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c-THEOREM AND SPECTRAL REPRESENTATION

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Zamolodchikov's c-theorem is reformulated by using the spectral representation for the two-point function of the stress tensor. This approach makes explicit the unitarity constraints on the field theory and implements a nice physical picture of the renormalization group flow. An attempt is made to generalize the theorem above two space-time dimensions. There are two candidate c-functions, the spectral densities for spin-zero and spin-two intermediate states. The latter one is ruled out by means of examples. The spin-zero density can satisfy a generalized c-theorem, if the corresponding "central charge" is well defined at the fixed points. A meaningful charge is obtained by defining the theory on curved hyperbolic space. However, its limit to flat space needs some assumptions which seem to hold for free theories only. As a by-product, the trace anomaly in four dimensions is related to the spectral densities.

1. Introduction

The flow of the renormalization group (RG) [1] is the one-parameter transformation $\mathrm{d}g^i = -\beta^i \mathrm{d}t$ in the space of field theories $\mathscr{Q}$ given by the vector field $\beta^i(g) = \Lambda\, \mathrm{d}g^i/\mathrm{d}\Lambda$. In the early days of this subject, Wallace and Zia [2] first raised the question whether this flow satisfies simple properties. They showed that the RG is a gradient flow

$$\beta^i(g) = G^{ij}(g)\, \partial\Phi(g)/\partial g^j , \tag{1.1}$$

where G^{ij} is a riemannian metric, in the multi-coupling $\lambda\varphi^4$ theory in $n = 4 - \epsilon$ dimensions, to three-loop order.

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Let us examine the consequences of the gradient flow. It forbids limit cycles and more involved recurrent behaviours, like strange attractors. There are only fixed points, which are the critical points of the potential Φ. Away from them, Φ is monotonically decreasing along the flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi \equiv -\beta^i \frac{\partial}{\partial g^i}\Phi = -(\nabla\Phi)^2 \leqslant 0. \tag{1.2}$$

Therefore the flow is irreversible. This fits our naive understanding of the RG transformation, as in the Kadanoff block-spin transformation. The degrees of freedom at short distance are averaged to define an effective theory at large distance, causing an irreversible loss of information. Then one seeks for an appropriate entropy function which monotonically increases. However, this intuitive argument is too rough, as its opposite sounds good as well. Namely, long distance physics may display a variety of patterns and contain much more "information" than short distance physics.

More evidence for the gradient flow is provided by the general nonlinear σ-model in two dimensions* [3]. This theory describes the propagation of a string in the presence of background fields, which represent its massless states, like the graviton $G_{\mu\nu}$ and the dilaton ϕ. These background fields are couplings of the σ-model, and indeed have a gradient flow, $\beta^\phi \propto \delta\Phi[G_{\mu\nu}, \phi, \dots]/\delta\phi, \dots$, to low order in perturbation theory [4]. There are arguments extending this property to all orders [5].

Moreover, the functional Φ turns out to be the effective action for the low energy limit of the string theory [4]. Then eq. (1.1) has a very nice physical interpretation. The condition for scale (and conformal) invariance $\beta^\phi = 0$, $\beta^G = 0, \dots$, is equivalent to the equation of motion for the low energy states of the string, $\delta_\phi\Phi = 0$, $\delta_G\Phi = 0, \dots$. As it is often said, two-dimensional conformal field theories correspond to classical vacua of the string. Moreover, irreversibility of the flow in eq. (1.2) corresponds to the minimum energy principle for the stable vacuum.

Irreversibility of the flow is proven in two dimensions by Zamolodchikov's c-theorem [6]. It says that there exists a function $c(g)$, which is monotonically decreasing along the flow

$$\frac{\mathrm{d}}{\mathrm{d}t}c \equiv -\beta^i \frac{\partial}{\partial g^i}c(g) = -6\pi^2\beta^i\beta^j G_{ij} \leqslant 0 \tag{1.3}$$

* Technically speaking, this discussion extends to $n = 2 + \epsilon$ dimensions, because in minimal subtraction $\beta^i_{(n=2+\epsilon)} = \epsilon g^i + \beta^i_{(n=2)}$.

and stationary for conformal invariant theories,

$$\beta^i = 0 \quad \Leftrightarrow \quad \partial c/\partial g^i = 0,$$

where it takes the value of the Virasoro central charge. The proof is very simple and uses two new technical ingredients, besides euclidean invariance: (i) the existence of the stress tensor $T_{\mu\nu}$, (ii) unitarity of the field theory.

Let us sketch it. Consider the correlators of the components of the stress tensor $\Theta \equiv T^{\mu}_{\mu}$ and $T \equiv T_{zz}$. By euclidean invariance and the absence of anomalous dimensions, they can be parametrized as follows,

$$\langle T(z,\bar{z})T(0,0)\rangle = \frac{F(z\bar{z}\Lambda)}{z^4}, \qquad \langle \Theta(z,\bar{z})T(0,0)\rangle = \frac{G(z\bar{z}\Lambda)}{z^3\bar{z}},$$

$$\langle \Theta(z,\bar{z})\Theta(0,0)\rangle = \frac{H(z\bar{z}\Lambda)}{z^2\bar{z}^2}, \tag{1.4}$$

where $z = x^1 + ix^2$ is the complex coordinate, and Λ is a mass scale of the theory. Euclidean invariance implies the conservation of the stress tensor, which gives differential relations among the scalar functions F, G, H, written in terms of $\dot{F} = z\bar{z}\, \mathrm{d}F/\mathrm{d}(z\bar{z})$. They give

$$\dot{C} = -\tfrac{3}{4}H \leqslant 0, \tag{1.5}$$

for the quantity $C = 2(F - \tfrac{1}{2}G - \tfrac{3}{16}H)$, which reduces to the central charge at the fixed point, where $\Theta = 0$.

Unitarity of the theory says that H is positive definite, and it gives the inequality in the r.h.s. of eq. (1.5).

Next, this equation is rewritten in terms of couplings and their derivatives, by using the Callan–Symanzik equation $(\Lambda\,\partial/\partial\Lambda + \beta^i\partial/\partial g^i)C(z\bar{z}\Lambda, g^i(\Lambda)) = 0$. The trace Θ is expressed in terms of the relevant fields ϕ_i of the theory around a fixed point, and their correlators define a riemannian metric in $\mathscr{C}$

$$\Theta = 2\pi\beta^i\phi_i, \qquad \langle\phi_i(x)\phi_j(0)\rangle|_{|x|=1} = G_{ij}. \tag{1.6}$$

Equivalent definitions are

$$\frac{\mathrm{d}}{\mathrm{d}t}S \equiv \beta[S] = \beta^i\int \mathrm{d}^2x\,\phi_i = \frac{1}{2\pi}\int \mathrm{d}^2x\,\Theta, \qquad S = S^* - \int \mathrm{d}^2x \int^g \mathrm{d}g^i\,\phi_i, \tag{1.7}$$

where S^* is the fixed-point action. Finally, eq. (1.3) is obtained for the c-function $c(g) = C|_{|z|=1}$.

Note that the c-theorem is weaker than the gradient flow (1.1), which can only be proven perturbatively in the neighbourhood of the fixed point. For this reason we cannot strictly identify the c-function with Φ.

There is so far no analogous theorem above two dimensions. The interest of such an achievement would be in charting the space of field theories. It would be an intrinsically nonperturbative property, and it would allow one to classify theories unambiguously. Actually, suppose there are fixed points only. Then their basins of attraction give well-defined universality classes. This is a fundamental issue in field theory. The c-theorem in two dimensions has been widely used in this way, yielding very important qualitative and quantitative results, both in statistical mechanics of spin chains [7], and in string-theory model building [8].

There have been attempts to formulate the c-theorem above two dimensions as a property of the stress tensor. Actually, this operator is defined for any theory and it couples to all the degrees of freedom. Moreover, it is believed that the c-function should be a measure of degrees of freedom. Yet it seems to be a real challenge to pin down its precise form.

First of all, Zamolodchikov's proof does not extend to higher dimensions. Due to the larger number of tensor structures for the correlator $\langle T_{\alpha\beta}(x)T_{\mu\nu}(0)\rangle$, it is impossible to find a quantity with negative definite scale derivative, as in eq. (1.5).

Then Cardy made the following observation [9]. In two dimensions, the central charge also parametrizes the trace anomaly in curved spaces. He then proposed a candidate for the c-function in four dimensions, as the trace anomaly computed on the sphere S^4. Though he did not prove that this quantity decreases along RG trajectories, he verified this fact to lowest order in perturbation theory. Later on, Osborn has elaborated on this idea [10, 11]. In particular he has shown that the RG equation satisfied by one coefficient of the $n=4$ trace anomaly is similar to eq. (1.3). However the metric G is not manifestly positive in this case.

At the outset, these efforts miss the basic ingredient of unitarity. We do think that this concept is at the heart of the c-theorem and, consequently, take a different line of work.

An alternative proof of the c-theorem in two dimensions was given by Friedan [12], using the Lehmann spectral representation of the correlator of two stress tensors. Unitarity is manifest in this approach, as it simply means that the spectral density is positive. Then the theorem follows by studying the behaviour of this density along the RG flow. Its shape gives the "number" of degrees of freedom of the theory as a function of the mass scale, and it shows the decoupling of massive states in the IR limit. This gives a better physical picture of the loss of degrees of freedom (and information) along the RG flow. This proof of the c-theorem is recalled in sect. 2, together with some examples.

In sect. 3, we turn to constructing the equivalent spectral representation in any dimension. Then the spectral density is made of two structures, for intermediate states of spin 0 and spin 2, the latter existing only above $n=2$. Both provide candidates for a c-function in higher dimensions, having different properties. On one hand, the spin-two density gives the correlator $\langle T_{\alpha\beta}T_{\mu\nu}\rangle$ in the conformal field theory, which has naturally associated a central charge. This charge is not manifestly decreasing along the RG flow. On the other hand, the spin-zero density gives a manifestly decreasing c-function, much like two dimensions, but the associated central charge is not well defined at the conformal field theory. These basic facts are summarized in sect. 4.

Sect. 5 is devoted to the study of the spin-two part. By means of perturbative calculations on a number of interacting theories in 4 and $4-\epsilon$ dimensions, we show that the spin-2 candidate is not always decreasing along the flow. Then it cannot satisfy a c-theorem. Particular attention is given to the $\lambda\varphi^4$ theory.

In sect. 6, we generalize the spectral representation in curved space. We consider hyperbolic space of constant negative curvature, because the physical criteria for building the Hilbert space are almost the usual ones in flat space [13, 14]. We show that the more natural candidate c-function of spin 0 is well defined there. Then we have been looking for a limit to flat space of this quantity, which could be independent of the specific theory. This limit exists for free massive bosonic and fermionic theories in two, three and four dimensions, due to a remarkable sum rule independent of the curvature scale. However, this sum rule does not hold in an interacting theory, the nonlinear σ-model in three dimensions. Therefore this program of going to curved space is so far unsuccessful. Nevertheless, we believe this idea is worth discussing, also in the light of some recent literature [15]. As a by-product, we clarify the meaning of the four-dimensional trace anomaly.

We remind the reader of the basics of spectral representations in appendix A. Details of the perturbative calculations in $\lambda\varphi^4$ theory are given in appendix B. The techniques for hyperbolic space are given in appendix C.

2. The c-theorem in two dimensions

The basic idea of a spectral representation [16] consists in constraining the form of a two-point correlator by enforcing Poincaré invariance of the propagating intermediate states. In appendix A we recall how to derive it. By analysing the correlator of two stress tensors, a simple and physically transparent proof of the c-theorem follows.

Let us suppose the theory can be defined on an n-dimensional curved space with riemannian metric $g_{\mu\nu}$, at least infinitesimally close to flat space. Then the stress tensor is defined by the metric variation

$$\frac{\sqrt{g(x)}}{2V}\langle T_{\mu\nu}(x)\phi(x_1)\dots\rangle \equiv \frac{\delta}{\delta g^{\mu\nu}(x)}\langle\phi(x_1)\dots\rangle_{g_{\mu\nu}}, \tag{2.1}$$

where V is the volume of the S^{n-1} sphere.

It follows from this definition that $T_{\mu\nu}$ has the canonical dimension n.

2.1. PROOF OF THE c-THEOREM

The spectral representation of a two-point correlator in euclidean field theory is obtained by inserting the identity operator written as

$$I = \int_0^\infty \mathrm{d}\mu^2 \mathscr{P}_{\mu^2} = \int \mathrm{d}\mu^2 \delta_+(\mathbb{P}^2 + \mu^2), \tag{2.2}$$

where $\mathscr{P}_{\mu^2}$ is the projection on unitary representations of the Poincaré group labelled by $\mathbb{P}^2 = -\mu^2$ ($\mathbb{P}_\alpha = -i\partial_\alpha$ is the euclidean momentum operator). The physical Hilbert space is made of irreducible representations with positive energy, selected by δ_+, also called highest weight representations.

Let us consider the correlator of two stress tensors on the plane,

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{\pi}{3}\int_0^\infty \mathrm{d}\mu\, c(\mu) \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\, \mathrm{e}^{ipx} \frac{\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left(g_{\rho\sigma}p^2 - p_\rho p_\sigma\right)}{p^2+\mu^2}. \tag{2.3}$$

In two dimensions there is only one Lorentz invariant made out of p_α with four indices and which is compatible with conservation of the stress tensor. Therefore we are only left with one unknown scalar function of the intermediate mass scale μ, the spectral density $c(\mu)$. This density also depends on a mass scale in the theory, that we call Λ, as well as on dimensionless couplings g^i.

The proof of the c-theorem goes on by establishing the properties of $c(\mu)$.

(i) Reflection positivity of the euclidean field theory [17], i.e. unitarity of the Hilbert space, implies that $c(\mu) \geqslant 0$.

(ii) Let us compute the dimension of $\mathrm{d}\mu\, c(\mu)$. It vanishes, due to $\dim(T_{\mu\nu}) = 2$. Then this spectral density is a dimensionless measure of degrees of freedom!

(iii) The form of $c(\mu)$ in a scale-invariant field theory is completely fixed by its dimensionality. There are no scales in the theory, therefore we must have either

$$c(\mu) = c_0\delta(\mu) \quad \text{or} \quad c(\mu) \propto 1/\mu. \tag{2.4}$$

The second possibility gives an infrared (IR) divergence in the spectral representation at $\mu = 0$. So only the first behaviour is allowed. Thus, in a scale-invariant theory the spectral function must be a delta function at 0.

(iv) Then, scale invariance $\Leftrightarrow$ conformal invariance in two dimensions. Let us substitute a delta function in the spectral representation

$$\langle \Theta(x)\Theta(0)\rangle = \frac{\pi}{3}\int_0^\infty \mathrm{d}\mu\, c(\mu) \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\, \mathrm{e}^{ipx} \frac{(p^2)^2}{p^2+\mu^2} = -\frac{\pi}{3} c_0 \partial^2 \delta^{(2)}(x). \tag{2.5}$$

The two-point function of the trace of the stress tensor vanishes for $|x| \neq 0$, so the trace annihilates the vacuum, so the trace is zero, so the theory is conformally invariant*. Note that this argument relies on the good definition of the spectral representation. This holds under our hypothesis (2.1), which also implies that the stress tensor is a local field of the theory.

(v) If the theory is not scale invariant, the general form of $c(\mu)$ is

$$c(\mu) = c_0 \delta(\mu) + c_1(\mu, \Lambda), \tag{2.6}$$

where c_1 is supported away from $\mu = 0$ and depends on the mass scale Λ of the theory.

(vi) Let us analyse the long (IR) and short (UV) distance behaviour of the two-point function (2.3). Using the complex coordinate $z = x^1 + ix^2$, we obtain** as $z \to 0$,

$$\langle T_{zz}(z) T_{zz}(0) \rangle \to \frac{1}{2z^4} \int_0^\infty \mathrm{d}\mu\, c(\mu); \tag{2.7}$$

as $z \to \infty$,

$$\langle T_{zz}(z) T_{zz}(0) \rangle \to \frac{1}{2z^4} \lim_{\epsilon \to 0} \int_0^\epsilon \mathrm{d}\mu\, c(\mu). \tag{2.8}$$

According to the RG, these asymptotic limits of the massive theory are described by the UV and IR scale-invariant theories which lie at the end points of the RG trajectory. Since these are conformal invariant, we can identify the coefficients of $1/2z^4$ as the corresponding central charges c_{UV} and c_{IR},

$$c_{\mathrm{UV}} = \int_0^\infty \mathrm{d}\mu\, c(\mu), \qquad c_{\mathrm{IR}} = \lim_{\epsilon \to 0} \int_0^\epsilon \mathrm{d}\mu\, c(\mu). \tag{2.9, 2.10}$$

We can write the spectral density as

$$c(\mu) = c_{\mathrm{IR}} \delta(\mu) + c_1(\mu, \Lambda) \tag{2.11}$$

and

$$c_{\mathrm{UV}} = \int_0^\infty \mathrm{d}\mu\, c(\mu) = c_{\mathrm{IR}} + \int_0^\infty \mathrm{d}\mu\, c_1(\mu). \tag{2.12}$$

* The contact term left in this correlator is unfamiliar in conformal field theory. In general, contact terms can be modified by adding local terms to the effective action, i.e. their form depends on the renormalization scheme. In the present case, it can be cancelled by redefining the stress tensor. Accordingly, the traceless part T_{zz} is no more a true tensor and, under conformal transformations, it takes an additional term, the schwartzian derivative. The stress tensor redefined in this way fulfills the properties of conformal field theories, as discussed in ref. [18].

** These formulae are easily obtained in momentum space, where the Fourier transform of $1/z^4$ is $(\pi/24)\bar{p}^3/p$.

By positivity of the spectral measure $c(\mu)$, the end of the proof follows

$$c_{\rm UV} \geqslant c_{\rm IR}\,. \tag{2.13}$$

Namely, the central charge decreases for conformal theories connected by a RG trajectory, and it is constant if the theory is conformal along the flow.

(vii) Conversely, if $c_{\rm UV} = c_{\rm IR}$ then the theory is conformal invariant. Using the wave equation satisfied by the propagator, we can write

$$\langle \Theta(x)\Theta(0)\rangle = \frac{\pi}{3}\int_0^\infty \mathrm{d}\mu\,\mu^4 c_1(\mu) \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{e}^{ipx}\,\frac{1}{p^2+\mu^2}\,, \quad \text{for} \quad |x| \neq 0. \tag{2.14}$$

If $c_{\rm UV} = c_{\rm IR}$, eq. (2.12) implies $c_1 = 0$ by positivity. Then, this correlator vanishes for $|x| \neq 0$.

