Nuclear Physics B274 (1986) 71–92
North-Holland, Amsterdam

ALL FREE STRING THEORIES ARE THEORIES OF FORMS

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Received 24 January 1986

We generalize the gauge-invariant theory of the free bosonic open string to treat closed strings and superstrings. All of these theories can be written as theories of string differential forms defined on suitable spaces. All of the bosonic theories have exactly the same structure; the Ramond theory takes an analogous first-order form. We show, explicitly, using simple and general manipulations, how to gauge-fix each action to the light-cone gauge and to the Feynman-Siegel gauge.

1. Introduction

After remaining for a long time incomplete and ill-understood, the covariant formulation of string theory is finally being completed to a gauge-invariant string field theory. The recent developments began with Siegel's formulation of a covariantly gauge-fixed bosonic string [1], based on the BRST first-quantization of the string [2]. Out of this work grew a gauge-invariant formulation of free bosonic strings, to which many authors have contributed [3–15, 40]. The geometrical foundation of this theory has been investigated in refs. [16, 17]. In addition, progress has been made on the pressing issue of identifying the gauge-invariant interaction terms for open bosonic string fields [3, 18–22, 40]. Some of this progress has been reflected

[1] On leave from Tel Aviv University. Supported in part by the US-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

[2] Work supported by the Department of Energy, contract DE-AC03-76F00515.

[3] Supported in part by the Department of Energy, contract DE-FG02-84ER45144, and by the Alfred P. Sloan Foundation.

[4] Supported in part by the Department of Energy, contract DE-AC02-76ER03072.

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(North-Holland Physics Publishing Division)

in the theory of supersymmetric strings. The Neveu-Schwarz-Ramond formulation of the superstring theory has been written as a covariantly gauge-fixed string field theory [23–25] and as a gauge-invariant theory [4, 6, 7, 26, 27]. However, as yet none of these gauge-invariant formulations of the theory contain the full set of Stueckelberg fields needed to make these theories equivalent to the covariant or light-cone gauge-fixed formulations. In addition, the technology involved in many of these papers is complex, in a way that obscures the relatively simple structure of these theories.

In this paper, we address these last two difficulties by presenting a unified formulation of the gauge invariant free string field theories associated with all known string models. We will construct lagrangians for these theories which have a common structure and which are simply given in terms of appropriate differential operators. A weakness of our formalism is that is does not properly treat the gauge invariances associated with the zero modes of closed strings; in particular, our formulation of the Ramond/Ramond sector of closed superstrings requires an externally-imposed dynamical constraint. Nevertheless, all of our string field actions are quantum-mechanically complete: We will show how to fix these lagrangians to the covariant Feynman-Siegel gauge and to the light-cone gauge by techniques applicable to all of the string theories, and we will prove that these manipulations lead to the known physical spectra. The differential operators which we require arise naturally from the structure of BRST transformations for each string, though, unfortunately, we have not been able in all cases to connect our gauge transformations precisely with BRST transformations*. In the Neveu-Schwarz-Ramond theory, we find intriguing relations, also noted by LeClair [23], between world-sheet and space-time statistics and BRST invariance. Many of the results we present have been obtained independently by other groups [3, 29, 30].

The paper is organized as follows: in sect. 2, we introduce the calculus of forms and exterior derivatives on which our formalism rests. We will identify appropriate operators for each string as components of the BRST charge Q and use the relation $Q^2 = 0$ to derive identities among these operators. This operator calculus generalizes that of ref. [7]. In sect. 3, we present the complete theory of free open bosonic strings [7–9] in a simplified formulation discovered independently by Restuccia and Taylor [3], Witten [18], Ramond [14], and Neveu, Nicolai, and West [15]. We display the action of this theory and discuss the gauge-fixing in a manner conducive to generalization. In sects. 4 and 5, we generalize this construction to all other known string theories, treating first theories of bosons and then theories of fermions.

2. The string exterior derivative and the BRST charge

In this section, we will set up our conventions for discussing string forms and differential operators. Essentially, this formalism works by considering the ghosts of

* Ooguri has claimed to have found this connection for the open Ramond string [28].

the first-quantized string theory as differentials, a point of view advocated in a more general setting by Baulieu and Thierry-Mieg [32]. Our notation will generally follow the notation for the ghosts of string theories presented in ref. [31]. Note that our conventions here differ somewhat from those of ref. [7] and, in fact, serve to simplify some of the identities given there.

