths of the S^1 be $\beta = (\beta_1, \ldots, \beta_d)$.

The partition function will be

$$Z(\beta) = Z_{\text{fluctuations}}(\beta) \sum_w e^{-S[\phi_w]} \tag{6.1}$$

where the ϕ_w are the classical minima of $S[\phi]$ with winding numbers $w_1, w_2, \ldots, w_d$ around the S^1

$$\phi_w(\tau, \vec{x}) = 2\pi R \sum_i w_i \beta_i^{-1} \tau_i \qquad w_i \in \mathbb{Z} \tag{6.2}$$

For $d > 2$,

$$S[\phi_w] = 0 \tag{6.3}$$

so $Z(\beta) = \infty$.

To get a meaningful partition function, we have to introduce chemical potentials $\theta - (\theta_1, \ldots, \theta_d)$ for the winding numbers

$$Z(\beta, \theta) = Z_{\text{fluctuations}}(\beta) \sum_w e^{-S[\phi_w] + i \sum_i \theta_i w_i} \tag{6.4}$$

Then

$$Z(\beta, \theta) = Z_{\text{fluctuations}}(\beta) \delta^d(\theta) \tag{6.5}$$

6.3 $j^*(x)$ is not conformally invariant

1. Whether or not $\phi(x)$ takes values in a circle, it is likely that there exist compact riemannian space-times for which the covariant version of $\Box^n$ has negative eigenvalues, so the functional integral over the fluctuations of $\phi(x)$ cannot be defined because the covariant version of the quadratic action functional $S[\phi]$ is not bounded below.

2. When $\phi(x)$ takes values in a circle, $\phi(x) \sim \phi(x) + 2\pi R$, the partition function is infinite on a space-time of the form $S^1 \times M_{d-1}$, because the classical solutions ϕ_0 have action $S[\phi_0] = 0$ for all winding numbers, so the sum over winding modes diverges.

The construction given here is incomplete in two respects:

1. The correlation functions of the free quantum field theory can be calculated once the 2-point functions of the currents are known. Here, we construct the two-point function $\langle\, j^*(x)\, j(x')\,\rangle$ on any space-time manifold. Then $\langle\, j^*(x)\, j^*(x')\,\rangle$ can be obtained by applying the appropriate differential operator. But we only construct $\langle\, j(x)\, j(x')\,\rangle$ on the d-sphere. It remains to construct it on an arbitrary space-time manifold.

2. The partition function remains to be constructed.

Only the local construction is given here. It gives a sensible conformal field theory in euclidean space-time. However, it is not obvious that the construction can be extended to the d-sphere or to other compact space-times. The extension to the d-sphere will depend on the behavior of the theory at infinity in euclidean space-time. The underlying problem is the non-unitarity of the theory.

There is a well-known example in d=2: the non-unitary 2-d cft of the massless periodic scalar with modified stress-energy tensor $T(z) = -(\partial\phi)^2 + Q\partial^2\phi$. It has sensible correlation functions on the plane, but there is a background charge Q at infinity. The partition function on the 2-sphere is zero. The non-zero correlation functions on the 2-sphere are of products of fields whose total charge is equal to Q.

Consider the basic case $k = 1$, the free conformal 1-form. The $\langle\, j^*(x) j(x')\,\rangle$ 2-point function goes as

$$\langle\, j^*(x) j(x')\,\rangle \sim |x - x'|^{-d} \xrightarrow[x\to\infty]{} |x|^{-d} \tag{6.6}$$

But $j^*(x)$ has scaling dimension $d-1$, so it should go as $|x|^{-2(d-1)}$. It looks as if there could be a background field at infinity of dimension $2-d$.

Another, probably related, discrepancy is in the operator products of the currents with the stress-energy tensor,

$$T_{\mu\nu}(x)\, j(x') \sim |x-x'|^{-d} j(x) + \cdots \tag{6.7}$$

but

$$T_{\mu\nu}(x)\, j^*(x') \sim |x-x'|^{-2(d-1)} j(x) + \cdots \tag{6.8}$$

We can still expect $j^*(x)$ to transform covariantly as a $(d-1)$-form under conformal symmetries. When we calculate an infinitesimal conformal transformation of j^* by integrating a conformal vector field against the $T_{\mu\nu}j^*$ ope, we might expect that only sub-leading terms contribute, giving the covariant conformal transformation of j^*. But the "anomalous" leading terms in the ope should have some implications.

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