More comments

(viii) The net change of central charge is a universal quantity, i.e. it is invariant under continuous deformations of the RG trajectory. It can be expressed as a sum rule, which coincides with the one derived from Zamolodchikov's proof by Cardy [19],

$$\Delta c \equiv c_{\rm UV} - c_{\rm IR} = \int_0^\infty \mathrm{d}\mu\, c_1(\mu) = \frac{3}{4\pi}\int_{|x|>\epsilon} \mathrm{d}^2x\,x^2 \langle \Theta(x)\Theta(0)\rangle\,. \tag{2.15}$$

Clearly, this sum rule is convergent if both $c_{\rm UV}$ and $c_{\rm IR}$ are finite. Let us verify this fact in the last form of an x-integral. For $\mu \to \infty$ we are close to the UV conformal field theory (CFT), and we can express Θ perturbatively as in eq. (1.6), $\Theta = 2\pi\beta^i\phi_i$, $\beta^i \sim \epsilon^i g^i + \mathrm{O}(g^2)$, $\epsilon^i = 2 - \dim(\phi_i) < 2$, in terms of the relevant fields of the UV CFT. Then $\langle\Theta\Theta\rangle \sim g_i^2|x|^{4-2\epsilon^i}$ and $c(\mu) \sim g_i^2\mu^{-1-2\epsilon^i}$ as $\mu \to \infty$, so it is integrable. A similar argument can be done for $\mu \to 0$ around the IR CFT. In the case of marginal deformations, $\langle\Theta\Theta\rangle = 0$ apart from contact terms which do not contribute to the sum rule.

(ix) We can make contact with Zamolodchikov's proof in the form of eq. (1.3) by defining a function $c(g)$ off criticality which interpolates monotonically between $c_{\rm UV}$ and $c_{\rm IR}$, and a metric $G_{ij}(g)$ in the space of couplings. These are obtained by integrating the density $c(\mu)$ against positive smearing functions. There are many such functions, all giving the same physical content. The c-function is defined by smearing the density with a function $f(\mu)$,

$$c(g(\Lambda)) = \int \mathrm{d}\mu\, c(\mu) f(\mu) = \int \mathrm{d}\mu\, c_1(\mu,\Lambda) f(\mu) + c_{\rm IR}\,, \tag{2.16}$$

fulfilling the properties $f > 0$, $f(0) = 1$, $f(\mu)$ decreases exponentially as $\mu \to \infty$, and $\mu\, \mathrm{d}f/\mathrm{d}\mu \leqslant 0$. Then the derivative along the flow is

$$-\beta^i \frac{\partial}{\partial g^i} c = \Lambda \frac{\partial}{\partial \Lambda} c = \int \mathrm{d}\mu\, c_1(\mu, \Lambda) \mu \frac{\mathrm{d}}{\mathrm{d}\mu} f(\mu) \leqslant 0, \tag{2.17}$$

where we have used the Callan–Symanzik equation and integrated by parts. This variation can be expressed in terms of a metric in the space of couplings. By expanding Θ in the set of relevant fields ϕ_i, this is defined as

$$\langle \Theta(x)\Theta(0)\rangle = 4\pi^2 \beta^i \beta^j \int \mathrm{d}\mu\, \rho_{ij}(\mu, \Lambda) G(\mu, x), \qquad G_{ij} = \frac{2}{\pi} \int \mathrm{d}\mu\, \rho_{ij} h(\mu), \tag{2.18}$$

with h another smearing function satisfying $h > 0$, $h(0) = \text{const.}$ and exponentially decreasing for $\mu \to \infty$. We can finally obtain eq. (1.3)

$$-\beta^i \partial c/\partial g^i = -6\pi^2 \beta^i \beta^j G_{ij}, \tag{2.19}$$

by matching the two smearing functions, $\mu\, \mathrm{d}f/\mathrm{d}\mu = -\mu^4 h(\mu)$. For example*, take

$$h = \tfrac{1}{6} \mathrm{e}^{-\mu}, \qquad f = \left(1 + \mu + \tfrac{1}{2}\mu^2 + \tfrac{1}{6}\mu^3\right) \mathrm{e}^{-\mu}. \tag{2.20}$$

2.2. EXAMPLES

The cases of free massive theories give interesting examples where $c(\mu)$ is exactly computed as the imaginary part of a one-loop Feynman diagram.

The Majorana fermion spectral density was given in ref. [20]

$$c_1(\mu, m) = 6 \frac{m^2}{\mu^3} \sqrt{1 - \frac{4m^2}{\mu^2}}\, \theta(\mu - 2m), \tag{2.21}$$

where m is the mass of the fermion and θ the step function. Note that $c_{\mathrm{IR}} = 0$ for a purely massive theory. Upon substitution of this density into the sum rule (2.15), one can compute $c_{\mathrm{UV}} = \frac{1}{2}$.

The massive perturbation of the $c_{\mathrm{UV}} = 1$ bosonic theory produces a trace of the stress tensor $\Theta = 2\pi m^2 \varphi^2$, $\varphi(x)$ being the bosonic field. The spectral density turns out to be

$$c_1(\mu, m) = 24 \frac{m^4}{\mu^5} \left(1 - \frac{4m^2}{\mu^2}\right)^{-1/2} \theta(\mu - 2m), \tag{2.22}$$

which again fits the sum rule (2.15).

* Zamolodchikov's choice corresponds to $h(\mu) = (\pi/2) G(|x| = 1, \mu)$, the propagator.

2.3. RENORMALIZATION GROUP FLOW OF THE SPECTRAL DENSITY

Here we would like to digress on the physical meaning of the spectral density

$$c(\mu) = c_{\mathrm{IR}}\delta(\mu) + c_1(\mu, \Lambda). \tag{2.23}$$

The interpretation is that the spectral density $c(\mu)\mathrm{d}\mu$ measures the density of degrees of freedom coupling to the stress tensor (which all degrees of freedom do). It consists of a delta function at $\mu = 0$ which represents the degrees of freedom at arbitrarily large distance. The rest of the spectral density, $c_1(\mu)\mathrm{d}\mu$, represents the density of degrees of freedom at distance μ^{-1}.

Let us see how this interpretation fits the RG philosophy. Since $c(\mu)\mathrm{d}\mu$ cannot develop anomalous dimensions, its behaviour under the RG flow is simply given by dimensional analysis. The effective density (solution of the Callan–Symanzik equation) is

$$c_\lambda(\mu)\,\mathrm{d}\mu = c(\lambda\mu)\lambda\,\mathrm{d}\mu. \tag{2.24}$$

The $\delta(\mu)$ terms does not flow and we shall only discuss the smooth part of the spectral density in the following. The sum rule may be written as

$$\Delta c = \int_0^\infty \mathrm{d}\mu\, c_1(\mu, \Lambda) = \int_0^\infty \lambda\,\mathrm{d}\mu\, c_1(\lambda\mu, \Lambda) = \int_0^\infty \mathrm{d}\mu\, c_1(\mu, \Lambda/\lambda). \tag{2.25}$$

The first equality tells us that Δc is a RG invariant and can be computed for any value of $\Lambda \neq 0, \infty$, i.e. at any intermediate point of the trajectory. The second equality makes it manifest that the UV asymptotic limit $\lambda \to \infty$ is equivalent to tuning the couplings to the UV fixed point ($\Lambda \to 0$, $\mathrm{d}g = \beta(g)\mathrm{d}\Lambda/\Lambda$) along the RG trajectory.

The function $c_1(\mu)$ is integrable and roughly bell-shaped, as it can be seen in the previous examples. Therefore in the UV limit we have

$$\lim_{\lambda\to\infty} c_\lambda(\mu)\,\mathrm{d}\mu = 0 \quad \text{for} \quad \mu \neq 0, \tag{2.26}$$

while its total integral remains finite. Then this function is a representation of the delta function, $c_\infty(\mu) = \Delta c\,\delta(\mu)$. In the UV limit, we recover

$$c(\mu) = (c_{\mathrm{IR}} + c_{\mathrm{UV}} - c_{\mathrm{IR}})\delta(\mu), \tag{2.27}$$

which is the correct expression for the UV CFT.

The compression of c_1 into a delta is interpreted as the flow of massive degrees of freedom into effectively massless ones. Conversely, as we move away from the UV critical point, part of the degrees of freedom get massive, forming a broad distribution peaked at a characteristic scale Λ, and eventually decouple in the IR limit $\Lambda \to \infty$, leaving only the delta term with a smaller coefficient. This is precisely the quantitative picture of the loss of degrees of freedom under the RG flow as it can be argued qualitatively in a Kadanoff block-spin transformation.

3. Spectral representation above two dimensions

3.1. DERIVATION AND GENERAL PROPERTIES OF THE SPECTRAL DENSITIES

We shall now discuss the spectral representation of the stress tensor correlator in more than two dimensions. The derivation proceeds as in the two-dimensional case except for the fact that there are two possible Lorentz structures for the intermediate states. They correspond to spin $s = 0$ states, which already appeared in two dimensions, and new $s = 2$ states. Let us first give the result and then discuss it:

$$\begin{aligned}\langle T_{\alpha\beta}(x)T_{\rho\sigma}(0)\rangle &= \langle T_{\alpha\beta}(x)T_{\rho\sigma}(0)\rangle_{s=0} + \langle T_{\alpha\beta}(x)T_{\rho\sigma}(0)\rangle_{s=2} \\ &= \frac{A_n}{(n-1)^2}\int_0^\infty \mathrm{d}\mu\, c^{(0)}(\mu)\Pi^{(0)}_{\alpha\beta,\rho\sigma}(\partial)G(x,\mu) \\ &\quad + \frac{A_n}{(n-1)^2}\int_0^\infty \mathrm{d}\mu\, c^{(2)}(\mu)\Pi^{(2)}_{\alpha\beta,\rho\sigma}(\partial)G(x,\mu), \end{aligned} \tag{3.1}$$

where the tensors $\Pi^{(s)}$ of spin s are

$$\Pi^{(0)}_{\alpha\beta,\rho\sigma}(\partial) = \frac{1}{\Gamma(n)}S_{\alpha\beta}S_{\rho\sigma}, \qquad \Pi^{(2)}_{\alpha\beta,\rho\sigma}(\partial) = \frac{1}{\Gamma(n-1)}\left(\frac{n-1}{2}S_{\alpha(\rho}S_{\beta\sigma)} - S_{\alpha\beta}S_{\rho\sigma}\right) \tag{3.2}$$

and where $S_{\alpha\beta} = \partial_\alpha\partial_\beta - \delta_{\alpha\beta}\partial^2$. Note that the tracelessness of the spin $s = 2$ part is explicit since $\Pi^{(2)}_{\alpha\alpha,\rho\sigma} = 0$. The propagator is

$$G(x,\mu) = \int \frac{\mathrm{d}^n p}{(2\pi)^n}\frac{\mathrm{e}^{ipx}}{p^2+\mu^2} = \frac{1}{2\pi}\left(\frac{\mu}{2\pi|x|}\right)^{(n-2)/2} K_{(n-2)/2}(\mu|x|). \tag{3.3}$$

The constants

$$A_n = \frac{V}{(n+1)2^{n-1}}, \qquad V \equiv \mathrm{Vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{3.4}$$

are introduced in order to simplify later expressions.

The traceless and transverse tensor $\Pi^{(2)}$ vanishes identically in two dimensions. In fact, it maps symmetric, traceless and transverse two-tensors $h^{\mathrm{TT}}_{\alpha\beta}(x)$ ($h^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\nu\mu}$, $h^{\mathrm{TT}}_{\mu\mu} = 0$ and $\partial_\mu h^{\mathrm{TT}}_{\alpha\beta} = 0$) into themselves, which do not exist in two dimensions. Introducing also a two-tensor of spin zero, $f_{\alpha\beta} = S_{\alpha\beta} g(x)$, we can show how the $\Pi^{(s)}$ tensors act on the two spaces

$$\Pi^{(0)} f = \frac{1}{\Gamma(n-1)} (\partial^2)^2 f, \qquad \Pi^{(0)} h^{\mathrm{TT}} = 0,$$

$$\Pi^{(2)} f = 0 \qquad \Pi^{(2)} h^{\mathrm{TT}} = \frac{n-1}{\Gamma(n-1)} (\partial^2)^2 h^{\mathrm{TT}}. \tag{3.5}$$

These are invariant subspaces. As a consequence, the densities $c^{(0)}$ and $c^{(2)}$ measure independent degrees of freedom. Reflection positivity implies again that both $c^{(s)}$ are non-negative functions*.

Form of the spectral densities at CFT. Next we establish the form of the densities for a scale-invariant theory. There is a qualitative change above two dimensions because $\dim(c^{(s)}) = n - 3 > -1$.

(i) A power law behaviour

$$c^{(s)}(\mu) \propto \mu^{n-3} \tag{3.6}$$

does not lead to IR singularities and is, therefore, possible.

(ii) Since $\delta(\mu)$ has dimension -1, it might appear in combination with a dimensionful quantity, like $c^{(s)}(\mu) \propto \Lambda^{n-2}\delta(\mu)$, but this is forbidden by scale invariance. On the other hand, the form

$$c^{(s)}(\mu) \propto \mu^{n-2}\delta(\mu) \tag{3.7}$$

vanishes identically. Recall that in two dimensions the central charge was identified as the coefficient of the delta term in the spectral measure at the fixed point. If such a term cannot appear above two dimensions, we have big trouble.

* Reflection positivity [17] states that $\langle \mathscr{R}_{x^n}(\mathscr{O})\mathscr{O} \rangle \geqslant 0$, where $\mathscr{R}_{x^n}$ is antilinear and it acts on tensor operators $\mathscr{O} = h_{\alpha\beta} T_{\alpha\beta}(0, x^n)$ ($h_{\alpha\beta}$ are coefficients), by implementing a reflection with respect to the plane orthogonal to the euclidean time x^n,

$$\mathscr{R}(\mathscr{O}) = h^*_{\alpha\beta} \tilde{T}_{\alpha\beta}(0, -x^n) = \tilde{h}^*_{\alpha\beta} T_{\alpha\beta}(0, -x^n),$$

$$\tilde{v}_{\alpha\beta} = (v_{ij}, -v_{ni}, -v_{in}, v_{nn}), \quad i, j = 1, \ldots, n-1.$$

By choosing $h_{\alpha\beta} = h^{\mathrm{TT}}_{\alpha\beta}$, e.g. $h^{\mathrm{TT}}_{nn} = h^{\mathrm{TT}}_{ni} = 0$ and $\Sigma_{i=1}^{n-1} h^{\mathrm{TT}}_{ii} = 0$, positivity of $c^{(2)}$ follows from $\tilde{h}^{\mathrm{TT}} \Pi^{(2)} h^{\mathrm{TT}} \geqslant 0$.

However, there is a possible way out, by reinterpreting the meaningless eq. (3.7) as follows. Consider $c^{(s)} = c^{(s)}(\mu, \Lambda)$ in the theory away from criticality and introduce another spectral density, which becomes a delta in the limit to the critical point,

$$\lim_{\Lambda \to 0} \frac{c^{(s)}(\mu, \Lambda)}{\mu^{n-2}} \mathrm{d}\mu \propto \delta(\mu)\, \mathrm{d}\mu\,. \tag{3.8}$$

This limit simply follows by dimensionality and scaling. Before going into further details we need to discuss spin zero and spin two separately.

Spin two. The power law behaviour defines the constant $c^{(2)}$

$$c^{(2)}(\mu) \overset{\mathrm{CFT}}{=} c^{(2)} \mu^{n-3}\,. \tag{3.9}$$

This parametrizes the leading short-distance singularity in the operator product expansion of two stress tensors and, therefore, is observable in flat space field theories. By substituting this scaling behaviour in the spectral representation, one finds

$$\begin{aligned}\langle T_{\alpha\beta}(x) T_{\rho\sigma}(0) \rangle_{s=2} &\overset{\mathrm{CFT}}{=} \frac{A_n}{(n-1)^2} c^{(2)} \Pi^{(2)}_{\alpha\beta,\rho\sigma}(\partial) \int_0^\infty \mathrm{d}\mu\, \mu^{n-3} G(\mu, x) \\ &= \frac{n c^{(2)}}{n-1} \frac{1}{|x|^{2n}} \Big(\tfrac{1}{2} \delta_{\alpha(\rho} \delta_{\beta\sigma)} - \frac{1}{n} \delta_{\alpha\beta} \delta_{\rho\sigma} - \delta_{\alpha(\rho} \hat{x}_\beta \hat{x}_{\sigma)} \\ &\qquad - \delta_{\beta(\sigma} \hat{x}_\alpha \hat{x}_{\rho)} + 4 \hat{x}_\alpha \hat{x}_\beta \hat{x}_\rho \hat{x}_\sigma \Big)\,, \end{aligned} \tag{3.10}$$

where $\hat{x}^\alpha \equiv x^\alpha / |x|$. In refs. [21,22] this result was obtained in coordinate space, by assuming that the theory is conformal invariant at the fixed point. Then tracelessness and conservation of $T_{\alpha\beta}$ fix completely the Lorentz structure in eq. (3.10). This structure reduces to $\langle T_{zz} T_{zz} \rangle = c^{(2)}/2z^4$ in two dimensions, but note that in that case it comes from the $s = 0$ density!

The leading short-distance singularity enters in many physical properties of the theory, which therefore are parametrized by the charge $c^{(2)}$. For example, it enters in anisotropic corrections to correlators in finite geometries [22]. This property as well as examples of the RG flow of this charge will be discussed in sect. 5.

In presence of a power-law term, it does not seem possible to define a delta term for the spin-two density, as the limit from off-criticality of eq. (3.8), and we do not discuss this possibility any more.

Spin zero. Improvement hypothesis. A power law behaviour $c^{(0)}(\mu) \propto \mu^{n-3}$ leads to a well-defined two-point function of the trace of the stress tensor

$$\langle \Theta(x)\Theta(0)\rangle \propto (\partial^2)^2 \int_0^\infty \mathrm{d}\mu\, \mu^{n-3} G(\mu, x). \tag{3.11}$$

This is not inconsistent with scale invariance because the latter requires the dilatation operator $D = \int \mathrm{d}^n x\, \Theta$ to annihilate the vacuum, then Θ may be a derivative of a nonvanishing operator. Indeed eq. (3.11) vanishes upon integration over the space.

The field Θ has different properties above two dimensions. In a local two-dimensional field theory, scale invariance implies $\Theta = 0$ and, then, conformal invariance. In higher dimensions, scale invariance implies $\Theta =$ (derivative operator), and does not imply (global) conformal invariance. For some theories, $T_{\alpha\beta}$ is not uniquely defined from the action in flat space, it can be modified by the term

$$T_{\alpha\beta} \to T_{\alpha\beta} + \text{const.} \times \left(\delta_{\alpha\beta}\partial^2 - \partial_\alpha\partial_\beta\right)\phi, \tag{3.12}$$

because a local field ϕ of appropriate dimension exists in the theory*. Θ is called improved if it can be put to zero in this way. Then scale invariance is promoted to global conformal invariance [23].

This somehow historical presentation of the improvement problem is rather misleading, because $T_{\alpha\beta}$ is unique for the theory defined in a curved space. Improvement redefinitions amount to adding terms in the action which vanish in flat space, like the term $R\phi$. A better presentation of this problem is as follows: $\Theta = 0$ if and only if the scale invariant theory can be defined in curved space such that it is Weyl invariant [13, 23, 24]. That is, iff the action displays the following invariance

$$S\left[g_{\alpha\beta}, \phi_i\right] = S\left[\mathrm{e}^{2\sigma(x)} g_{\alpha\beta}, \mathrm{e}^{-\Delta_i\sigma}\phi_i\right] \tag{3.13}$$

for some constants Δ_i.