The general string fields which we will use in this paper will be functions of the string coordinates $x^\mu(\sigma)$ and the reparametrizations ghosts $b(\sigma), c(\sigma)$. In the covariant superstring theories, we need also the supersymmetric partners of these objects, the fermionic coordinates $\psi^\mu(\sigma)$ and the superconformal ghosts $\beta(\sigma)$, $\gamma(\sigma)$. In general, we will work with the normal mode creation and annihilation operators associated with these fields. It is useful to group these operators into supersymmetry multiplets. Letting α_n^μ represent the normal mode operators for the field x^μ, we may group together:

$$A_N^\mu = (\alpha_n^\mu, \psi_{\hat{n}}^\mu),$$

$$B_N = (b_n, \beta_{\hat{n}}),$$

$$C^N = (c^n, \gamma^{\hat{n}}). \tag{2.1}$$

We will refer to the first member of each pair as bosonic and the second as fermionic. The index N runs over the appropriate set of normal modes – e.g. integers for the bosonic open string variables, or left-and right-moving integers for bosonic closed string variables. The notation $\hat{n}$ is intended to remind the reader that the bosonic and fermionic variables may run over different sets, as happens in the Neveu-Schwarz string. We will need the notation:

$$(-1)^{MN} = \begin{cases} -1, & \text{if } M \text{ and } N \text{ are fermionic} \\ +1, & \text{otherwise} \end{cases}. \tag{2.2}$$

We assign the commutation relations of the operators (2.1) as follows:

$$A_M^\mu A_N^\nu - (-1)^{MN} A_N^\nu A_M^\mu = \eta^{\mu\nu}\delta(M+N), \tag{2.3}$$

$$B_M C^N + (-1)^{MN} C^N B_M = \delta(M+N). \tag{2.4}$$

These relations are preserved by the hermitian conjugation ($M \neq 0$)

$$(A_M^\mu)^\dagger = A_{-M}^\mu, \qquad (B_M)^\dagger = B_{-M}, \qquad (C^M)^\dagger = (-1)^M C^{-M}. \tag{2.5}$$

In these conventions, the A_M^μ are real- or Grassmann-valued operators, the creation parts (the $M<0$ components) of B_M and C^M are real- or Grassmann-valued differentials, and the annihilation parts ($M>0$) of B_M and C^M are operators which remove these differentials. The fermionic components of A^{μ}_{M} then commute with the bosonic components of B_M and C_M. It is possible, by a Klein transformation, to redefine b, c, β, and γ so that b and c behave as Grassmann-valued operators and β and γ as real-valued operators; that is the formulation of the theory chosen in ref. [31].

We now define a string differential form as an object containing a product of string differentials. To be more specific, define a vacuum state $|0\rangle$ such that

$$A^{\mu}_{M}|0\rangle = 0\,, \qquad B_M|0\rangle = 0\,, \qquad C^{M}|0\rangle = 0\,, \tag{2.6}$$

for $M > 0$, and such that $|0\rangle$ has no dependence on the zero modes of the coordinate or ghost operators*. (We will later discuss a more complete vacuum $|\Omega\rangle$ which includes this dependence.) Then, keeping the restriction that integers M, N take only positive values, we can represent a string differential form as a state in the Fock space built on $|0\rangle$:

$$|\phi\rangle = C^{-N_b} \dots C^{-N_1} B_{M_1} \dots B_{M_a} \phi[A]^{M_1 \dots M_a}{}_{N_1 \dots N_b}|0\rangle\,. \tag{2.7}$$

$\phi[A]^{M_1 \dots}{}_{N_1 \dots}$ may be expanded in a sum of local fields times products of A^{μ} creation operators. We will refer to a state with a B's and b C's as an $\binom{a}{b}$-form. It will be useful to focus on the difference between b and a, the difference, that is, between the number of ghosts and antighosts. We will label a string form with $(b - a) = g$ as a g-form and refer to g as the *ghost number*.

Since we have identified differentials with ghost operators, the BRST charge Q takes the form of an exterior differential operator. Then the central identity $Q^2 = 0$ can be recast as an identity, or as a set of identities, defining the cohomology of string exterior derivatives. Let us now work out these identities explicitly for the various string theories.