Counterexamples are given by gauge theories for $n \neq 4$, and spontaneously broken theories. They resist to being improved because the local field ϕ in eq. (3.12) is forbidden by a symmetry. This is gauge symmetry in the former case and shift symmetry for the Goldstone bosons, $\pi \to \pi + \text{const.}$, which have only derivative interactions.

To summarize, we shall only consider theories which are conformal invariant at fixed points,

$$\Theta(x) \overset{\text{CFT}}{=} 0 \quad \Leftrightarrow \quad c^{(0)}(\mu) \overset{\text{CFT}}{=} 0 \quad \text{for} \quad n > 2, \quad \text{by hypothesis}. \tag{3.14}$$

* It can never exist in two dimensions by the Mermin–Wagner theorem.

Let us also note that improvement redefinitions do not affect $c^{(2)}(\mu)$. Actually, $\langle T_{\alpha\beta}T_{\rho\sigma}\rangle$ gets other terms whose Lorentz structure is necessarily the same as $\Pi^{(0)}_{\alpha\beta,\rho\sigma}$.

Candidate c-function. Away from criticality $c^{(0)}(\mu,\Lambda)$ is a smooth function supported away from $\mu=0$. By dividing it by μ^{n-2} we can introduce a "reduced" density which behaves almost as the two-dimensional density. In the limit to the UV CFT, it becomes a delta function

$$\lim_{\Lambda\to 0}\frac{c^{(0)}(\mu,\Lambda)}{\mu^{n-2}}=\Delta c^{(0)}\delta(\mu)\,. \tag{3.15}$$

The new constant $\Delta c^{(0)}$ can be computed by a sum rule of the type $\int \mathrm{d}^n x\, x^n\langle\Theta(x)\Theta(0)\rangle$ away from criticality.

The trivial fixed point has $c^{(0)}=0$ and the nontrivial ones have a positive value, which can be computed in principle by following backward the RG trajectory. If this passes through intermediate fixed points, $c^{(0)}$ is defined piecewise, as the sum of $\Delta c_i^{(0)}$ along these intermediate steps. In our normalization, $c^{(0)}=1$ for the free bosonic theory in any dimensions, later computed by a mass perturbation.

As in two dimensions, we can construct a function $\Phi(g^i)$ monotonically decreasing along the flow, and stationary at fixed points, where it takes the value $\Phi=c^{(0)}$. This is defined by smearing the reduced density with a function $f(\mu)$,

$$\Phi(\Lambda)=\int \mathrm{d}\mu\,\frac{c^{(0)}(\mu,\Lambda)}{\mu^{n-2}}f(\mu)+c^{(0)}_{\mathrm{IR}}\,, \tag{3.16}$$

fulfilling the properties $f>0$, $f(0)=1$, $f(\mu)$ decreases exponentially as $\mu\to\infty$, and $\mu\,\mathrm{d}f/\mathrm{d}\mu\leqslant 0$. Then the derivative along the flow is

$$\Lambda\frac{\mathrm{d}}{\mathrm{d}\Lambda}\Phi=\int \mathrm{d}\mu\,\frac{c^{(0)}(\mu,\Lambda)}{\mu^{n-2}}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}f(\mu)\leqslant 0\,. \tag{3.17}$$

This variation can be expressed again in terms of Zamolodchikov's metric in the space of couplings. Expand Θ in the set of relevant fields ϕ_i around a conformal-invariant fixed point,

$$\Theta(x)=V\beta^i(g)\phi_i\,, \tag{3.18}$$

$$\langle\Theta(x)\Theta(0)\rangle=V^2\beta^i\beta^j\int \mathrm{d}\mu\,\rho_{ij}(\mu,\Lambda)G(\mu,x)\,. \tag{3.19}$$

The metric in the space of couplings is defined by

$$G_{ij} = \left(V^2/A_n\right)\int \mathrm{d}\mu\, \rho_{ij}(\mu, \Lambda) h(\mu) \tag{3.20}$$

with h another smearing function satisfying $h > 0$, $h(0) = \text{const.}$, $h \sim \exp(-\mu)$ for $\mu \to \infty$. Then we can write

$$\begin{aligned} \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Phi &\equiv -\beta^i \frac{\partial}{\partial g^i} \Phi \\ &= -\beta^i \beta^j G_{ij}, \end{aligned} \tag{3.21}$$

because the two smearing functions can match*, $\mu\, \mathrm{d}f/\mathrm{d}\mu = -\mu^{n+2} h(\mu)$.

This discussion looks as if we had found a generalization of the c-theorem above two dimensions. However there is a problem. This construction does not tell us the physical meaning, if any, of $\Phi \equiv c^{(0)}$ in the conformal invariant theory. This raises the doubt that it might not be well-defined or single-valued at a multicritical fixed point. A fixed point can be reached flowing backwards along many different RG trajectories, and the value of $c^{(0)}$ could depend a priori on the details of the RG flow. A c-function manifestly decreasing along the flow, but double-valued at a fixed point, would still allow a closed RG trajectory. We are missing an intrinsic definition of $c^{(0)}$, i.e. based on the conformal theory only.

Let us explain better this point, for example in the case of the trivial (0), critical (1) and tricritical (2) points. One has the flow (2) → (1) → (0), where (2) → (1) is a critical line, and the flows (2) → (0) in the massive phase. Any point in the massive phase is carried to the IR trivial fixed point by a unique trajectory. Eq. (3.21) shows that Φ can be integrated along them, and its limit to the UV fixed point (2) exists. Actually, in a neighbour of this point, we can expand perturbatively eq. (3.21), $\beta^i = -\epsilon^i g^i$, $\partial_i \Phi \propto \beta_i$, then Φ is constant on closed smooth surfaces surrounding the fixed point. Therefore $c^{(0)}(2)$ is consistently defined for all trajectories which avoid the critical trajectory (2) → (1).

Let us now suppose that the space of theories $\mathscr{C}$ is divided in two parts by a critical surface, such that trajectories in one massive phase cannot be deformed into the other phase without passing through it. In this case, $c^{(0)}$ may be double-valued at the UV fixed point.

In summary, in more than two dimensions the proof of the c-theorem along these lines misses a physical interpretation for $c^{(0)}$ in the conformal field theory. In sect. 6, we shall present an attempt to find it by defining the theory on a curved space.

* The power μ^{n+2} is necessary for convergence of smearing. Note also that massless fields do exist above two dimensions, but cannot contribute to ρ_{ij}.

3.2. EXAMPLES. SPECTRAL DENSITIES FOR FREE MASSIVE BOSONS AND FERMIONS

General remarks. The spectral representation discussed so far applies to the renormalized field theory and there are no infinities to worry about. The spectral densities are matrix elements of the physical Hilbert space, then they are finite functions in the renormalized theory. They reconstruct the full stress-tensor correlator in coordinate space because the integral over μ in eq. (3.1) is finite for any given $|x| > 0$.

Conversely, the two spectral measures may be extracted from the correlator $\langle T_{\alpha\beta}(x)T_{\rho\sigma}(0)\rangle$ by taking two independent traces. For $|x| \neq 0$ we can also use the equation of motion for $G(x, \mu)$ and obtain

$$\langle\Theta(x)\Theta(0)\rangle = \frac{A_n}{\Gamma(n)}\int_0^\infty \mathrm{d}\mu\, c^{(0)}(\mu)\mu^4 G(x,\mu), \tag{3.22}$$

$$\langle T_{\alpha\beta}(x)T_{\alpha\beta}(0)\rangle - \frac{1}{n-1}\langle\Theta(x)\Theta(0)\rangle = \frac{V(n-2)}{2^n\Gamma(n)}\int_0^\infty \mathrm{d}\mu\, c^{(2)}(\mu)\mu^4 G(x,\mu). \tag{3.23}$$

This can be inverted in momentum space by means of a dispersion relation

$$\Pi(p^2) = \int \frac{\mathrm{d}\mu^2}{\pi}\,\frac{\operatorname{Im}\Pi(p^2 = -\mu^2)}{p^2+\mu^2}, \tag{3.24}$$

where $\operatorname{Im}\Pi(p^2) = (\Pi(p^2 = -\mu^2 - i\epsilon) - \Pi(p^2 = -\mu^2 + i\epsilon))/2i$. Therefore,

$$c^{(0)}(\mu) = \frac{2\Gamma(n)}{\pi A_n}\frac{1}{\mu^3}\operatorname{Im}\langle\Theta(p)\Theta(-p)\rangle|_{p^2=-\mu^2}, \tag{3.25}$$

$$c^{(2)} = \frac{2^{n+1}\Gamma(n)}{\pi V(n-2)}\frac{1}{\mu^3}\operatorname{Im}\left(\langle T_{\alpha\beta}(p)T_{\alpha\beta}(-p)\rangle - \frac{1}{n-1}\langle\Theta(p)\Theta(-p)\rangle\right)\Bigg|_{p^2=-\mu^2}. \tag{3.26}$$

Let us add two comments.

First, given a lagrangian of a renormalizable theory, how is $c^{(s)}(\mu)$ expressed in terms of bare fields, i.e. how is the renormalized $T_{\alpha\beta}$ constructed? By eq. (2.1), the stress tensor has the canonical dimension, so it cannot have wave function renormalization*. By taking the vacuum expectation value of the bare $T_{\alpha\beta}$ on the

* If $T_{\alpha\beta}$ can have an improvement term, the improvement coefficient must be renormalized. Later, we shall see this point for $\lambda\phi^4$.

r.h.s. of eqs. (3.25) and (3.26), and expressing the result in terms of the renormalized coupling, the densities must be automatically finite.

Second, correlators of composite operators have extra singularities in momentum space in addition to those coming from the elementary fields. Renormalized composite operators, like $T_{\alpha\beta}$, have correlations which admit spectral representations as eq. (3.1). Extra singularities appear when we interchange the order of integrations $\int \mathrm{d}\mu \int \mathrm{d}^n p \leftrightarrow \int \mathrm{d}^n p \int \mathrm{d}\mu$ in order to define correlators in momentum space. This is not a valid operation. In momentum space, the integral over the spectral representation is UV logarithmic divergent in four dimensions. Therefore it must be subtracted one time, i.e. the finite $c^{(s)}(\mu)$, related to imaginary parts, reconstruct the real parts up to a constant.

Examples. Let us recall the actions of a free massive boson and fermion [13,24]. They can be extended to curved space in such a way that they are Weyl invariant in the massless case.

For the boson we have

$$S = \tfrac{1}{2}\int \mathrm{d}^n x \sqrt{g}\left(g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + m^2\varphi^2 + \frac{n-2}{4(n-1)} R\varphi^2 \right). \tag{3.27}$$

The stress tensor is, in the flat space limit,

$$\frac{T_{\alpha\beta}}{V} = -\frac{1}{2(n-1)}\left(n\,\partial_\alpha\varphi\,\partial_\beta\varphi - (n-2)\varphi\,\partial_\alpha\partial_\beta\varphi - \delta_{\alpha\beta}\left((\partial\varphi)^2 - \frac{n-2}{n}\varphi\,\partial^2\varphi \right)\right)$$
$$+\frac{1}{n}\delta_{\alpha\beta} m^2\varphi^2 . \tag{3.28}$$

For the Dirac fermion, the action and the stress tensor read

$$S = \int \mathrm{d}^n x \sqrt{g}\left(\tfrac{1}{2}\bar{\psi}\overleftrightarrow{\not{D}}\psi + m\bar{\psi}\psi \right), \tag{3.29}$$

$$\frac{T_{\alpha\beta}}{V} = -\left(\tfrac{1}{4}\bar{\psi}\gamma_{(\alpha}\overleftrightarrow{\partial}_{\beta)}\psi - \frac{1}{n}\delta_{\alpha\beta}\left(\tfrac{1}{2}\bar{\psi}\gamma^\alpha\overleftrightarrow{\partial}_\alpha\psi + m\bar{\psi}\psi \right)\right). \tag{3.30}$$

These stress tensors are conserved and, for $m = 0$, manifestly traceless. Conservation can be checked, at the classical level, by using the equation of motion. The computation of the spectral densities involves the calculation of the imaginary part of a one-loop diagram. The result for the boson is

$$c^{(0)}(\mu,m)=8(n+1)(n-1)m^4\mu^{n-7}\left(1-\frac{4m^2}{\mu^2}\right)^{(n-3)/2}\theta(\mu-2m),\quad(3.31)$$

$$c^{(2)}(\mu,m)=\mu^{n-3}\left(1-\frac{4m^2}{\mu^2}\right)^{(n+1)/2}\theta(\mu-2m).\quad(3.32)$$

For the fermion we find

$$c^{(0)}(\mu,m)=2^{[n/2]}2(n+1)(n-1)m^2\mu^{n-5}\left(1-\frac{4m^2}{\mu^2}\right)^{(n-1)/2}\theta(\mu-2m),\quad(3.33)$$

$$c^{(2)}(\mu,m)=2^{[n/2]}\frac{n-1}{2}\mu^{n-3}\left(1-\frac{4m^2}{\mu^2}\right)^{(n-1)/2}\left(1+\frac{2}{n-1}\frac{4m^2}{\mu^2}\right)\theta(\mu-2m).$$
$$(3.34)$$

Some remarks are in order.

(i) In the UV limit $m\to0$ one finds $c^{(2)}=1$ for the boson (by convention) and $c^{(2)}=2^{[n/2]}(n-1)/2$ for the Dirac fermion [24], where $2^{[n/2]}$ is the dimension of the Clifford algebra, $c^{(2)}=1,2,6,\ldots$ for $n=2,3,4,\ldots$.

(ii) The density $c^{(2)}$ is monotonically increasing in μ for the boson in any dimension and for the fermion if $n\geqslant3$.

(iii) When $m\to0$ the spin-zero density gets compressed into a delta,

$$\frac{c^{(0)}(\mu,m)}{\mu^{n-2}}\to c^{(0)}\delta(\mu),$$

as discussed before. Moreover, for these free theories one finds $c^{(0)}=c^{(2)}$ in any dimension!

4. Natural candidates for a c-theorem

Let us summarize the previous section results on the two candidates for a c-theorem above two dimensions, and introduce the next two sections devoted to more deep investigations.

(I) The first candidate is $c^{(2)}$, the coefficient in front of the powerlike behaviour of the spin-two spectral density. This number is well-defined at the CFT. Nevertheless, it cannot be proven to decrease by simple arguments. The c-theorem holds if the quantity

$$\Delta c^{(2)} = c^{(2)}_{\mathrm{UV}} - c^{(2)}_{\mathrm{IR}} = \left(\lim_{\mu \to \infty} - \lim_{\mu \to 0} \right) \frac{c^{(2)}(\mu)}{\mu^{n-3}} \tag{4.1}$$

is always positive*. Unitarity of the theory implies that $c^{(2)}(\mu) \geqslant 0$, but this is not enough to establish its monotonicity and, then, the sign of $\Delta c^{(2)}$. Therefore, a c-theorem for $c^{(2)}$ in higher dimensions, if it exists, involves the dynamical properties of the theory. In this respect, we do not have any general argument to present. In sect. 5 we discuss a physical interpretation of the spin-two density, which suggests that this quantity does not measure the response of the theory to dilatations.

Then, we compute $\Delta c^{(2)}$ for some examples of RG flow in four and $4-\epsilon$ dimensions. We find that $\Delta c^{(2)}$ is not positive definite and behaves in a way which is not clearly related to the known dynamical properties of the theories considered.

We believe that there is enough evidence for ruling out this candidate for a c-theorem.

(II) The second candidate is $c^{(0)}$, related to the spin-zero spectral density. It is given by the limit of the "reduced" spectral density, which is defined away from criticality

$$\lim_{\Lambda \to 0} \frac{c^{(0)}(\mu, \Lambda)}{\mu^{n-2}} \, \mathrm{d}\mu = \Delta c^{(0)} \delta(\mu) \, \mathrm{d}\mu \,. \tag{4.2}$$

The quantity $c^{(0)}$ is positive, monotonically decreasing along the RG flows and stationary at fixed points by construction, exactly as in two dimensions. However we do not know if it has a meaning in the CFT, because $\langle T_{\alpha\beta}(x) T_{\rho\sigma}(0) \rangle_{\mathrm{CFT}}$ is independent of $c^{(0)}$ for any $|x|$. Then $c^{(0)}$ could be multiple valued for a multicritical point, because it could depend on which RG trajectory we follow to reach this point.

Therefore the proof of the c-theorem for $c^{(0)}$ is missing a better definition of $c^{(0)}$ at CFT. In sect. 6 we attack this problem in the following way:

(i) The field theory is defined on a (classical) curved space, the hyperbolic space H_n of constant negative curvature $R = -a^2 n(n-1)$. Then a new mass scale a is present, playing the role of an IR cut-off. By generalizing the spectral analysis, the spin-zero spectral function contains more data and preserves its positivity properties. In particular, the correlator $\langle \Theta \Theta \rangle_{\mathrm{CFT}}$ does not vanish in curved space, due to the trace anomaly, and the spectral density reads

$$c^{(0)}(\mu, a) = \Gamma(n) \rho_0 a^{n-2} \delta\left(\mu - a \frac{n+1}{2} \right) \quad \text{(at CFT in hyperbolic space)}. \tag{4.3}$$

* It can be shown that the generalization of Zamolodchikov's argument to higher dimension discussed in ref. [9] reduces to this tautology for the $c^{(2)}$-charge. Namely, the spectral representation makes explicit the unitarity constraint, which is not sufficient to prove the decreasing of $c^{(2)}$.

This is a well-defined distribution, selecting the massless state at the bottom of the spectrum (i.e. the Weyl invariant state). Note that the dimensional factor μ^{n-2} is replaced by another one made out of the curvature.

How ρ_0 is related to the trace anomaly will be discussed at length. In particular, in four dimensions we show how the two spectral measures at the conformal point correspond to the two structures in the trace anomaly.

(ii) In the second part we discuss the conditions which allows to take the flat limit in a careful way, such that ρ_0 is related to the ill-defined $c^{(0)}$. In particular, for free bosonic and fermionic theories in 2, 3 and 4 dimensions (probably any) we prove that $c^{(0)} = \rho_0$. However, a simple argument relating these two numbers for any interacting theory was not found. Moreover, $c^{(0)} \neq \rho_0$ in the case of the O(N) model in three dimensions, for large N.

Therefore this approach was so far unsuccessful. Nevertheless, we think the idea of going to curved space is worth discussing.

5. Study of the spin-two spectral density

Let us turn our attention to the spin-two spectral density $c^{(2)}(\mu)$. First we give some physical intuition on this quantity, following ref. [22]. Then we discuss the relation between $c^{(2)}(\mu)$ at CFT and the trace anomaly at $n = 4$. We prove that the charge $c^{(2)}$ is equal to the coefficient of one term in the anomaly, the so-called F-term, F being the square of the Weyl tensor [13].

Next we study nontrivial examples for the RG flow of the charge $c^{(2)}$ by means of perturbation theory. The comparison of $c^{(2)}_{\rm UV}$ and $c^{(2)}_{\rm IR}$ is consistent with the perturbative approximation if the theory possesses a "close" IR fixed point, i.e. in the perturbative region. This is the case of the $\lambda\varphi^4$ theory in $n = 4 - \epsilon$, which we examine in detail. Let us summarize our results as follows

$$\begin{array}{ll} \multicolumn{2}{c}{\lambda\varphi^4} \\ \hline n = 4 - \epsilon, & c^{(2)}(g) = 1 - \frac{5}{144}g^2 + \mathrm{O}(g^3, \epsilon g^2), \\ \beta(g) = -\epsilon g - \frac{3}{2}g^2 + \mathrm{O}(g^3), & \Delta c^{(2)} = \frac{5}{324}\epsilon^2 > 0, \\ g^* = -\frac{2}{3}\epsilon. & \end{array} \tag{5.1}$$

The nontrivial zero of the beta function gives the IR fixed point $g^* = \mathrm{O}(\epsilon) \ll 1$ [1]. We have computed $c^{(2)}(g)$ to the lowest nonvanishing order, and obtained

$$\Delta c^{(2)} = c^{(2)}_{\rm UV} - c^{(2)}_{\rm IR} = c^{(2)}(g = 0) - c^{(2)}(g^*) . \tag{5.2}$$

We have found $\Delta c^{(2)} > 0$ in this case, i.e. decreasing along the flow. However, it is not stationary at the new fixed point, $\partial c^{(2)}/\partial g\,|_{g^*} \neq 0$. This suggests that the spin-two spectral density does not feel the behaviour of the theory under scale transformations.

In other examples, we find that $c^{(2)}$ is not stationary as well, and moreover it is increasing along the flow. We use the perturbative calculations of the $n=4$ trace anomaly, existing in the literature for QED [25], pure QCD [26], and QCD with fermions [11]. The general relation between this anomaly and $c^{(2)}$, proven in subsect. 5.2, gives the first perturbative correction $c^{(2)}(\alpha)=c^{(2)}(0)+\text{const.}\times\alpha$ for all these theories in four dimensions. This form of the leading perturbative term is unchanged for $n=4-\epsilon$, where some of these theories have a flow to a close fixed point. These results are summarized as follows (see later for details).

QCD (SU(N) colours, f quarks, $f=11N/2-k,\ k\sim 1,\ N\gg 1$)	Pure QCD (SU(N) colours, $C_{\mathrm{V}}=N,\ n_{\mathrm{V}}=N^2-1$)
$n=4$	$n=4-\epsilon$
$\beta(\alpha)=-\frac{4}{3}k\frac{\alpha^2}{4\pi}+25N^2\frac{\alpha^3}{(4\pi)^2}$	$\beta(\alpha)=-\epsilon\alpha-\frac{22}{3}C_{\mathrm{V}}\frac{\alpha^2}{4\pi}$
$\frac{\alpha^*}{4\pi}=\frac{4}{75}\frac{k}{N^2}$	$\frac{\alpha^*}{4\pi}=-\frac{3}{22C_{\mathrm{V}}}\epsilon$
$c^{(2)}(\alpha)=\mathrm{O}(N^2)\left(1+\text{const.}\times\frac{\alpha}{4\pi}\right)$	$c^{(2)}(\alpha)=12n_{\mathrm{V}}\left(1-\frac{20}{9}C_{\mathrm{V}}\frac{\alpha}{4\pi}\right)$
$\Delta c^{(2)}=N^2\left(-2\frac{k}{N}\right)<0$	$\Delta c^{(2)}=-\frac{40}{11}n_{\mathrm{V}}C_{\mathrm{V}}\epsilon<0$ (5.3)

QED (one fermion)

$n=4-\epsilon$	$c^{(2)}(\alpha)=18+\frac{70}{3}\frac{\alpha}{4\pi}$,
$\beta(\alpha)=-\epsilon\alpha+\frac{8}{3}\frac{\alpha^2}{4\pi}$,	$\Delta c^{(2)}=-\frac{35}{4}\epsilon<0$
$\frac{\alpha^*}{4\pi}=\frac{3}{8}\epsilon.$	(5.4)

These three examples display an increasing $c^{(2)}$ function along the RG flow.

As a conclusion, the behaviour of the $c^{(2)}$ density along the RG flow is not completely constrained by unitarity. Examples in 4 and $4-\epsilon$ dimensions have shown that neither the dynamics constrains it.

We consider this enough evidence to reject $c^{(2)}$ as a candidate for the c-theorem.

5.1. PHYSICAL MEANING OF $c^{(2)}(\mu)$

Above two dimensions, the charge $c^{(2)}$ parametrizes the leading short-distance singularity of $\langle T_{\mu\nu}T_{\rho\sigma}\rangle$, which only has traceless components*

$$T^{\mathrm{TT}}_{\mu\nu}=T_{\mu\nu}-(1/n)\delta_{\mu\nu}\Theta.$$

* $\Theta=0$ is assumed at CFT, i.e. at short distance.

This generates reparametrizations which are orthogonal to conformal transformations, i.e. $x^\mu \to x^\mu + \xi^\mu$ such that $\partial^\nu \xi^\mu - \partial^\mu \xi^\nu - (2/n)\delta^{\mu\nu}\partial \cdot \xi \neq 0$, called shear transformations, which exist above two dimensions.

In ref. [22], the following physical application was given. For systems defined on finite geometries, the two-point correlators have an anisotropic correction parametrized by $c^{(2)}$. As an example, let us suppose we take periodic boundary conditions, of period L, in one direction, whereas the other directions remain infinite. Then, the correlator of a scalar field $\phi(x)$ has a correction $O((|x|/L)^n) \ll 1$

$$\langle \phi(x)\phi(0)\rangle = \frac{1}{|x|^{2\Delta}}\left[1 - \frac{A}{c^{(2)}} V\Delta\left(\frac{|x|}{L}\right)^n \frac{(x^0)^2 - \Sigma_{i=1}^{n-1}(x^i)^2}{x^2} + \dots\right]. \quad (5.5)$$

An infinitesimal change of L is equivalent to a shear transformation of coordinates, which couples to $T_{\mu\nu}^{\mathrm{TT}}$ and affects $\langle\phi\phi\rangle$ by the operator product expansion $T\varphi$. The charge $c^{(2)}$ enters as a coefficient of the anisotropy correction, in combination with the finite-size free energy, $F/\mathrm{Volume} \sim -A/L^n$. This gives a practical way to compute $c^{(2)}$ in physical systems or in computer simulations.

On the other hand, the trace Θ generates dilatations and the RG flow describes how the theory responds to these transformations. A decreasing $c^{(2)}$ along the flow would mean that the behaviour of the theory under shear transformations is related to those under scale transformation in a way independent of the theory.

5.2. RELATION TO THE TRACE ANOMALY IN FOUR DIMENSIONS

The central charge parametrizes the trace anomaly of CFTs in two-dimensional curved spaces

$$\langle\Theta\rangle = -\tfrac{1}{12}cR, \quad (n=2), \quad (5.6)$$

where R is the scalar curvature.

It is remarkable that a similar relation holds in four dimensions between the trace anomaly and the spin-two spectral density at CFT. The trace anomaly can be written as*

$$\langle\Theta\rangle = \frac{1}{2880}(-3\alpha F + \gamma G), \quad (n=4), \quad (5.7)$$

where F is the square of the Weyl tensor and G is the Euler density $\int\sqrt{g}\,G \propto \chi$,

$$F = C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \tfrac{1}{3}R^2, \qquad G = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2. \quad (5.8)$$

* Two other terms cannot appear. R^2 is forbidden by the Wess–Zumino consistency condition of Weyl covariance of the effective potential Γ. ΔR can be put to zero by adding a local term to Γ; moreover, it does not appear in the integrated anomaly [24].

In the following, we shall show that

$$\alpha = c^{(2)}. \tag{5.9}$$

The values of α and γ have been computed for various theories. For free particles of spin 0, $\frac{1}{2}$ (Dirac) and 1 they read [13]*

	α	γ
boson	1	1
Dirac	6	11
vector	12	62

Indeed, the numbers for α agree with the ones previously given for $c^{(2)}$ for bosons and fermions. We have also checked the value $c^{(2)} = 12$ for the massless vector by an analogous one-loop computation. Moreover, a further item of agreement is found in the context of the ϵ-expansion of the $\lambda\phi^4$ theory (see next section).

For two-dimensional theories, the proof of eq. (5.6) [27] goes as follows: (i) take a metric variation of $\langle\Theta\rangle$ and obtain a contact term for $\langle\Theta\Theta\rangle$, (ii) check consistency with the short-distance behavior of $\langle TT\rangle$ using the conservation law. Note that comparison of contact terms makes sense in a given renormalization scheme.

In four dimensions, one compares contact terms appearing in 3-point functions in flat space which are computed in two different ways. We use

$$\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{g}\,G = 0, \qquad g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int\sqrt{g}\,F = 0 \qquad (n=4). \tag{5.10}$$

Therefore a metric variation of the integrated anomaly $\int\sqrt{g}\langle\Theta\rangle$ only takes contribution from the F-term, if the variation is traceless. The double variation is proportional, in the flat limit, to the $s=2$ structure $\Pi^{(2)}_{\mu\nu,\rho\sigma}$ in eq. (3.1)

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\rho\sigma}(y)}\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}(x)}\int\sqrt{g}\langle\Theta\rangle|_{\text{flat}} &= -\frac{\alpha}{1440}\Pi^{(2)}_{\mu\nu,\rho\sigma}(\partial)\delta^{(4)}(x-y)\\ &= \frac{1}{4V^2}\int \mathrm{d}^2z\langle\Theta(z)T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle.\end{aligned} \tag{5.11}$$

The second equality follows from the Ward identity produced by $\delta S = -(1/2V)\int\sqrt{g}\,T_{\mu\nu}\delta g^{\mu\nu}$ and assuming $\langle\Theta\Theta\rangle = 0$ in flat space (Weyl invariance),

*See also ref. [38].

which is consistent with $\langle\Theta\rangle$ being purely anomalous. On the other hand, a finite contact term in $\langle\Theta TT\rangle$ tells us that $\langle TT\rangle$ is not scale invariant. Actually, it has a logarithmic singular contact term with a cut-off dependence Λ. Its amplitude can be computed in terms of $c^{(2)}$ by inserting $c^{(2)}(\mu)=c^{(2)}\mu^{n-3}$ into eq. (3.1). Then

$$\begin{aligned}\frac{1}{4V^2}\left\langle\int\Theta T_{\mu\nu}(x)T_{\rho\sigma}(y)\right\rangle &= -\frac{1}{2V}\Lambda^2\frac{\mathrm{d}}{\mathrm{d}\Lambda^2}\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle_{\Lambda^2}\\ &= -\frac{1}{1440}c^{(2)}\Pi^{(2)}_{\mu\nu,\rho\sigma}(\partial)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{ipx}\,\Lambda^2\frac{\mathrm{d}}{\mathrm{d}\Lambda^2}\int_0^{\Lambda^2}\frac{\mathrm{d}\mu^2}{p^2+\mu^2}\\ &= -\frac{1}{1440}c^{(2)}\Pi^{(2)}_{\mu\nu,\rho\sigma}(\partial)\delta^{(4)}(x-y)\,. \qquad (5.12)\end{aligned}$$

Notice that we have again commuted the order of the μ and p-integrals in order to uncover the singular contact term and we have taken the $\Lambda\rightarrow$ limit in the last finite expression.

This proves the announced result. The α-term in the anomaly exactly corresponds to the coefficient of the spin-two spectral density in flat space. As a by product, positivity of the spectral density means that α must be positive.

5.3. FLOW OF $c^{(2)}$ IN $\lambda\varphi^4$ THEORY

The renormalized $\lambda\varphi^4$ theory in $4-\epsilon$ dimensions is obtained by using dimensional regularization and minimal subtraction [1,28]. We almost follow the minimal subtraction scheme of ref. [29]. Details of the calculation are presented in appendix B.

The bare action is

$$S=\int\mathrm{d}^nx\sqrt{g}\left(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi_0\partial_\nu\varphi_0+\tfrac{1}{2}\xi_0R\varphi_0^2-\frac{\lambda_0}{4!}\varphi^4\right)$$

$$+\text{ counterterms depending only on the metric}. \qquad (5.13)$$

The stress tensor reads

$$\frac{T_{\mu\nu}}{V}=\frac{T^{\mathrm{B}}_{\mu\nu}}{V}-\frac{1}{n}g_{\mu\nu}\epsilon\frac{\lambda_0}{4!}\varphi_0^4+\left(\xi_0-\frac{n-2}{4(n-1)}\right)\left(\partial_\mu\partial_\nu-\delta_{\mu\nu}\partial^2\right)\varphi_0^2, \qquad (5.14)$$

where $T^{\mathrm{B}}_{\mu\nu}$ is the free massless boson tensor of eq. (3.28). The renormalized coupling g is defined by

$$\lambda_0=\left(Z_g/SZ^2\right)\kappa^\epsilon g\,, \qquad (5.15)$$

where $S = V/(2\pi)^n$, κ is the renormalization scale, Z_g and Z are the renormalization constants for the coupling and the field respectively. Its flow is given by the beta function

$$\kappa\, \mathrm{d}g/\mathrm{d}\kappa = \beta(g) = -\epsilon g - \tfrac{3}{2}g^2 + \mathrm{O}(g^3). \tag{5.16}$$

The nontrivial zero of the beta function gives the IR fixed point $g^* = -\frac{2}{3}\epsilon + \mathrm{O}(\epsilon^2)$.

These preliminaries are already enough to state the result for the charge $c^{(2)}$,

$$c^{(2)}(g) = 1 - \frac{5}{144}g^2 + \mathrm{O}(g^3, \epsilon g^2) \quad (n = 4 - \epsilon). \tag{5.17}$$

Notice that the leading correction $\mathrm{O}(g)$ vanishes identically for any n. This is because the theory is free in the large-N limit, where N is the number of components of φ. Actually, if we consider an N-component field, φ_a, $a = 1, \dots, N$, with $\mathrm{O}(N)$-symmetric quartic interaction, then $c^{(2)} = N(1 + \mathrm{O}(1/N))$, because the theory is free to leading order. In the perturbative series, the Feynman diagrams surviving this limit are multi-bubble diagrams, and indeed they all vanish. In particular, the leading $\mathrm{O}(g)$ term is the two-bubble diagram.

The computation of the $\mathrm{O}(g^2)$ is rather cumbersome, as it is equivalent to a three-loop computation. This is performed in appendix B by using the technique of refs. [30, 31].

The result we have obtained contains some interesting physical information. First, $c^{(2)}$ is decreasing along the flow

$$\Delta c^{(2)} = c^{(2)}_{\mathrm{UV}} - c^{(2)}_{\mathrm{IR}} = \tfrac{5}{324}\epsilon^2 > 0. \tag{5.18}$$

Nevertheless $c^{(2)}$ is not stationary at the new fixed point*

$$\frac{\partial}{\partial g} c^{(2)}(g)|_{g^*} \neq 0. \tag{5.19}$$

Instead, the two-dimensional Zamolodchikov's c-function is stationary at fixed points, because the c-theorem eq. (1.3) implies $\partial c/\partial g \sim \beta$ in our case of slightly relevant perturbation, i.e. a function of the type $c(g) = c(0) + \mathrm{const.} \times (\epsilon g^2 + g^3)$. We conclude that $c^{(2)}$ is rather different from the two-dimensional c-function.

Let us add some more technical comments.

(i) The leading singularity at order g^2 is $1/\epsilon^2$. Therefore, the nontrivial cancellations which take place confirm that $c^{(2)}$ is finite in perturbation theory, as we argued before.

* Note that the variation along the flow $-\beta\, \partial c^{(2)}/\partial g = 0$ at fixed points, but this is just a generic property of finite RG functions.

(ii) Invariance of $c^{(2)}$ under improvement was checked by repeating the calculation with a non-improved $T_{\mu\nu}$, actually simplifying the computation.

(iii) Reparametrization invariance was also checked to the order of the computation.

(iv) There are no IR singularities in dimensional regularization for the massless theory, for nonvanishing external momentum (see ref. [28]).

(v) When discussing the stress tensor, it is compulsory to consider the action in curved space. Then, we are faced with the existence of an additional coupling, ξ_0, which fixes the improvement of $T_{\mu\nu}$. But recall that $c^{(2)}$ is improvement independent, so it does not depend on ξ_0. Moreover, since ξ (the renormalized coupling associated to ξ_0) does not enter in $\beta(g)$, the leading UV behaviour does not feel curvature corrections. Therefore $\Delta c^{(2)}$ is free of ambiguities. What is left to see is whether there is a choice of ξ_0 such that Θ is improved at both fixed points, i.e. if it holds $\Theta = \beta(g)\Psi$, where $\Psi = \varphi^4/4! + \dots$ is a renormalized field. The answer to $O(g^2)$ we are working on is yes and it is explained in appendix B. It turns out that $\xi_0 = (n-2)/4(n-1)$, the free theory value, fits the requirements and the corresponding flow of $\xi(g)$ is consistent with the findings of Lüscher in ref. [32].

5.4. OTHER EXAMPLES OF $\Delta c^{(2)}$ FOR $n=4$ AND $n=4-\epsilon$

$n=4$ QCD. Let us consider Quantum Chromodynamics with gauge group $SU(N)$ and f quarks (Dirac fermions in the fundamental representation). The beta function reads

$$\kappa\frac{d\alpha}{d\kappa} = \beta(\alpha) = \alpha\left(-b_0\frac{\alpha}{4\pi} - b_1\left(\frac{\alpha}{4\pi}\right)^2 - b_2\left(\frac{\alpha}{4\pi}\right)^3 - \dots\right). \tag{5.20}$$

Recall the well-known value of $b_0 = \frac{2}{3}(11N - 2f)$. If we let N and f be large with

$$f = \tfrac{11}{2}N - k, \qquad N \gg 1, \qquad k \sim O(1), \tag{5.21}$$

such that asymptotic freedom is close to breaking down, the coefficients b_i are [11]

$$b_0 = \tfrac{4}{3}k, \qquad b_1 \sim -25N^2\left(1 + O\left(\frac{k}{N}\right)\right), \qquad b_2 \sim -\tfrac{701}{6}N^3\left(1 + O\left(\frac{k}{N}\right)\right). \tag{5.22}$$

Therefore, there is a close fixed point at

$$\frac{\alpha^*}{4\pi} = \frac{4}{75}\frac{k}{N^2}\left(1 + O\left(\frac{k}{N}\right)\right). \tag{5.23}$$

Let us compute $c^{(2)}$ in this case. At the UV fixed point, the theory is free, so from the table in eq. (5.3) we read

$$c_{\mathrm{UV}}^{(2)} = 6n_{\mathrm{F}} + 12n_{\mathrm{V}} = 6N\left(\tfrac{11}{2}N - k\right) + 12(N^2 - 1) = 45N^2\left(1 + \mathrm{O}\left(\frac{k}{N}\right)\right) = \mathrm{O}(N^2)\,. \tag{5.24}$$

From the perturbative calculation of the trace anomaly in ref. [11] (see their eq. (5.12)), and the relation given by eq. (5.9) we get the form $c^{(2)}(\alpha) = c^{(2)}(0) + \text{const.} \times \alpha/4\pi$. By substituting all the constants, we find

$$\Delta c^{(2)} = c_{\mathrm{UV}}^{(2)} - c_{\mathrm{IR}}^{(2)} = c^{(2)}(0) - c^{(2)}(\alpha^*) = N^2\left(-2\frac{k}{N}\right) < 0\,. \tag{5.25}$$

Thus, $c^{(2)}$ is not stationary at neither fixed points, moreover, it is increasing along the renormalization group flow. Therefore the results of ref. [11] provide a counterexample to the existence of a c-theorem for $c^{(2)}$.