For any string theory, the BRST charge can be written in the following form: Let the L_N be the generators of the appropriate reparametrization algebra – the Virasoro, Neveu-Schwarz, or Ramond algebra [33] – and write this algebra as

$$[L_M, L_N]_{\pm} = F_{MN}{}^{K} L_K + \tfrac{1}{2}\delta(M+N)\Gamma_M\,, \tag{2.8}$$

where Γ_M is the central charge. Let $\hat{L}_N = L_N - \delta_{N,0} l$, where $l = (1, \frac{1}{2}, 0)$ for bosonic, Neveu-Schwarz, and Ramond strings, respectively. Then

$$Q = C^{-N}\hat{L}_N + F_{IJ}{}^{K} : C^{-J} C^{-I} B_K : \tag{2.9}$$

* The state labeled $|0\rangle$ in ref. [31] differs from this one by including dependence on the zero mode, and shifts are necessary to make the state $SL(2, R)$-invariant. It is shown there that these shifts account for the shift l of L_0 given below eq. (2.8).

satisfies $Q^2 = 0$ in the critical dimensionality. The form (2.9) is precisely that given by Kato and Ogawa [2] for the bosonic string. The BRST charge for the fermionic string, found by Friedan, Martinec, and Shenker [35] and discussed more recently by a number of authors [23, 25, 27, 34], can be readily cast into this form. The critical dimensionality is required in order that terms in the square of the second term of (2.9) with two contractions can cancel terms in the square of the first term arising from the central charge. All other terms in $Q^2 = 0$ vanish by virtue of the Jacobi identities of the reparametrization algebra.

Let us now decompose Q into terms with distinct action on forms. In addition, we should extract terms in Q which depend explicitly on the ghost zero modes, which we have ignored up to now. For the bosonic open string, and for the Neveu-Schwarz string, the only ghost zero mode operators are b_0 and c^0. When Q acts on a form $|\phi\rangle$, every term in Q raises the ghost number g by 1 unit. However, the terms with explicit zero mode operators do not affect the total number of B_{-N} and C^{-N} operators in $|\phi\rangle$, while the other terms may create an additional C^{-N} or destroy a B_{-N}. Let us, then, break Q into pieces as follows:

$$Q = c^0 K + d + \delta - 2 \Downarrow b_0 , \tag{2.10}$$

where $d = C^{-N} L_N + \cdots$ contains all terms which create a net C, $\delta = C^N L_{-N} + \cdots$ contains all terms which destroy a net B, and all zero modes operators are indicated explicitly. The action of d and δ on forms reproduces the definition of the string exterior derivatives given in ref. [7], up to some overall minus signs (which are simply a matter of convention). The operators K and $\Downarrow$ may be evaluated from (2.9). K is given by

$$K = L_0 - l + \mathcal{N} , \tag{2.11}$$

where $\mathcal{N} = M(C^{-M} B_M + B_{-M}(-1)^M C^M)$ is the sum of the indices of the differentials in $|\phi\rangle$. This is precisely the kinetic energy operator of ref. [7]. $\Downarrow$ is given by

$$\Downarrow = \eta_{MN} C^{-M}(-1)^N C^N , \tag{2.12}$$

where

$$\eta_{MN} = \delta_{M,N} \begin{cases} M, & M \text{ bosonic} \\ 1, & M \text{ fermionic} \end{cases} . \tag{2.13}$$

This differs from the definition of $\Downarrow$ in ref. [7] by signs and a normalization factor. Using (2.5), one can see that $d^\dagger = \delta$, and that K and $\Downarrow$ are self-adjoint.

Now we have defined the basic operators of the bosonic and Neveu-Schwarz string theories. Since these operators appear as components of Q, they will obey identities which follow from the relation $Q^2 = 0$. To find these relations, square (2.10), separate the result into terms which create a fixed number of C's and destroy a fixed number of B's, and set each of these terms equal to zero. From the terms which create two C's and those which destroy two B's, we find:

$$d^2 = \delta^2 = 0 . \tag{2.14}$$

The term which preserves the number of the various differentials and contains no zero mode operators gives

$$(d\delta + \delta d) = 2\boldsymbol{K} \Downarrow . \tag{2.15}$$

Eqs. (2.14) and (2.15) are the fundamental relations for string differential operators applied in ref. [7]. The remaining pieces of the identity $Q^2 = 0$ imply that $\Downarrow$ and $\boldsymbol{K}$ commute with d and δ, and with one another.

We note parenthetically that the index-raising operator of ref. [7], written in this new notation, takes the form

$$\Uparrow = \eta^{MN} B_{-M} B_{N} , \tag{2.16}$$

where $\eta^{MN} = (\eta_{MN})^{-1}$. $\Uparrow$ can be used to invert $\Downarrow$ by virtue of the relation

$$[\Downarrow , \Uparrow] = C_{-}{}^{M} B_{M} - B_{-M} (-1)^{M} C^{M} = g . \tag{2.17}$$

In the Ramond theory, the fermionic ghosts β and γ are also integer-moded, so the BRST charge contains two new zero mode operators β_0 and γ^0. Let us write the decomposition of Q in this case as follows:

$$Q = c^0 \boldsymbol{K} + \gamma^0 \boldsymbol{F} + d + \delta - 2 \Downarrow b_0 - 2 \downarrow \beta_0 + (\gamma_0)^2 b_0 . \tag{2.18}$$

The separation between d and δ is defined just as before, and $\boldsymbol{K}$ and $\Downarrow$ are again given by (2.11) and (2.12). (2.18) also contains the Grassmann operators

$$\boldsymbol{F} = F_0 + f^{M}{}_{\overline{M}} \left[C_{-}{}^{\overline{M}} B_{M} - B_{-M} (-1)^{\overline{M}} C^{\overline{M}} \right] , \tag{2.19}$$

where the bar on an index changes it from bosonic to fermionic or vice versa, and where

$$f^{M}{}_{\overline{M}} = \begin{cases} 2, & M \text{ bosonic} \\ \frac{1}{2} M, & M \text{ fermionic} \end{cases} , \tag{2.20}$$

$$\downarrow = \tfrac{1}{4} M C_{-}{}^{M} C^{\overline{M}} . \tag{2.21}$$

$\boldsymbol{F}$ is the generalization of the Dirac-Ramond operator to the space of string forms; one may easily check that

$$\boldsymbol{F}^2 = \boldsymbol{K} . \tag{2.22}$$

Writing out the square of (2.18) and equating it to zero term by term yields a myriad of relations among the various operators we have introduced. One of these relations identifies $\downarrow$:

$$2\downarrow = [F, \Downarrow]; \tag{2.23}$$

others imply that K, F and $\Downarrow$ commute with d and δ, and that $K = F^2$ commutes with $\Downarrow$. Finally, one finds as before

$$d^2 = \delta^2 = 0, \tag{2.24}$$

and, making use of (2.23) and (2.22),

$$(d\delta + \delta d) = F(F\Downarrow + \Downarrow F). \tag{2.25}$$

Note that the two factors on the right-hand side of (2.25) commute with one another.

3. The open bosonic string

The operators defined in the previous section provide the basic components needed to build gauge-invariant free string actions, as has been shown already in ref. [7]. Recently, however, a form of the open bosonic string theory considerably simpler than the original formulation of ref. [7–9] has been discovered by Restuccia and Taylor [3], Witten [18], Ramond [14], Neveu, Nicolai, and West [15], Aratyn and Zimermann [27] and Baulieu and Ouvry [40]. It is most convenient to take this formulation as our starting point for a construction of the other free string theories. In this section, then, we will review this formulation and derive some of its properties.

In its simplified form, the gauge-invariant open-string action can be written as an expectation value of the corresponding BRST charge Q. To describe this construction, we must first adjoin to $|0)$ a wave function for the ghost zero mode. Define, then,

$$|\Omega) = |0) \otimes |\omega), \tag{3.1}$$

where $|\omega)$ is the zero mode wave function obeying $b_0|\omega) = 0$. $|\Omega)$ has the properties of the open string ground state, the tachyon. It is useful to recall that, in the covariant quantization of ref. [31], this state has the properties:

$$(\Omega|\Omega) = 0, \qquad (\Omega|c^0|\Omega) = 1. \tag{3.2}$$

In this paper, we will simply assume (3.2) as our starting point.