A more speculative argument follows if we assume that QCD is confining and chiral symmetry breaking takes place at the IR fixed point. The breaking of $\mathrm{SU}(f)_{\mathrm{L}} \times \mathrm{SU}(f)_{\mathrm{R}}$ to $\mathrm{SU}(f)_{\mathrm{V}}$ generates $f^2 - 1$ Goldstone bosons which are free in the IR limit (they can only have derivative couplings). Then we can guess $c_{\mathrm{IR}}^{(2)} = f^2 - 1$. The condition $\Delta c^{(2)} > 0$ can be discussed as a function of f and N for large values of f, where the bound is saturated. One finds

$$\Delta c^{(2)} = c_{\mathrm{UV}}^{(2)} - c_{\mathrm{IR}}^{(2)} > 0 \quad \text{for} \quad f \leqslant 7.6N\,. \tag{5.26}$$

The assumed flow requires asymptotic freedom, i.e. $f < \frac{11}{2}N$. Therefore $\Delta c^{(2)} > 0$ is verified for the range of parameters which give a confining QCD. It is interesting that the two bounds are close. However, we should stress that this argument is very rough*. On the contrary, the previous perturbative calculation stands as a more solid counterexample.

$\lambda\phi^4$ for $n = 4$. Let us reconsider our result for $c^{(2)}$ in $\lambda\phi^4$ theory, eq. (5.17). It is unchanged as we let $\epsilon \to 0$, then we can compare it to the trace anomaly in four dimensions by using eq. (5.9) again. Indeed, this anomaly has been computed in ref. [25] and their result agrees with our one. This is an a posteriori check of our computation. Had we obtained a stationary charge $c^{(2)}(g) = c^{(2)}(0) + \epsilon g^2 + g^3$, the $\mathrm{O}(\epsilon)$ terms would have been lost.

* The same argument for confining QED and QCD in three dimensions instead shows a region of parameters for which $\Delta c^{(2)} < 0$, thus providing a three-dimensional counterexample (Banks and Shenker, unpublished).

6. Spectral representation on hyperbolic space

6.1. FIELD THEORY ON HYPERBOLIC SPACE H_n

Let us summarize some basic geometrical properties*. Hyperbolic space is the maximally symmetric space of negative curvature $R = -n(n-1)a^2$ and riemannian metric. It can be embedded in $\mathbb{R}^{n+1}$ as

$$x^A g_{AB} x^B = -(x^0)^2 + \sum_{\mu=1}^{n} (x^\mu)^2 = -\frac{1}{a^2}, \qquad x^0 > 0, \tag{6.1}$$

where x^0 is the auxiliary variable. By picking up an origin $o^A = (1/a, 0^\mu)$, we introduce intrinsic polar coordinates

$$\sqrt{(x^\mu)^2} = \frac{\sinh ar}{a}, \qquad x^0 = \frac{\cosh ar}{a}, \qquad r = \mathrm{dist}(x, o), \tag{6.2}$$

where r is the geodesic distance from the origin. The metric is

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + \left(\frac{\sinh ar}{a}\right)^2 (\mathrm{d}\Omega_{n-1})^2. \tag{6.3}$$

where $(\mathrm{d}\Omega_{n-1})^2$ is the metric of the sphere S^{n-1}.

Since r can grow to infinity, this space is unbounded. Moreover, its size increases exponentially with the distance, $\mathrm{d\,Vol} = V(S^{n-1})((1/a)\sinh ar)^{n-1}\,\mathrm{d}r$, causing peculiar IR properties in the field theory.

This space has a boundary, because it has the topology of the euclidean ball $D^n = \{\xi^\mu, \mu = 1,\dots,n;\ \xi^2 \leqslant 1\}$, to which it can be mapped by the stereographical projection

$$x^\mu = \frac{2}{a}\frac{\xi^\mu}{1-\xi^2}, \qquad \mathrm{d}s^2 = \left(\frac{2}{a(1-\xi^2)}\right)^2 (\mathrm{d}\xi^\mu)^2. \tag{6.4}$$

Therefore the boundary conditions at infinity need to be discussed in the following.

As we anticipated before, our interest in a space with curvature is instrumental for introducing an extra (infrared) scale in our problem, hopefully in a theory-independent way, such that a well-defined spectral density is obtained at the fixed point. This is shown in the first part of this section. As a second step, we are interested in taking the flat limit. Curvature effects have not a direct physical interest for us and they complicate the RG flow. Hyperbolic space is the best

* Details are given in appendix C, see also ref. [33].

choice for the first step of extending the spectral analysis, but then it is hard to disentangle flat physics from it.

In the spectral analysis, we need to class the intermediate states appearing in correlation functions according to their quantum numbers. A maximally symmetric space is convenient, because particles and excited states are associated to irreducible unitary representations of the isometry group, and our problem is purely group theoretical. In short, the Hilbert space has a global meaning. Moreover, the hyperboloid of negative curvature is better than the sphere, because it is unbounded. Then correlators must decay at infinity, as in euclidean space,

$$\begin{aligned}\langle\phi(x)\phi(o)\rangle &= \langle 0|\phi(0)\mathrm{e}^{-arH}\phi(0)|0\rangle \\ &\sim |\langle\phi|\lambda\rangle|^2\,\mathrm{e}^{-\lambda ar}\to 0, \quad \text{as} \quad r\to\infty. \end{aligned}\tag{6.5}$$

The energy must be bounded below, $H|\lambda\rangle=\lambda|\lambda\rangle$, $\lambda>0$, therefore the states in the Hilbert space correspond to unitary highest weight representations of the isometry group. They are constructed in the following.

Potential drawbacks of hyperbolic space come from its peculiar boundary condition and IR behaviour. Free particles are not completely determined by their actions, because the solutions of the wave equation need a choice of boundary conditions, and the usual treatment is not sufficient here. For example, there are two bosons for small mass, i.e. two unitary representations in the Hilbert space satisfying the same wave equation. The introduction of boundary terms in the action has been discussed in the refs. [34].

Moreover, the IR limit on hyperbolic space, $r\gg 1/a$, is far beyond the curvature scale, which acts as an IR cut-off. Therefore, it is completely different from the IR limit in flat space, $1/\Lambda\ll r\ll 1/a$, where you first let the IR cut-off go to infinity, and then look at distances larger than the correlation length $1/\Lambda$. Therefore, on H_n the behaviour of massless states $\Lambda\sim a$ and the phase diagram is not interesting for flat physics. An exception could be the case of purely massive theories, like confined QCD, if the mass scale is $\Lambda\gg a$, and asymptotic distances are not considered, as discussed in ref. [15].

6.2. SPECTRAL REPRESENTATION

The generators L_{AB} of the isometry group $\mathrm{SO}(1,n)$ satisfy the algebra,

$$[L_{AB},L_{CD}]=g_{AC}L_{BD}+g_{BD}L_{AC}-g_{BC}L_{AD}-g_{AD}L_{BC}, \tag{6.6}$$

and act on scalar fields as follows

$$[L_{AB},\phi(x)]=(-x_A\partial_B+x_B\partial_A)\phi(x). \tag{6.7}$$

The Hilbert space description of the field theory requires the distinction between the time direction τ, $\tau \sim x^n$, for $x^n \sim 0$, and space x^i, $i = 1,\dots,n-1$. Time must be Wick rotated and unitarity of generators and representations are defined w.r.t. the pseudo-riemannian signature of anti-deSitter space. Therefore we seek for unitary h.w. (highest weight) representations of the group $\mathrm{SO}(2, n-1)$, more precisely its universal covering. The generators are divided as follows

$$
\begin{array}{lll}
L_{ij} = -L_{ij}^{\dagger} & \text{rotations} & (L_{ij}) \to \mathrm{SO}(n-1)\,, \\[2ex]
L_{in} \equiv B_i = B_i^{\dagger} & \text{boosts} & (L_{ij}, B_i) \to \mathrm{SO}(1, n-1)\,, \\[2ex]
L_{0n} \equiv H = H^{\dagger} & \text{hamiltonian,} & \\[2ex]
L_{0i} \equiv P_i = -P_i^{\dagger} & \text{momenta} & (L_{ij}, P_i, B_i, H) \to \mathrm{SO}(2, n-1)\,.
\end{array}
$$

Rotations and boosts form the Lorentz group, then particles have the usual spin quantum number s. The hamiltonian and the momenta generate pseudotranslations, which mix with Lorentz transformations. Highest weight representations are built by applying lowering and rising operators

$$
L_i \equiv \frac{1}{\sqrt{2}}(B_i - P_i)\,, \qquad L_i^{\dagger} \equiv \frac{1}{\sqrt{2}}(B_i + P_i)\,, \qquad [H, L_i] = -L_i\,, \qquad \left[H, L_i^{\dagger}\right] = L_i^{\dagger}\,, \tag{6.8}
$$

to the highest weight state $|\lambda\rangle$, defined by

$$
\begin{aligned}
&H|\lambda\rangle = \lambda|\lambda\rangle\,, \qquad L_i|\lambda\rangle = 0\,, \\
&L_{ij}|\lambda\rangle = \rho_{ij}^{(s)}|\lambda\rangle\,, \qquad \lambda = \sigma + (n-1)/2\,.
\end{aligned} \tag{6.9}
$$

Therefore the mass quantum number is replaced by σ. The representations are unitary for $\sigma \geqslant \max(-(n-1)/2, -1)$ and $\sigma = -(n-1)/2$, corresponding to the vacuum $|0\rangle$.

The quadratic Casimir is $C_2 = -\frac{1}{2}L_{AB}L^{AB}$, with eigenvalues

$$
c_2(\sigma) = \sigma^2 - \left(\frac{n-1}{2}\right)^2 + s(s+n-3)\,.
$$

When it acts on functions, it is represented by the covariant laplacian $\Delta = -\nabla^{\mu}\nabla_{\mu} = -a^2 C_2$. In the scalar case, this can be checked by using eq. (6.7). It follows that the projector $\mathscr{P}_{\sigma, s=0}$ on h.w. representations of weight σ and $s = 0$

can be represented as a sum of scalar wave functions, as we did in appendix A for flat space. Similarly, by inserting this projector in the correlator of general scalar fields, the spectral representation on H_n was derived in ref. [14].

Here we give a straightforward derivation by using group theory, which avoids reference to the wave equation, and discussion of boundary conditions. No ambiguities appear in the spectral representation. In appendix C, we derive from group theory the correlator of the scalar particle φ_σ, of "mass" σ, by summing up the intermediate excited states

$$\begin{aligned}\langle\varphi_\sigma(x)\varphi_\sigma(o)\rangle &= \langle 0|\varphi_\sigma(o)\mathrm{e}^{-arH}\varphi_\sigma(o)|0\rangle = \langle 0|\varphi_\sigma(o)\mathrm{e}^{-aRH}\mathscr{P}_{\sigma,s=0}\varphi_\sigma(o)|0\rangle \\ &= |\langle 0|\varphi_\sigma(o)|\sigma\rangle|^2 G(\sigma,r)\,. \end{aligned}\tag{6.10}$$

The constant of proportionality above is the projection of the scalar state on the h.w. state $|\sigma\rangle$. By using the Casimir operator, one can check that this correlator satisfies the wave equation, whose normalized solution is

$$G(\sigma,r) = \frac{1}{\Gamma(n/2)V}\left(\frac{a^2}{2\,\mathrm{e}^{i\pi}\sinh ar}\right)^{(n-2)/2} Q^{(n-2)/2}_{\sigma-1/2}(\cosh ar)\,,$$

$$\left[\Delta + a^2\left(\sigma^2 - \left(\frac{n-1}{2}\right)^2\right)\right]G(\sigma,r) = \frac{\delta^{(n)}(x)}{\sqrt{g(x)}}\,,\tag{6.11}$$

where $Q^\mu_\nu(z)$ are associate Legendre functions [35]. Notice however that there are two unitary solutions for $|\sigma| \leqslant 1$, differing for the behaviour at infinity (see later also).

The derivation of the spectral representation for the correlator of a general scalar field $A(x)$ is now easy to obtain. The sum over all h.w. representations gives a resolution of the identity

$$I = \int_{\text{h.w.s. of SO}(2,n-1)} \mathrm{d}\sigma\,\mathscr{P}_{\sigma,s=0}\tag{6.12}$$

to be inserted in correlators. The spectral representation is therefore

$$\langle A(r)A(o)\rangle = \int \mathrm{d}\sigma\,\rho_A(\sigma)G(\sigma,r)\,,\tag{6.13}$$

where $\rho_A(\sigma) = \Sigma_{(\alpha)}|\langle A(o)|\sigma,(\alpha)\rangle|^2$ is the projection on the h.w. state, possibly summed over other quantum numbers (α) which we do not need explicitly.

Next we represent the spin-zero part of $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$. The Lorentz structure of the state $|T_{\mu\nu}(x)\rangle$ is obtained by acting with the covariant derivative ∇_μ on a scalar state and requiring conservation

$$|T_{\mu\nu}\rangle \propto \left[\nabla_\mu\nabla_\nu + g_{\mu\nu}(-\nabla^2 + (n-1)a^2)\right]|\phi\rangle,$$

$$\langle\sigma|\Theta\rangle \propto (n-1)(na^2+\Delta)\langle\sigma|\phi\rangle. \tag{6.14}$$

Finally, the spectral representation is

$$\langle\Theta(x)\Theta(o)\rangle_{s=0} = A_n a^{n-2}\int_{a\left(\frac{n+1}{2}\right)}^{\infty} d\sigma\rho(\sigma)(\Delta+na^2)^2 G(\sigma,r). \tag{6.15}$$

Let us now find the form of the spectral density at conformal field theory. Since $\Theta = 0$, this correlator should vanish apart from a contract term. Actually, this happens for the values $\sigma = \pm(n+1)/2$, for which the differential operator inside the spectral representation is equal to the square of the wave equation. The negative value is discarded, being below the unitarity bound and we finally find

$$\rho(\sigma) = \rho_0\delta\left(\sigma - \frac{n+1}{2}\right),$$

$$\langle\Theta(x)\Theta(o)\rangle_{s=0} = A_n\rho_0 a^{n-2}(\Delta+na^2)\frac{\delta^{(n)}(x)}{\sqrt{g}} \quad \text{(at CFT on } \mathrm{H}_n\text{)}. \tag{6.16}$$

This is the result anticipated in sect. 4. In hyperbolic space, the spin-zero density is a delta function at CFT in any dimension. The number ρ_0 is the well-defined charge we were looking for. Let us add some comments.

(i) This delta term is at the bottom of the spectrum, $\sigma = (n+1)/2$. The bound is obtained by requiring that the dilatation operator $D = \int d^n x\sqrt{g}\,\Theta$ is well defined at infinity. It is convenient to map the correlator to the ball D^n, where $\sqrt{g(x)} \propto (1-\xi^2)^{-n}$, then

$$d^n x_1\sqrt{g(x_1)}\langle\Theta(x_1)\Theta(x_2)\rangle d^n x_2\sqrt{g(x_2)} \overset{x_1\to\infty}{\sim} e^{nar}e^{-(\sigma+(n-1)/2)ar},$$

$$r = \mathrm{dist}(x_1, x_2).$$

(ii) The spectral density is in flat space notations

$$c^{(0)}(\mu = \sigma a, \Lambda, a)\, d\mu = \Gamma(n)a^{n-2}\rho\left(\sigma, \frac{\Lambda}{a}\right)d\sigma.$$

6.3. EXAMPLE–FREE MASSIVE BOSON

Let us say something more about the doubling of bosons for low mass. Let σ_0 label the representation carried by the boson, and σ the intermediate mass. The action (3.27) specifies the type of bosonic particle through the wave equation, then one identifies

$$m^2 - \frac{n(n-2)}{4}a^2 = a^2 c_2(\sigma_0) \to \frac{m^2}{a^2} = \sigma_0^2 - \tfrac{1}{4}. \tag{6.17}$$

This equation has two solutions, compatible with unitarity, in the range $|\sigma_0| \leqslant \min(1,(n-1)/2)$. Therefore the action does not specify completely the theory, and it must be supplemented with a choice of boundary conditions. Actually, the propagators $G(\pm\sigma_0, r)$ have different behaviour at infinity. This is clear for the massless particles, $\sigma_0 = \pm\frac{1}{2}$, because the theory is Weyl invariant and we can conformally map the propagators to D^n. $G(\sigma_0 = \frac{1}{2}, x)$ corresponds to Dirichlet boundary condition on $\partial\mathrm{D}^n, G(\sigma_0 = -\frac{1}{2}, x)$ to Neumann one [34]. These are the two possible boundary conditions consistent with conformal invariance. The conformal propagators have an explicit form,

$$G(\pm\tfrac{1}{2}, r) \overset{\mathrm{H}_n}{=} \frac{1}{V(n-2)}\left[\left(\frac{a^2}{2(\cosh ar - 1)}\right)^{(n-2)/2} \mp \left(\frac{a^2}{2(\cosh ar + 1)}\right)^{(n-2)/2}\right]$$

$$\overset{\mathrm{D}^n}{=} \frac{1}{V(n-2)}\left(\frac{a^2}{4}(1-\xi^2)\right)^{(n-2)/2}\left[\frac{1}{(\xi^2)^{(n-2)/2}} \mp 1\right]. \tag{6.18}$$

The Dirichlet b.c. on D^n corresponds to fast exponential decrease at infinity in H_n, $G(1/2, r) \sim \exp(-nar/2)$, faster than the growth of the volume element $\sqrt{\mathrm{Vol}} \sim \exp(-(n-1)ar/2)$. The Neumann b.c. gives a slower one $G(-1/2, r) \sim \exp(-(n-2)ar/2)$. In absence of further physical intuition, both choices of boundary conditions are possible. In the case of the Neumann particle, certain quantities may need boundary terms to correct their behaviour at infinity.

Let us compute the spectral representation (6.15) for the massive boson. For $|x| \neq 0$, $\langle\Theta\Theta\rangle$ can be computed as in flat space

$$\langle\Theta\Theta\rangle = 2m^4 V^2 G^2(\sigma_0, x)$$

$$= \frac{2a^4(\sigma_0^2 - \frac{1}{4})^2}{\Gamma(n/2)^2}\left(\frac{a^2}{2\,\mathrm{e}^{i\pi}\sinh ar}\right)^{n-2}\left(Q_{\sigma_0-1/2}^{(n-2)/2}(\cosh ar)\right)^2. \tag{6.19}$$

The spectral density can be obtained by comparing this equation to eq. (6.15). Unfortunately, we do not have a simple way to invert the spectral representation in curved space. The Fourier space of the hyperboloid has been introduced in refs. [33, 14], but it is not easy to handle. Nevertheless, $\rho(\sigma, \sigma_0)$ can be obtained in odd dimensions, because $G(\sigma, r)$ is an elementary function. For example, in three dimensions

$$G(\sigma_0, r) = \frac{a}{V \sinh ar} \, \mathrm{e}^{-a\sigma_0 r}, \tag{6.20}$$

and

$$\rho(\sigma, \sigma_0) = 4 \sum_{k=0}^{\infty} \delta(\sigma - 2\sigma_0 - 2k - 1) \left[\frac{(2\sigma_0)^2 - 1}{(2k + 2\sigma_0 + 1)^2 - 4} \right]^2 . \tag{6.21}$$

We shall mainly consider the bosonic theory with the Dirichlet b.c., which exists for $\frac{1}{2} < \sigma < \infty$, and allows to consider the RG flow in the massive phase. The massless limit is $\sigma_0 \to \frac{1}{2} + \epsilon$

$$\rho(\sigma, \sigma_0) \sim \delta(\sigma - 2) + \mathrm{O}(\epsilon^2) \tag{6.22}$$

in agreement with our previous claim in eq. (6.16). Therefore we have computed $\rho_0 = 1$ for the boson, which fixes our conventions. Here are more comments.