Now let $|\Phi)$ be an arbitrary 0-form on this extended space:

$$|\Phi) = \big(|\phi) + c^{0}|\eta)\big)|\omega) , \tag{3.3}$$

where ϕ is a 0-form and η is a (-1)-form of the structure (2.7). The free open bosonic string action can then be written as

$$S = -\tfrac{1}{2}(\Phi|Q|\Phi) . \tag{3.4}$$

This action has the obvious gauge invariance:

$$\delta_E|\Phi) = Q|E) , \tag{3.5}$$

where E is an arbitrary (-1)-form. To understand the structure of this transformation a bit better, let us write

$$|E) = \big(|\varepsilon) + c^{0}|\theta)\big)|\omega) . \tag{3.6}$$

Then the gauge transformation is

$$\begin{aligned} \delta_E \phi &= (d+\delta)\varepsilon - 2 \Downarrow \theta , \\ \delta_E \eta &= \boldsymbol{K}\varepsilon - (d+\delta)\theta . \end{aligned} \tag{3.7}$$

The ε transformation of ϕ displayed here is the gauge symmetry identified in refs. [7–9].

The action presented in refs. [7–9] may be obtained from (3.4) by gauge-fixing some of the auxiliary fields which this action contains. Let us first expand (3.4) in the component forms ϕ, η. This yields:

$$\begin{aligned} S = &-\tfrac{1}{2}(\phi|\boldsymbol{K}|\phi) + (\eta|\Downarrow|\eta) \\ &- \tfrac{1}{2}(\phi|d+\delta|\eta) - \tfrac{1}{2}(\eta|d+\delta|\phi) . \end{aligned} \tag{3.8}$$

To go further, we should recall from ref. [7] the concept of a *maximally symmetrized* form. Consider the coefficient $\phi^{M_1 \dots M_k}{}_{N_1 \dots N_k}$ as a tensor with upper and lower indices, separately antisymmetrized. Imagine lowering the upper indices using the metric (2.13) and then projecting the full set of indices onto combinations of definite symmetry. Because of the separate antisymmetrization, one may find only representations of the permutation symmetry corresponding to Young tableaux with two columns. The maximally symmetrized combination is defined to be the combination in which the second column is as long as possible, that is, in which as many lower indices as possible are symmetrized with upper indices, and vice versa. In a 0-form such as ϕ, with equal numbers of upper and lower indices, the maximally symmetrized component is that in which every upper index is symmetrized with a lower index in the process of Young symmetrization. In general, maximally symmetrized forms with $g \geq 0$ are annihilated by $\Downarrow$.

Let us, then, partially gauge-fix (3.8) by imposing $\Downarrow|\phi) = 0$. The resulting Faddeev-Popov determinant is nondynamical. Since $\Downarrow$ commutes with d and δ, we can see that $(d + \delta|\phi)$ is a maximally symmetrized 1-form: thus, only the maximally symmetrized component of $|\eta)$ couples to the remaining components of $|\phi)$. Since $|\eta)$ is in any event nondynamical, we can freely drop (or integrate out) the other components, leaving only the maximally symmetrized one. This component is annihilated by $\Uparrow$; thus (2.17) implies $\Uparrow \Downarrow |\eta) = |\eta)$. Using this relation to integrate out this last piece of $|\eta)$, we find at last

$$S = -\tfrac{1}{2}(\phi|K|\phi) + \tfrac{1}{4}(\phi|(d+\delta)\Uparrow(d+\delta)|\phi), \tag{3.9}$$

which is the action of ref. [7], written in our new conventions. Our gauge-fixing left the residual gauge invariance:

$$\delta_E|\phi) = (d+\delta)|\epsilon), \tag{3.10}$$

where $|\epsilon)$ is restricted to be maximally symmetrized; this is precisely the gauge invariance of refs. [7–9].

In principle, one could now complete the gauge-fixing of this action along the line given in ref. [7], for the covariant Feynman-Siegel gauge, or ref. [12], for the light-cone gauge. However, it is much simpler to begin again from (3.4).