(i) For $\sigma_0 \to \frac{1}{2} + \epsilon$, the intermediate state contributing to ρ_0 is the two-particle state of lowest energy $\lambda = 2\lambda_0$, i.e. $\sigma = 2\sigma_0 + (n-1)/2$. This is true in any dimension. Then ρ_0 can be computed from the matrix element $|\langle 2\lambda_0 | \Theta \rangle|^2 \propto |\langle \lambda_0 | \langle \lambda_0 | | \varphi \rangle | \varphi \rangle|^2$ by group theory in the appendix C. It follows

$$\rho_0 = 1 \quad (\text{boson, any } n) . \tag{6.23}$$

(ii) In the cases of the "irregular" Neumann boson $-1 < \sigma_0 < -\frac{1}{2}$, and the "tachyons" $|\sigma_0| < \frac{1}{2}$, $\rho(\sigma)$ has states below the bound $\sigma > (n+1)/2$. Then the dilatation operator D is not well defined at ∞, it probably needs a boundary term, to be included in the action. We do not develop further this point here, and refer to the literature [34]. In the following, we shall consider theories for which the bound can be respected.

(iii) Nevertheless, $\langle \Theta\Theta \rangle$ makes sense for the Neumann particle, and it takes the same value in the massless limit, computed as before,

$$\rho(\sigma, -\tfrac{1}{2}) = \rho_0^- \delta\left(\sigma - \frac{n+1}{2} \right), \quad \rho_0^- = 1 \qquad (\text{Neumann boson, any } n) . \tag{6.24}$$

The fact that ρ_0 is independent of boundary conditions is welcome. Actually, if this charge is expected to have a physical meaning in flat space, then it should not be sensible to long distance effects in hyperbolic space.

6.4. RELATION TO THE TRACE ANOMALY

The Weyl variations $\delta g_{\mu\nu} = f g_{\mu\nu}$ of the effective action define the v.e.v.'s of the trace of the stress tensor*

$$\langle \Theta \rangle = -\frac{2V}{\sqrt{g}} \frac{\delta}{\delta f} \log Z, \qquad \langle \Theta\Theta \rangle = -\frac{2V}{\sqrt{g}} \frac{\delta}{\delta f} \langle \Theta \rangle. \tag{6.25}$$

By taking a Weyl variation of the trace anomaly in eqs. (5.6) and (5.7) and comparing the result of the spectral representation (6.16) at CFT, we can relate ρ_0 to the coefficients in the trace anomaly. In two dimensions we find

$$\langle \Theta(x)\Theta(0) \rangle_{\mathrm{CFT}} = c\frac{V}{6}(\Delta + 2a^2)\frac{\delta^{(2)}(x)}{\sqrt{g}}, \tag{6.26}$$

therefore

$$\rho_0 = c \qquad (n=2). \tag{6.27}$$

This important relation motivated our approach to curved space.

However, things get involved in higher dimensions. For $n=3$ and any odd dimension there is no trace anomaly, therefore

$$\langle \Theta(x)\Theta(0) \rangle_{\mathrm{CFT}} = 0 \qquad (n = 2k+1). \tag{6.28}$$

For $n=4$, the F-term in eq. (5.7) is Weyl invariant and drops out, while the G-term gives

$$\langle \Theta(x)\Theta(0) \rangle_{\mathrm{CFT}} = a^2\gamma\frac{V}{120}(\Delta + 4a^2)\frac{\delta^{(4)}(x)}{\sqrt{g}}, \tag{6.29}$$

i.e. $\rho_0 \Leftrightarrow \gamma/3$. These results from the trace anomaly seem inconsistent with the spectral representation giving $\rho_0 \neq 0$ in any dimensions, and it is known for the $n=4$ boson that $\rho_0 = \gamma = 1$ [13]. The solution of this puzzle is as follows

$$\langle \Theta\Theta \rangle = \int \mathrm{d}\sigma \langle \Theta \mathscr{P}_{\sigma, s=0} \Theta \rangle + \langle \Theta\Theta \rangle_{\mathrm{contact}}, \tag{6.30}$$

where the last term is a contact term $\mathrm{O}(a^{n-2})$ which exists in hyperbolic space for $n \geqslant 3$. It cannot be seen in the spectral representation, because it does not correspond to propagating intermediate states. At CFT, it has the same form as

* These definitions are consistent with conservation of $T_{\mu\nu}$.

the spectral representation in eq. (6.16),

$$\langle \Theta\Theta \rangle_{\text{contact}} = a\rho_c A_n(\Delta + na^2)\frac{\delta^{(n)}(x)}{\sqrt{g}}\,. \tag{6.31}$$

A careful analysis of the case of bosonic theory in three dimensions supports this interpretation. The heat kernel can be found explicitly and a renormalized effective action on H_3 can be derived, paying attention to contact terms. By taking its variations one finds

$$\lim_{m\to 0}\langle \Theta \rangle = 0, \qquad \lim_{m\to 0}\langle \Theta\Theta \rangle = 0. \tag{6.32}$$

Therefore $\rho_c = -\rho_0$ for $n=3$. For the boson in $n=4$, one has $3(\rho_c+\rho_0)= 3\rho_c + 3 = \gamma = 1$. This contact term can be thought to arise from a local term in the effective action

$$S_{\text{eff}} \to S_{\text{eff}} - \rho_c \frac{A_n}{2V^2(n-2)(n-1)} a^{n-2}\int \sqrt{g}R\,. \tag{6.33}$$

This in turn implies that ρ_c also contributes to the spin-two part and can be computed independently. For $n > 4$ additional local terms with powers of the Riemann tensor can appear. We have computed the bosonic trace anomaly for higher dimensions $n = 2k$, and found that it has nontrivial coefficients in the normalization of $\rho_0 = 1$. This fact indicates that the relation gets more and more involved. For the Dirac fermion in four dimensions the numbers are $\rho_0 = 6$, $\rho_c = -\frac{7}{3}$, $\gamma = 11$.

Note that $\rho_0 > 0$ by positivity of $\langle \Theta(x)\Theta(0)\rangle$ for $|x| \neq 0$, while the sign of purely contact terms is arbitrary. Therefore we cannot prove that $\gamma > 0$, even if it seems to be so in known examples (see the table in subsect. 5.2). Hopefully, further conditions exist, relating ρ_c to ρ_0. In such a case, ρ_0 would be related to the anomaly, which is a short-distance property of the field theory in curved space.

6.5. SUM RULE HYPOTHESIS ON HYPERBOLIC SPACE

Let us discuss the possibility of a c-theorem for ρ_0 and its relation to $c^{(0)}$ defined in flat space in sect. 3. ρ_0 is a well-defined positive number of CFT in curved space, but its RG flow is not simple, due to the extra dimensionful parameter of the curvature. In this section we present a possible way to avoid the curvature dependence. If it holds, it also implies $\rho_0 = c^{(0)}$ by taking a careful limit to flat space. Recall the form of the spectral density

$$\rho\left(\sigma, \frac{\Lambda}{a}\right) = \rho_0\left(\frac{\Lambda}{a}\right)\delta\left(\sigma - \frac{n+1}{2}\right) + \rho_1\left(\sigma, \frac{\Lambda}{a}\right), \tag{6.34}$$

where we singled out the state at the bottom of the spectrum. At fixed points one finds

$$\lim_{\Lambda/a \to 0} \rho\left(\sigma, \frac{\Lambda}{a}\right) = (\rho_0)_{\mathrm{UV}}\delta\left(\sigma - \frac{n+1}{2}\right),$$

$$\lim_{\Lambda/a \to \infty} \rho\left(\sigma, \frac{\Lambda}{a}\right) = (\rho_0)_{\mathrm{IR}}\delta\left(\sigma - \frac{n+1}{2}\right). \tag{6.35}$$

Note that in presence of two scales, Λ and a, there are no scaling arguments relating these two limits. However, in two dimensions, we just proved that $\rho_0 = c$, thus there should be a hidden relation. This suggests that the sum rule is actually independent of the curvature scale

$$s\left(\frac{\Lambda}{a}\right) = \int_{3/2}^{\infty} \mathrm{d}\sigma \rho\left(\sigma, \frac{\Lambda}{a}\right) \stackrel{?}{=} c_{\mathrm{UV}} \quad (n=2). \tag{6.36}$$

Actually, $s(\Lambda/a)$ is equal to c_{UV} as $\Lambda/a \to 0$, the UV limit, where the density reduces to a delta, eq. (6.35), and also in the opposite regime $\Lambda/a \to \infty$, the flat limit! We do not have a general proof of this fact. We have verified it in the case of free massive bosons and fermions.

Remarkably enough, a similar sum rule exists for these free theories in higher dimensions

$$c_{\mathrm{UV}}^{(0)} = \int_{(n+1)/2}^{\infty} \mathrm{d}\sigma \rho\left(\sigma, \frac{\Lambda}{a}\right)\hat{f}(\sigma), \tag{6.37}$$

independent of Λ/a, for free massive bosons and fermions, where

$$\hat{f}(\sigma) = \frac{\Gamma(n)}{\left(\sigma + \frac{n-3}{2}\right)\left(\sigma + \frac{n-5}{2}\right)\cdots\left(\sigma - \frac{n-3}{2}\right)}. \tag{6.38}$$

It was explicitly checked for $n = 3, 4$, and probably holds for any n. Our calculation used rather nontrivial identities of Legendre functions (see appendix C). The three-dimensional case can be verified by using the formulas given before.

This RG-invariant sum rule is a sufficient condition for the correct definition of $c^{(0)}$ at CFT, as a limit of a quantity defined in curved space. If $\hat{f}(\sigma)$ is the same for any theory, as for the free ones, then $c^{(0)}$ satisfies a c-theorem. Otherwise ρ_0 satisfies a c-theorem, but $\Delta\rho_0$ has only a qualitative meaning in flat space physics. These speculations take the form of a sufficient condition.

Sufficient condition. Let us suppose that there exists a positive function $\hat{f}(\sigma)$ (normalized by $\hat{f}((n+1)/2)=1$) such that the following sum rule holds:

$$s\left(\frac{\Lambda}{a}\right)=\int_{(n+1)/2}^{\infty}\mathrm{d}\sigma\,\rho\left(\sigma,\frac{\Lambda}{a}\right)\hat{f}(\sigma)\quad\text{is independent of }\frac{\Lambda}{a},\tag{6.39}$$

then

(i) $s=(\rho_0)_{\mathrm{UV}}$,

(ii) the c-theorem holds for ρ_0:

$$(\rho_0)_{\mathrm{UV}}\geqslant(\rho_0)_{\mathrm{IR}},\tag{6.40}$$

(iii) in the flat limit $\sigma\to\infty$, with $\mu=\sigma a$ and $\Lambda\neq 0$ finite, let $\hat{f}(\sigma)\to b/\sigma^{n-2}$. Then

$$\rho_1\left(\sigma,\frac{\Lambda}{a}\right)\hat{f}(\sigma)\,\mathrm{d}\sigma\to\frac{b}{\Gamma(n)}\,\frac{c^{(0)}(\mu,\Lambda)}{\mu^{n-2}}\,\mathrm{d}\mu.\tag{6.41}$$

(iv) In particular $\Delta\rho_0=(b/\Gamma(n))\,\Delta c^{(0)}$. If $\hat{f}(\sigma)$ has the same asymptotic behaviour for all theories, this implies the c-theorem for $c^{(0)}$ in flat space. Actually, this is equivalent to eq. (6.40). In the case of free massive bosons and fermions for $n=2,3,4$, this yields $(\rho_0)_{\mathrm{UV}}=(c^{(0)})_{\mathrm{UV}}$.

Proof. If the sum rule holds, it can be computed at any point of the RG trajectory, exactly as it happens in flat space. Take $\Lambda\to 0$, in this limit $\rho(\sigma,\Lambda/a)$ reduces to the lowest delta-term, and (i) follows. For $\Lambda>0$ we can write the sum rule as

$$(\rho_0)_{\mathrm{UV}}=\rho_0\left(\frac{\Lambda}{a}\right)+\int_{(n+1)/2+\epsilon}^{\infty}\mathrm{d}\sigma\,\rho\left(\sigma,\frac{\Lambda}{a}\right)\hat{f}(\sigma)>\rho_0\left(\frac{\Lambda}{a}\right).\tag{6.42}$$

This bound goes to the limit $\Lambda\to\infty$, where the massive states decouple, and (ii) follows, because $(\rho_0)_{\mathrm{UV}}>\rho_0(\infty)=(\rho_0)_{\mathrm{IR}}$.

Let us now consider the flat limit off criticality, $\Lambda\neq 0$. ρ is a collection of deltas at values $\mu=(n+1)/2$, $\mu=\mathrm{O}(\Lambda)+ka,\dots$. As $a\to 0$, the massive states stay at finite distance from the massless one, and form a smooth function times the asymptotic behaviour of $\hat{f}(\sigma)$

$$\mathrm{d}\sigma\,\rho\left(\sigma,\frac{\Lambda}{a}\right)\hat{f}(\sigma)\to\mathrm{d}\mu\left(\rho_0(\infty)\delta(\mu-O(a))+\lim_{a\to 0}b\frac{\rho_1(\mu/a,\Lambda)a^{n-3}}{\mu^{n-2}}\right).\tag{6.43}$$

Then point (iii) follows, and $\Delta\rho_0=(b/\Gamma(n))\,\Delta c^{(0)}$.

6.6. THE O(N) σ-MODEL IN THREE DIMENSIONS

This theory provides a counterexample to the previous hypothesis that the sum rule holds in a way independent of the theory, i.e. eqs. (6.37), (6.38) with $\hat{f}$ given by the free theories. This sufficient condition is too strict. On the other hand, the possibility that $\hat{f}$ depends on the theory was not investigated, as well as checks of the sum rule in four dimensional theories.

The σ-model is defined by the bare action

$$S_0[\phi, \alpha] = \frac{1}{2}\int \frac{\mathrm{d}^3x}{4\pi}\left[\phi^i(\Delta + \mu^2)\phi^i + \alpha(\phi^i\phi^i - N\mu_0) + \frac{N}{\sqrt{\pi}}\left(\frac{\Lambda_0^3}{3} - \Lambda_0\mu^2\right)\right], \tag{6.44}$$

where the Lagrange multiplier α gives the constraint $\phi^i\phi^i = N\mu_0$, and Λ_0 is the UV cut-off. The theory is solved in the large-N limit by the saddle point approximation. The value $\mu_0 = \Lambda_0/\sqrt{\pi} - \mu$ removes the singularity in the saddle point equation and define the renormalized mass-coupling parameter μ. A phase transition takes place at $\mu = 0$, which separates the disordered phase ($\mu > 0$), from the spontaneously broken one ($\mu < 0$). These basic facts are recalled in ref. [36].

We have extended the saddle point calculation to curved hyperbolic space. The form of the action is unchanged, only replace $\Delta \to p^2 = \Delta + R/6$. In the unbroken phase, the integration over ϕ gives the effective action

$$\begin{aligned} S_{\text{eff}}[\alpha] &= \frac{N}{2}\ln\det_{\Lambda_0}(p^2 + \mu^2 + \alpha) \\ &\quad + \frac{N}{2}\int\frac{\mathrm{d}^3x\sqrt{g}}{4\pi}\left(-\mu_0\alpha(x) + \frac{1}{\sqrt{\pi}}\left(\frac{\Lambda_0^3}{3} - \Lambda_0\mu^2\right)\right). \end{aligned} \tag{6.45}$$

The determinant is regularized by introducing the heat kernel

$$\ln\det_{\Lambda_0}(p^2 + \mu^2 + \alpha) = -\int \mathrm{d}^3x\sqrt{g}\int_{\Lambda_0^{-2}}^{\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t(p^2+\mu^2+\alpha)}(x,x). \tag{6.46}$$

The saddle-point equation and the correlator $\langle\Theta\Theta\rangle$ are computable because the heat kernel has an explicit expression in the three-dimensional hyperbolic space. The flow to the massive phase $\mu > 0$ provides a nontrivial test of the sum rule in eq. (6.37) satisfied by the free theories. This was checked in two limits

$$\begin{aligned} s\left(\frac{\Lambda}{a}\right) &\to (\rho_0)_{\text{UV}} = N\frac{\pi^2}{16} = N(0.61685\dots), \quad &&\text{for } \frac{\Lambda}{a}\to\infty; \\ s(0) &= N(0.5863\dots), &&\text{for } \frac{\Lambda}{a}\to 0. \end{aligned} \tag{6.47}$$

Therefore, it is not verified.

The calculation was extended to the broken phase, yielding a rather convincing picture of the phase diagram in hyperbolic space. This side issue, as well as details of the previous calculation, will be discussed elsewhere.

6.7. THE CARDY–OSBORN CONJECTURE IN FOUR DIMENSIONS

These authors have conjectured that the coefficient γ of the anomaly might satisfy a c-theorem in four dimensions [9–11]. Osborn did a careful analysis of renormalization in the minimal subtraction scheme in curved space. He considered various RG functions which describe the flow of couplings parametrizing metric counterterms. Renormalizability of the theory implies consistency conditions for these functions.

One among these has an appealing similarity with the Zamolodchikov theorem, eq. (1.3) [10]. There exists a function* $\tilde{\gamma}(g^i)$ of the couplings g^i such that $\tilde{\gamma} = \gamma$ at fixed points, and its flow is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma} = -\beta^i \partial_i \tilde{\gamma} = -\beta^i \beta^j \chi_{ij} . \tag{6.48}$$

The c-theorem follows if $\chi > 0$. While in two dimensions χ is positive definite by construction, Osborn's metric is not. This approach misses the main ingredient of unitarity and is, in some sense, complementary to ours. Nevertheless, Cardy and Osborn provided examples of theories in which $\chi > 0$ in lowest order of the perturbation expansion. These are (i) QCD with a close fixed point, discussed in subsect. 5.4, for which $\Delta\tilde{\gamma} = \Delta\gamma = \mathrm{O}(\alpha^{*2}) > 0$; (ii) bosonic theory with a slightly relevant perturbation $\lambda\phi$, $\dim \lambda = y \ll 1$, such that it also possesses a close fixed point in four dimensions, at $\lambda = \mathrm{O}(y)$: $\Delta\gamma = \mathrm{O}(y^3) > 0$.

Moreover, $\chi > 0$ was also checked to lowest order in the theories $\lambda\phi^4$ and QED, which do not possess a close IR fixed point. These cases are less convincing because χ can change sign in higher perturbative orders, and nothing can be said about the monotonicity of the flow of γ.

In summary, there are positive examples supporting this conjecture. In the light of the discussion in the previous section, the quantity which is manifestly positive and analogous to the two-dimensional case is ρ_0, not γ. The problem of contact terms hides the relation between these two numbers. No relation was yet found specific to four dimensions, nor nontrivial examples for the flow of ρ_0.

It is an open possibility, that the positivity conditions for the Cardy–Osborn conjecture might be found by generalizing the spectral representation to other curved space-times possessing anisotropy. This will be discussed elsewhere.

* β_b in his notation.

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Note added in proof

Positivity of the coefficient α in the trace anomaly eq. (5.7) was already proven in ref. [39], by an argument similar to the one in subsect. 5.2. We thank M. Duff for informing us of his result.

Appendix A

SPECTRAL REPRESENTATION

The Lehmann spectral representation of a two-point Green function amounts to an expansion in propagating intermediate states [16]. These belong to the physical Hilbert space of the theory and correspond to the representations of the Poincaré group. The propagation of states with an intermediate mass μ is given by the propagator of a free scalar particle of the same mass.

In order to construct the spectral representation of a two-point correlator we shall proceed in three steps: (i) find a basis of the Hilbert space of the theory, (ii) use it to produce a resolution of the identity and (iii) insert this resolution of the identity in the correlator.

Let us consider an n-dimensional space-time with euclidean signature. The Hilbert space is made of eigenfunctions of the Laplace operator, $\Delta = -(\partial_\alpha \partial^\alpha)$,

$$\Delta | \boldsymbol{p}, \mu \rangle = -\mu^2 | \boldsymbol{p}, \mu \rangle , \qquad \Phi_{\mu, \boldsymbol{p}}(x) = \langle x | \boldsymbol{p}, \mu \rangle , \tag{A.1}$$

where we use minkowskian quantum numbers $p_\alpha = (\boldsymbol{p}, p_n = -ip_0 = -i\sqrt{\boldsymbol{p}^2 + \mu^2})$ with positive energy. These eigenfunctions form a complete $(n-1)$-dimensional basis of the Hilbert space

$$\int \mathrm{d}^{n-1}\boldsymbol{x}\, \Phi^*_{\mu,\boldsymbol{p}}(\boldsymbol{x}, -x^n) \frac{\overleftrightarrow{\partial}}{\partial x^n} \Phi_{\mu,\boldsymbol{q}}(\boldsymbol{x}, x^n) = \delta^{(n-1)}(\boldsymbol{p} - \boldsymbol{q}) , \tag{A.2}$$

$$-\int \mathrm{d}^{n-1}\boldsymbol{p}\, \Phi_{\mu,\boldsymbol{p}}(\boldsymbol{y}, x^n) \frac{\overleftrightarrow{\partial}}{\partial x^n} \Phi^*_{\mu,\boldsymbol{p}}(\boldsymbol{x}, -x^n) = \delta^{(n-1)}(\boldsymbol{x} - \boldsymbol{y}) , \tag{A.3}$$

$$\Phi_{\mu,\boldsymbol{p}}(\boldsymbol{x}, x^n) = \frac{\mathrm{e}^{i\boldsymbol{x}\boldsymbol{p} + x^n\sqrt{\boldsymbol{p}^2+\mu^2}}}{\left(2\sqrt{\boldsymbol{p}^2 + \mu^2}\,(2\pi)^{n-1}\right)^{1/2}} , \tag{A.4}$$

where we have singled out an euclidean time direction x^n, and the reflection of the euclidean bracket is $\langle \boldsymbol{p}, \mu | x \rangle = \Phi^*_{\mu,\boldsymbol{p}}(\boldsymbol{x}, -x^n)$.

This space gives a unitary (highest weight) representation of the Poincaré group ISO(1, $n-1$). The momentum operator acts as $\mathbb{P}_\mu = -i\partial_\mu$ and the eigenvalue of the Casimir is $\mathbb{P}^2 = \Delta = -\mu^2$. Therefore, the projector on representations of squared mass μ^2 is built as

$$\mathscr{P}_{\mu^2} = \int d^{n-1}p |p, \mu\rangle\langle p, \mu| . \tag{A.5}$$

The sum over all the representations of the Poincaré group gives a resolution of the identity

$$I = \int d\mu^2 \mathscr{P}_{\mu^2} , \tag{A.6}$$

which can be inserted into correlators.

As a first exercise, the propagator of a free scalar particle of mass μ can be obtained by inserting the projector $\mathscr{P}_{\mu^2}$ in the correlation function

$$\begin{aligned} G(x, \mu) &\equiv \langle \varphi(x)\varphi(0)\rangle = \langle \varphi(x)\mathscr{P}_{\mu^2}\varphi(0)\rangle \\ &= \theta(x^0)\langle \varphi(0, \mathbf{x})e^{-x^0 H}\mathscr{P}_{\mu^2}\varphi(0)\rangle + \theta(-x^0)\langle \varphi(0)e^{x^0 H}\mathscr{P}_{\mu^2}\varphi(0, \mathbf{x})\rangle \\ &= \int \frac{d^n p}{(2\pi)^n} \frac{e^{ipx}}{p^2 + \mu^2} . \end{aligned} \tag{A.7}$$

The normalization is fixed by $\langle \mathbf{p}, \mu|\varphi(x)|0\rangle = \Phi_{\mu, \mathbf{p}}(\mathbf{x}, x^n)$.

Next, let us consider the correlator of an arbitrary scalar field $\langle A(x)A(0)\rangle$. Inserting the resolution of the identity previously constructed we get

$$\langle A(x)A(0)\rangle = \int d\mu^2 \langle A(x)\mathscr{P}_{\mu^2}A(0)\rangle . \tag{A.8}$$

The amplitudes $\langle A(x)|\mathbf{p}, \mu\rangle$ and $\langle \varphi(x)|\mathbf{p}, \mu\rangle$ transform in the same way under the action of the Poincaré group, which is given by $[\mathbb{P}_\alpha, A] = -i\partial_\alpha A$ and $[\mathbb{J}_{\mu\nu}, A] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)A$, for a scalar field A or φ. By acting with $\mathbb{P}_\mu$ and $\mathbb{J}_{\mu\nu}$, one can check that they both have the same quantum numbers. Therefore they are equal up to a normalization, which can only depend on the invariant quantum numbers, i.e. the Casimirs of the group, mass and spin ($s = 0$ in this case)

$$\langle A(x)|\mathbf{p}, \mu\rangle = N_A(\mu^2)\langle \varphi(x)|\mathbf{p}, \mu\rangle . \tag{A.9}$$

By inserting this equation in eq. (A.8), the sum over eigenfunctions at fixed mass reproduces the propagator and the spectral representation follows

$$\langle A(x)A(0)\rangle = \int d\mu^2 |N_A(\mu^2)|^2 G(x, \mu) = \int d\mu^2 \rho_A(\mu^2) G(x, \mu) . \tag{A.10}$$

The function $\rho_A(\mu^2)$ is called the spectral density. It has been defined by just using the Hilbert space of the theory, so it is nonperturbative and finite. Moreover, if the theory is unitary, it is a positive function.

By expressing $G(\mu, x)$ in momentum space, the spectral representation takes the form of a dispersion relation (see ref. [16]) which can be inverted as

$$\rho_A(\mu^2) = (1/\pi)\mathrm{Im}\langle A(p)A(-p)\rangle|_{p^2=-\mu^2}, \tag{A.11}$$

where $\mathrm{Im}\, f(p^2)|_{p^2=-\mu^2} = (f(p^2=-\mu^2-i\epsilon) - f(p^2=-\mu^2+i\epsilon))/2i$. The analytical properties assumed in this context correspond, in our derivation, to the restriction to the positive energy spectrum, i.e. to the unitary highest weight representations. These, in turn, are sufficient for analysing physical correlators falling off exponentially at infinity.

A quick derivation. In the previous argument, we have constructed explicitly the projector on Poincaré representations in terms of wave-functions. This was meant for pedagogical reasons, but it is not necessary. Actually, we only need their combination giving the propagator, more precisely the sum over highest weight representations of given μ. This can be obtained directly, as follows. Let us rewrite the projector on Poincaré representations as

$$\begin{aligned} I &= \int \mathrm{d}\mu^2 \mathscr{P}_{\mu^2} = \int \mathrm{d}\mu^2 \delta_+(\mathbb{P}^2+\mu^2) \\ &= \int \mathrm{d}\mu^2 \int_{\text{Minkowski}} \mathrm{d}^n p\, \delta(p^2+\mu^2)\delta^{(n)}(\mathbb{P}-p)\theta(p_0), \end{aligned} \tag{A.12}$$

where $\mathbb{P}_\alpha$ is the momentum operator and $p_\alpha = (p_0, \boldsymbol{p})$ is its minkowskian eigenvalue. Let us insert it in $\langle A(x)A(0)\rangle$, e.g. for $x^n < 0$,

$$\begin{aligned} \langle A(x)A(0)\rangle &= \langle 0|A(0)A(x)|0\rangle = \langle 0|A(0)\mathrm{e}^{i\mathbb{P}x}A(0)|0\rangle \quad (x^n<0) \\ &= \int \mathrm{d}\mu^2 \langle 0|A(0)\mathscr{P}_{\mu^2}\,\mathrm{e}^{i\mathbb{P}x}A(0)|0\rangle \\ &= \int \mathrm{d}\mu^2 \int_{\text{Minkowski}} \mathrm{d}^n p\, \mathrm{e}^{ipx}\delta(p^2+\mu^2)\theta(p_0)\langle 0|A(0)\delta^{(n)}(\mathbb{P}-p)A(0)|0\rangle. \end{aligned} \tag{A.13}$$

The matrix element in the last equation does not depend on x, thus it is a function of the Casimirs μ^2 and s. The integral over p factorizes

$$\int_M \mathrm{d}^n p\, \mathrm{e}^{ipx}\, \delta(p^2+\mu^2)\theta(p_0) = \int \mathrm{d}^{n-1}\boldsymbol{p}\, \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \int_0^\infty \mathrm{d}p_0\, \mathrm{e}^{p_0 x^n}\, \delta(-p_0^2+\mu^2+\boldsymbol{p}^2)$$

$$= \int \mathrm{d}^{n-1}\boldsymbol{p}\, \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_n}{2\pi}\, \mathrm{e}^{ip_n x^n} \frac{\theta(-x^n)}{p_n^2+\boldsymbol{p}^2+\mu^2}\,. \quad \text{(A.14)}$$

The propagator in euclidean space follows by adding the term for $x^n > 0$. Therefore we obtain again the spectral representation (A.10), with spectral measure

$$\rho_A(\mu^2) = (2\pi)^{n-1}\langle 0|A(0)\delta^{(n)}(\mathbb{P}-p)A(0)|0\rangle|_{p^2=-\mu^2}\,. \quad \text{(A.15)}$$

Appendix B

PERTURBATIVE CALCULATIONS IN $\lambda\varphi^4$ THEORY

In this appendix, we sketch the perturbative calculation of the spin-two spectral density reported in subsect. 5.3. We also show that $\Theta = 0$ at the IR fixed point, by a suitable choice of the improvement parameter ξ in the action (5.13). Therefore this theory gives a nontrivial example for the improvement hypothesis we made in subsect. 3.1.

From the action (5.13), we set up the renormalization in $n = 4-\epsilon$ dimensions, using dimensional regularization and minimal subtraction. Our scheme agrees with ref. [29], chapter 9. The renormalization constants Z and Z_g are defined by

$$\varphi_0 = \sqrt{Z}\varphi\,, \qquad S\lambda_0 = (Z_g/Z^2)\kappa^\epsilon g\,, \quad \text{(B.1)}$$

where $S = V/(2\pi)^n$ is a geometrical factor and κ the renormalization scale. These constants are determined by the requirements that (i) they are the sum of poles in ϵ, and (ii) the two- and four-vertex functions $\Gamma^{(2)}(p^2=\kappa^2)$ and $\Gamma^{(4)}(p^i)|_{\mathrm{S.P.}}$ are finite as $\epsilon \to 0$. At the two-loop order one finds [29]

$$Z = 1 - \frac{1}{48}\frac{g^2}{\epsilon}\,, \qquad Z_g^{-1} = 1 + \frac{3}{2}\frac{g}{\epsilon} + \frac{3}{4}\frac{g^2}{\epsilon}\,,$$

$$S\lambda_0 = \kappa^\epsilon g\left(1 - \frac{3}{2}\frac{g}{\epsilon} + g^2\left(\frac{9}{4\epsilon^2} - \frac{17}{24\epsilon}\right)\right). \quad \text{(B.2)}$$

By requiring $\kappa\, \mathrm{d}\lambda_0/\mathrm{d}\kappa = 0$ one finds the beta function (5.16). Feynman integrals are computed with the formula

$$b(\alpha,\beta)=\int\frac{\mathrm{d}q^{n}}{(2\pi)^{n}(q^{2})^{\alpha}\left((p-q)^{2}\right)^{\beta}}$$

$$=\frac{S\Gamma(2-\epsilon/2)}{2}\frac{\gamma(\alpha+\beta-2+\epsilon/2)}{\gamma(\alpha)\gamma(\beta)}(p^{2})^{2-\epsilon/2-\alpha-\beta}, \tag{B.3}$$

where $\gamma(x)=\Gamma(x)/\Gamma(2-\epsilon/2-x)$.

Let us notice that there are no IR divergences in these massless diagrams, because the external momenta provides a cut-off. IR divergences can only appear at very high perturbative order $O(1/\epsilon)$, (this can be seen by using the operator product expansion). Moreover there are no IR divergences associated to mass renormalization, because tadpoles diagrams vanish in dimensional regularization.

The density $c^{(2)}$ is computed by the imaginary part of the correlator $\langle T(p)T(-p)\rangle$ as in eq. (3.26), with T given in eq. (5.14). This has the same Feynman series as $\langle :\varphi^{2}::\varphi^{2}:\rangle$, but the T-vertex gives a complicate tensor algebra. The perturbative expansion of the spectral density is

$$c^{(2)}(\mu)=\mu^{1-\epsilon}\epsilon\left(\frac{1}{\epsilon}+\mathcal{N}_{1}(\epsilon)\lambda_{0}\mu^{-\epsilon}+\mathcal{N}_{2}(\epsilon)(\lambda_{0}\mu^{-\mu})^{2}\right). \tag{B.4}$$

In this equation, the overall ϵ is obtained by taking the imaginary part, and $\mathcal{N}_i$ are the result of the Feynman diagrams. $\mathcal{N}_1$ vanishes, as explained in the text. $\mathcal{N}_2$ is given by 3 three-loop diagrams, naively divergent as $O(1/\epsilon^{3})$. One of these diagrams, the "cat-eye", is not known in closed form. In ref. [30] it was expressed as a double series, by using Gegenbauer polynomials. We need it in the form

$$I(\alpha)=\int\frac{\mathrm{d}q^{n}\,\mathrm{d}k^{n}}{(2\pi)^{2n}q^{2}(p-q)^{2}k^{2}(p-k)^{2}\left((q-k)^{2}\right)^{\alpha}}$$

$$=(p^{2})^{-\epsilon-\alpha}\frac{S^{2}\Gamma(2-\epsilon/2)^{2}\Gamma(1-\epsilon/2)^{3}}{2\Gamma(2-\epsilon)\gamma(\alpha)}$$

$$\times\sum_{n,m=0}^{\infty}\frac{(-1)^{m}\Gamma(n+2-\epsilon)\Gamma(m+n+\alpha+\epsilon)}{m!n!(n+1-\epsilon/2)\Gamma(m+n+2-\epsilon/2)\Gamma(2-m-3\epsilon/2-\alpha)}$$

$$\times\left(\left(\frac{1}{(n+\alpha)}+\frac{1}{(m+n+1)}\right)\frac{1}{(m+n+\alpha+\epsilon/2)}\right.$$

$$\left.+\frac{1}{(m+n+1)(n+2-\alpha-\epsilon)}\right), \tag{B.5}$$

together with the remarkable recurrence relation [31]

$$I(\alpha) = -\frac{\alpha+\epsilon/2}{\alpha-1+\epsilon} I(\alpha+1) - \frac{2-4\alpha-3\epsilon}{\alpha-1+\epsilon} b(1,1+\alpha) b(1,1+\alpha+\epsilon/2) . \quad \text{(B.6)}$$

Actually, in our diagrams the nontrivial numerator, coming from the T-vertices, can be simplified with the denominator, leaving three irreducible terms $I(\alpha)$, for $\alpha = \epsilon/2$, $\epsilon/2-1$, $\epsilon/2-2$, and some b-integrals. Then the part of $I(\alpha)$ singular in ϵ is computed by a finite number of terms in eq. (B.5). It turns out that this is all we need. All the algebra was checked with the REDUCE program.

The result is $\mathcal{I}_2 = O(1/\epsilon)$, i.e. the leading and subleading singularities cancel. Therefore the spectral measure is finite, as expected,

$$c^{(2)}(\mu, g)|_{\mu=\kappa} = \kappa^{1-\epsilon} c^{(2)}(g) ,$$

$$c^{(2)}(g) = 1 - \tfrac{5}{144}(\lambda_0 S_\kappa^{-\epsilon})^2 + \dots = 1 - \tfrac{5}{144} g^2 + O(g^3, g^2\epsilon) . \quad \text{(B.7)}$$

Improvement at the IR fixed point. In discussing the stress tensor, it is compulsory to consider the theory in curved space. Then the bosonic theory possesses an additional coupling ξ_0 which fixes the improvement term. At the UV fixed point $\lambda_0 = 0$ the trace Θ vanishes if

$$\xi_0 = \frac{n-2}{4(n-1)} \quad (\text{for} \quad \lambda_0 = 0) . \quad \text{(B.8)}$$

In the interacting theory, this parameter gets renormalized like a mass parameter. Lüscher has obtained [32]

$$\xi_0 = \tfrac{1}{6} + (\xi - \tfrac{1}{6}) \left[1 - \frac{g}{2\epsilon} + \frac{g^2}{2} \left(\frac{1}{\epsilon^2} - \frac{5}{24\epsilon} \right) \right] + \frac{g^2}{144\epsilon} + O(g^3) \quad (n = 4 - \epsilon) . \quad \text{(B.9)}$$

Clearly ξ cannot affect $\beta(g)$, at least to low order, because this is determined by leading UV singularities, but it has a flow in g, and it determines the properties of Θ at the IR fixed point. By tuning it, we have a one-parameter family of IR fixed points, for the theory in curved space.

However:

(i) The spin-two measure is independent of ξ, as argued in the text. Actually the computation of $c^{(2)}$ was repeated with a different choice of ξ_0, obtaining the same result.

(ii) There is a unique choice of ξ, such that $\Theta = 0$ at the IR fixed point, to $O(g^2)$. This corresponds to keeping fixed ξ_0 to the free-theory value, i.e. adding no corrections in λ_0, λ_0^2.

Indeed, by inverting eq. (B.9) with ξ_0 given by eq. (B.8), one finds that ξ is finite as $\epsilon \to 0$, i.e. it is an admissible parameter for the renormalized Green functions in curved space. Analysis of the RG flow also shows that this is the unique perturbative form of $\xi = \xi(g)$ satisfying the UV boundary condition $\xi(0) = \xi_0$.

Moreover for this value of ξ_0 the bare expression of the trace is

$$\Theta = -\epsilon V \lambda_0 \varphi_0^4/4! . \tag{B.10}$$

The coupling and the composite field undergo a renormalization, and the result is

$$\Theta = V\beta(g)\Psi , \tag{B.11}$$

where $\Psi = \varphi^4/4! + \dots$ is a renormalized field. This formula is the improvement hypothesis of subsect. 3.1. It was verified to $O(g^2)$ in two relatively easy cases. They are

$$\begin{aligned}\mathrm{Im}\langle\Theta(p)\Theta(-p)\rangle|_{p^2=\kappa^2} &= (\epsilon\lambda_0 V)^2\,\mathrm{Im}\left\langle\frac{\varphi_0^4}{4!}\frac{\varphi_0^4}{4!}\right\rangle \sim \epsilon^2(\lambda_0 S)^2\left(1+6\frac{\lambda_0 S\kappa^{-\epsilon}}{\epsilon}\right)\\ &\sim \epsilon^2 g^2\left(1-3\frac{g}{\epsilon}\right)\left(1+6\frac{g}{\epsilon}\right) \sim \left(\epsilon g+\frac{3}{2}g^2\right)^2 \sim \beta(g)^2,\end{aligned} \tag{B.12}$$

$$\begin{aligned}\mathrm{Im}\left\langle\Theta(-p)\varphi\left(\frac{p}{2}\right)\varphi\left(\frac{p}{2}\right)\right\rangle\bigg|_{p^2=\kappa^2} &= \epsilon\lambda_0 V\frac{1}{Z}\,\mathrm{Im}\left\langle\frac{\varphi_0^4}{4!}\varphi_0\varphi_0\right\rangle \sim \epsilon(\lambda_0 S)^2\left(1+\frac{9}{2}\frac{\lambda_0 S\kappa^{-\epsilon}}{\epsilon}\right)\\ &\sim \epsilon g^2\left(1-3\frac{g}{\epsilon}\right)\left(1+\frac{9}{2}\frac{g}{\epsilon}\right) \sim g\beta(g) .\end{aligned} \tag{B.13}$$

Indeed these correlators are finite as $\epsilon \to 0$ and proportional to the beta function, then eq. (B.11) is checked with a renormalized field Ψ. We took the imaginary part, because correlators of composite fields have additional singularities not taken into account by conventional renormalization (say $\langle :\varphi^2::\varphi^2: \rangle$ is logarithmic singular in the free theory). Therefore eq. (B.11) is perfectly applicable to spectral functions, which are imaginary parts. They reconstruct finite correlators in coordinate space. On the other hand the real part of correlators in momentum space is divergent, and it is given by a subtracted dispersion relation.

Appendix C

THE HYPERBOLOID IN n DIMENSIONS

C.1. GENERALITIES

The hyperboloid in n dimensions [33], which is denoted as H_n, can be covariantly described as a hypersurface in $n+1$ dimensions

$$x\cdot x \equiv g_{AB}x^A x^B = -(x^0)^2+(x^1)^2+\dots+(x^n)^2 = -\frac{1}{a^2} \quad \text{with} \quad x^0>0, \tag{C.1}$$

It corresponds to the x^n Wick rotation of anti-de Sitter space. It is convenient to define the covariant derivative on the hypersurface as

$$\nabla_A f \equiv \partial_A f - \frac{x_A x^C}{x \cdot x} \partial_C f. \tag{C.2}$$

The curvature is obtained as follows

$$[\nabla_A, \nabla_B] x_C = -\frac{1}{x \cdot x}(x_B g_{AC} - x_A g_{BC}) \equiv R_{ABC}{}^D x_D. \tag{C.3}$$

It follows that $R = -n(n-1)a^2$, by comparing eq. (C.3) with the form of the Riemann tensor in a maximally symmetric spaces.

There are several intrinsic ways of representing H_n. Polar coordinates are introduced by picking an origin $o^A = (1/a, 0^\mu)$ and eliminating the auxiliary variable x^0

$$\sqrt{(x^\mu)^2} = \frac{\sinh ar}{a}, \qquad x^0 = \frac{\cosh ar}{a}, \qquad r = \mathrm{dist}(x, o), \tag{C.4}$$

where r is the geodesic distance from the origin. The metric is

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + \left(\frac{\sinh ar}{a}\right)^2 (\mathrm{d}\Omega_{n-1})^2. \tag{C.5}$$

where $(\mathrm{d}\Omega_{n-1})^2$ in the metric of the sphere S^{n-1}. We shall also use the variables

$$z = \cosh ar > 1, \qquad \tau = ar = \log\left(z + \sqrt{z^2 - 1}\right). \tag{C.6}$$

The volume element can be obtained from the metric eq. (C.5). The integration of invariant functions $f = f(r)$ is

$$\int \mathrm{d}^n x \sqrt{g} f(x) = a^{-n} V \int_0^\infty \mathrm{d}z (z^2 - 1)^{(n-2)/2} f(z). \tag{C.7}$$

where $V = \mathrm{Vol}(S^{n-1})$.

C.2. SO(2, $N-1$) UNITARY HIGHEST WEIGHT REPRESENTATIONS

As indicated in subsect. 6.2, we consider the generators L_{AB} of SO(1, n), the isometry group of H_n, but unitarity is chosen according to the x^n Wick-rotated metric, which is, in standard notations, whose of the group SO(2, $n-1$). We recall from the text that the generators are divided into the sets

$$
\begin{aligned}
&L_{ij} = -L_{ij}^{\dagger}, && \text{rotations}, && (L_{ij}) \to \mathrm{SO}(n-1),\\
&L_{in} \equiv B_i = B_i^{\dagger}, && \text{boosts}, && (L_{ij}, B_i) \to \mathrm{SO}(1, n-1),\\
&L_{0n} \equiv H = H^{\dagger}, && \text{hamiltonian}, && \\
&L_{0i} \equiv P_i = -P_i^{\dagger}, && \text{momenta}, && (L_{ij}, P_i, B_i, H) \to \mathrm{SO}(2, n-1),
\end{aligned}
$$

and the lowering and raising operators are

$$L_i \equiv \frac{1}{\sqrt{2}}(B_i - P_i), \qquad L_i^{\dagger} \equiv \frac{1}{\sqrt{2}}(B_i + P_i), \qquad [H, L_i] = -L_i, \qquad [H, L_i^{\dagger}] = L_i^{\dagger}. \tag{C.8}$$

The highest weight states $|\lambda, s\rangle$ are defined by

$$
\begin{aligned}
&H|\lambda, s\rangle = \lambda|\lambda, s\rangle, \qquad L_i|\lambda, s\rangle = 0,\\
&L_{ij}|\lambda, s\rangle = \rho_{ij}^{(s)}|\lambda, s\rangle, \qquad \lambda = \sigma + \frac{n-1}{2}.
\end{aligned} \tag{C.9}
$$

Then, the excited states are of the form $L_{i_1}^{\dagger} \ldots L_{i_r}^{\dagger}|\lambda, s\rangle$. We shall be mainly interested in spin-zero representations, which are indicated without the index s, $|\lambda, 0\rangle \equiv |\lambda\rangle$.

For $s = 0$, the unitarity bound on λ comes from

$$
\begin{aligned}
&\| L_i^{\dagger}|\lambda\rangle \|^2 \geqslant 0 \to \lambda \geqslant 0,\\
&\Big\| \sum_i L_i^{\dagger} L_i^{\dagger} |\lambda\rangle \Big\|^2 \geqslant 0 \to \lambda \geqslant \frac{n-3}{2} \text{ or } \lambda = 0.
\end{aligned} \tag{C.10}
$$

The quadratic Casimir operator is defined by

$$C_2 \equiv -\tfrac{1}{2} L_{AB} L^{AB} = -\tfrac{1}{2} L_{ij} L^{ij} + H^2 + P_i^2 - B_i^2 = J^2 - L_i^{\dagger} L_i + H^2 - (n-1)H, \tag{C.11}$$

where J^2 is the Casimir of $\mathrm{O}(n-1)$, with eigenvalue $s(s+n-3)$. Therefore

$$C_2|\lambda, s\rangle = [\lambda(\lambda - n + 1) + s(s + n - 3)]|\lambda, s\rangle. \tag{C.12}$$

On scalar fields ϕ, the L_{AB} are represented by the derivatives

$$[L_{AB}, \phi(x)] = (-x_A \partial_B + x_B \partial_A)\phi(x) \equiv K_{AB}\phi(x). \qquad \text{(C.13)}$$

Therefore, the Casimir is represented on scalar functions $f(x)$ by the covariant laplacian, $\Delta f = -a^2 C_2 f$. This can be derived by using eq. (C.2), as follows

$$-a^2 C_2 f = \frac{a^2}{2} K^{AB} K_{AB} f = -\left(\partial_A^2 - \frac{x^A x^B}{x \cdot x}\partial_A \partial_B - n\frac{x^B}{x \cdot x}\partial_B\right) f = -\nabla^A \nabla_A f = \Delta f. \qquad \text{(C.14)}$$

On invariant functions $f(r)$, this has the intrinsic expression

$$-\Delta f(r) = \left[\partial_r^2 + a(n-1)\frac{\cosh ar}{\sinh ar}\partial_r\right] f = a^2\left[(z^2-1)\partial_z^2 + nz\partial_z\right] f. \qquad \text{(C.15)}$$

The propagator (6.10) of the scalar field φ_λ carrying the representation of weight $\lambda = \sigma + (n-1)/2$ is an eigenstate of the Casimir operator. By applying it as a differential operator, the wave equation (6.11) can be obtained. Its solution is an invariant function, therefore the laplacian is given by eq. (C.15), which can be put into the form of the Legendre equation for $z > 1$, giving solution (6.11) in the text [37].

Actually, we should specify the choice of boundary conditions which led to this solution. As discussed in the literature [15], there is a one-parameter family of solutions singular at the origin, which reduces to two solutions compatible with Weyl invariance in the massless case. Therefore, the propagator is not completely reconstructed by the fact that it is a singular solution of the equation of motion.

We avoid this problem by giving a straightforward derivation of the propagator based on group theory. We proceed in two steps: (i) construct a scalar state, (ii) sandwich $e^{-\tau H}$ between two scalar states. The first step is done as follows. The general form for a scalar state is

$$|\varphi_\lambda\rangle \equiv \varphi_\lambda(0)|0\rangle = \sum_{N=1}^{\infty} a_N (K^\dagger)^N |\lambda\rangle, \qquad \text{(C.16)}$$

where $K = \frac{1}{2} L_i L_i$ and the vacuum $|0\rangle$ is the h.w. state with $\lambda = 0$. A scalar state must be annihilated by the boosts B_i, yielding a recursion relation

$$a_N = -a_{N-1}\frac{1}{N\left(\lambda + \frac{3-n}{2} + N - 1\right)} = a_0(-)^N \frac{\Gamma\left(\lambda + \frac{3-n}{2}\right)}{\Gamma(N+1)\Gamma\left(\lambda + \frac{3-n}{2} + N\right)}. \qquad \text{(C.17)}$$

The propagator is then obtained as

$$
\begin{aligned}
G(x,\lambda) &= \langle \varphi_\lambda | \mathrm{e}^{-\tau H} | \varphi_\lambda \rangle \\
&= \sum_{N=0}^{\infty} |a_N|^2 \, \mathrm{e}^{-\tau(\lambda+2N)} \langle \lambda | (K)^N (K^\dagger)^N | \lambda \rangle \\
&= |a_0|^2 \frac{\Gamma\left(\lambda + \frac{3-n}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{n-1}{2}\right)} \sum_{N=0}^{\infty} \mathrm{e}^{-\tau(\lambda+2N)} \frac{\Gamma(N+\lambda)\Gamma\left(N+\frac{n-1}{2}\right)}{\Gamma(N+1)\Gamma\left(N+\lambda+\frac{3-n}{2}\right)} \\
&= |a_0|^2 \, \mathrm{e}^{-\tau\lambda} F\left(\frac{n-1}{2}, \lambda; \lambda + \frac{3-n}{2}, \mathrm{e}^{-2\tau}\right),
\end{aligned}
\tag{C.18}
$$

which agrees, once the normalization is fixed

$$
|a_0|^2 = a^{n-2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\lambda + \frac{3-n}{2}\right)V}
\tag{C.19}
$$

with the solution of the wave equation (6.11).

C.3. FIRST TERM OF THE SPECTRAL REPRESENTATION FOR FREE MASSIVE BOSONS

Let us recall that the mass of the free boson is related to the highest weight by the equation $m^2/a^2 = \sigma_0^2 - \frac{1}{4}$. Let us consider the positive branch $\sigma_0 > \frac{1}{2}$, and compute the first term in the spectral representation of $\langle \Theta\Theta \rangle$. Since $\Theta = Vm^4\varphi^2$, the intermediate states in $\langle \Theta\Theta \rangle$ are two-particle states, of the form

$$
\varphi_{\lambda_0}^2 |0\rangle = \sum_{N,M=0}^{\infty} a_N a_M (K^\dagger)^N |\lambda_0\rangle \otimes (K^\dagger)^M |\lambda_0\rangle,
\tag{C.20}
$$

in agreement with eq. (C.16). The lowest intermediate state is the two-particle state $\lambda = 2\lambda_0$

$$
\mathscr{P}_{\lambda=2\lambda_0} \varphi_{\lambda_0}^2 |0\rangle = a_0^2 |\lambda_0\rangle \otimes |\lambda_0\rangle .
\tag{C.21}
$$

This produces the term $\rho_0 \delta(\lambda - 2\lambda_0)$ in the spectral representation $\rho(\lambda)$. In order to compute ρ_0, we need the normalization of the two-particle state in eq. (C.21) w.r.t. the standard one given in the spectral representation (6.15). This latter one is, for $a = 1$ and $r \neq 0$,

$$\langle \Theta \mathscr{P}_{\lambda=2\lambda_0} \Theta \rangle = \frac{A_n}{V} \rho_0 \frac{\left(\sigma^2 - \left(\frac{n+1}{2} \right)^2 \right)^2}{\Gamma\left(\frac{n}{2} \right) (2 \mathrm{e}^{i\pi})^{\mu}} (z^2 - 1)^{-\mu/2} Q_{\nu}^{\mu}(z), \quad \text{(C.22)}$$

where $\mu = (n-2)/2$ and $\nu = \sigma - \frac{1}{2}$. The former one is, by eqs. (C.21) and (C.18)

$$\langle \phi^2 \mathscr{P}_{\lambda=2\lambda_0} \phi^2 \rangle = 2|a_0|^4 \frac{\Gamma\left(2\lambda_0 + \frac{3-n}{2} \right)}{\Gamma(\frac{1}{2}) \Gamma(2\lambda_0) (2 \mathrm{e}^{i\pi})^{\mu}} (z^2 - 1)^{-\mu/2} Q_{\nu}^{\mu}(z). \quad \text{(C.23)}$$

By comparing the last two equations and using eq. (C.19), it follows

$$\frac{A_n}{V} \rho_0 = \frac{(\sigma_0 + \frac{1}{2})^2}{2(2\sigma_0 + n)^2} \frac{\Gamma(\frac{1}{2}) \Gamma^2\left(\sigma_0 + \frac{n-1}{2} \right)}{\Gamma\left(\frac{n}{2} \right) \Gamma^2(\sigma_0 + 1)} \frac{\Gamma\left(2\sigma_0 + \frac{n+1}{2} \right)}{\Gamma(2\sigma_0 + n - 1)}. \quad \text{(C.24)}$$

In the massless limit $\sigma_0 \to \frac{1}{2}$ one obtains

$$\frac{A_n}{V} \rho_0 = \frac{1}{(n+1) 2^{n-1}} \quad \to \quad \rho_0 = 1, \quad \text{(C.25)}$$

while the higher states in the spectral representation vanish.

This computation of ρ_0 can be repeated for the Neumann massless boson, $\sigma = -\frac{1}{2}$, obtaining the same result. In this case, the contribution comes from the first excitation of the two particle state, $\lambda = 2\lambda_0 + 2$. Other states vanish again, due to Weyl invariance.

C.4. SUM RULES INDEPENDENT OF THE CURVATURE

The sum rule, eqs. (6.37) and (6.[illegible] was first shown to be independent of σ_0 in the form of an integral over H_n

$$1 = \int_{\mathrm{H}_n} \mathrm{d}^n x \sqrt{g} f_n(x, a) \langle \Theta(x) \Theta(0) \rangle_{\text{boson}}$$

$$= V \int_1^{\infty} \mathrm{d}z f_n(z) \frac{(4\sigma_0^2 - 1)^2}{\Gamma\left(\frac{n}{2} \right) 2^{n+1} \mathrm{e}^{2i\pi\mu}} (Q_{\nu_0}^{\mu})^2, \quad \text{(C.26)}$$

where $\mu = (n-2)/2$ and $\nu_0 = \sigma_0 - \frac{1}{2}$.

The function f_n was found as follows. The square of the function Q was rewritten by using the integral representations 8.715 and 7.137.1 in ref. [35]. Then the identity in eq. (C.26) was interpreted as a condition on the Laplace transform

$$1=\left(4\sigma_0^2-1\right)^2\int_0^\infty d\tau\, e^{-2\sigma_0\tau}F(\tau) \quad \to \quad F=2\tau\cosh\tau-2\sinh\tau\,, \qquad (C.27)$$

where F is related to f_n by an integral operator, $F=K_n f_n$, whose action is computed by series expansion. The result of cumbersome computations is

$$f_2=\frac{1}{a^2V}2\left(z\log\frac{1+z}{2}+z-1\right),$$

$$f_3=\frac{1}{a^3V}4(\tau\cosh\tau-\sinh\tau)\,, \qquad z=\cosh\tau$$

$$f_4=\frac{1}{a^4V}8\left(z\log\frac{1+z}{2}-\frac{z-1}{z+1}\right). \qquad (C.28)$$

Note that the leading behaviour at short distance is $f_n\sim r^n$, by dimensionality.

These functions can be turned into functions of the spectral parameter $\hat{f}(\sigma)$ by inserting the spectral representation (6.15) in eq. (C.26), and using the wave equation satisfied by $G(\sigma,r)$. The answers are

$$\hat{f}_2(\sigma)=1\,, \qquad \hat{f}_3(\sigma)=\frac{2}{\sigma}\,, \qquad \hat{f}_4(\sigma)=\frac{6}{\sigma^2-\frac{1}{4}}\,. \qquad (C.29)$$

They led to conjecture the general form of f_n in any dimensions in eq. (6.38).

These sum rules were verified in the case of free massive Dirac fermions following similar steps. In ref. [37], the correlation of the fermionic stress tensor was written as a sum of two terms of the bosonic type,

$$\langle\Theta\Theta\rangle_{\text{fermion}}=\frac{2^{[n/2]}a^{2n}\nu_0^2}{\Gamma\left(\frac{n}{2}\right)^2 2^{n-2}e^{2i\pi\mu}}\left(z^2-1\right)^{-\mu}\left[\left(Q_{\nu_0}^{\mu+1}\right)^2-(\nu-\mu)^2\left(Q_{\nu_0}^{\mu}\right)^2\right], \qquad (C.30)$$

The previous integral kernel was inspected for each term.

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