To reach the Feynman-Siegel gauge, we use the gauge-fixing condition

$$b_0|\Phi) = |\eta) = 0. \tag{3.11}$$

The associated Faddeev-Popov ghost action is

$$S_E = (\bar{E}|\delta_E b_0 \Phi) = -(\bar{E}|b_0 Q|E). \tag{3.12}$$

In this expression, the ghost E is a general (-1)-form, and the antighost is a general 2-form. In a manner familiar from the analysis of refs. [7–9], this ghost action has in turn its own gauge invariances which require the introduction of higher-order ghosts. In particular, (3.12) is clearly invariant to the motion

$$\delta_G|E) = Q|G), \qquad \delta_G|\bar{E}) = 0, \tag{3.13}$$

reflecting the fact that a gauge parameter of the form $|E) = Q|G)$ leaves $|\eta)$ invariant. Note that only the c^0 component of the antighost survives in (3.12), and that this component has no corresponding gauge transformation. In addition, since every component of $|\eta)$ transforms under some gauge motion, fixing the functional integral with a $\delta(|\eta))$ produces no hidden ghosts. These two statements have analogues at all higher levels. The fully gauge-fixed action is, then,

$$S = -\tfrac{1}{2}(\Phi|Q|\Phi) - (\bar{E}|b_0 Q|E) - (\bar{G}|b_0 Q|G) - (\bar{E}'|b_0|E') - \cdots, \quad (3.14)$$

where $|\Phi), |E), \ldots,$ are constrained to be annihilated by b_0. If we decompose each field into components, according to (3.3), (3.6),

$$|G) = (|\gamma) + c^0|\zeta))|\omega), \quad (3.15)$$

etc., (3.14) takes the form

$$S = -\tfrac{1}{2}(\phi|K|\phi) - (\bar{\theta}|K|\varepsilon) - (\bar{\zeta}|K|\gamma) - (\bar{\theta}'|K|\varepsilon') - \cdots. \quad (3.16)$$

In this expression, $|\phi)$ is a general 0-form, $|\bar{\theta})$ is a general 1-form, $|\varepsilon)$ is a general (-1)-form, etc. The action (3.16) is thus exactly the covariant-gauge open string action (Feynman-Siegel gauge) derived by Siegel in ref. [1].

So that the reader can compare this analysis to the more intricate gauge-fixing procedure of ref. [7], let us give a second gauge-fixing prescription closer to the spirit of that analysis. As we noted above, the non-maximally symmetrized components of $|\eta)$ in eq. (3.8) are nondynamical and do not couple to $|\phi)$; thus they may be discarded. This allows us to use the alternative gauge-fixing condition of setting to zero $\Downarrow|\phi)$ and the maximally symmetrized component of $|\eta)$. The reader may verify that this leads to a Faddeev-Popov action for $|\varepsilon)$, $|\theta)$, $|\bar{\varepsilon})$, and $|\bar{\theta})$ which reproduces the form of (3.8), and to the appearance of hidden ghosts. Continuing this procedure, following the logic of ref. [7], one also eventually arrives at the action (3.16).

Let us now discuss the gauge-fixing of (3.4) to the light-cone gauge. The action of the open string in the light-cone gauge is given by

$$S = -\tfrac{1}{2}(\phi_T|K|\phi_T), \quad (3.17)$$

where ϕ_T contains only transverse states. To characterize these states, let us denote the light-cone components of A_N^μ by

$$K_N = A_N^+, \qquad M_N = A_N^-. \quad (3.18)$$

With this notation, the transverse states are those which include no K, M, B, or C creation operators acting on $|0)$. We must, then, show that all states other than the transverse states may be removed from (3.4) by a choice of gauge. To do this, we will use a counting argument similar in form to the one developed in ref. [12] to discuss the gauge fixing of the action of refs. [7–9]. (The reader who finds this argument a bit sketchy should consult ref. [12] for a more discursive presentation.)

Represent the classes of states we must gauge away as:

$$K^p C^q M^r B^s|0)\,, \tag{3.19}$$

where p, q, r, s denote the number of creation operators of the given type which act on $|0)$, $p+q+r+s=\mathscr{N}>0$. Since at any given mass level, $\mathscr{N}$ has a maximum value, we can confine our attention to states with a fixed value of $\mathscr{N}$, beginning at the maximum, and sequentially remove all of these states from (3.4). We can remove these fields without generating Faddeev-Popov determinants if we shift by terms in (3.7) which involve no factors of p^-. We will, in fact, use only terms in (3.7) involving $(d+\delta)$. We will only need to consider the term in d of the form

$$d = C^{-N}\cdot p^+ M_N + \cdots \tag{3.20}$$

and the term in δ of the form

$$\delta = C^{N}\cdot p^+ M_{-N} + \cdots\,; \tag{3.21}$$

we may imagine, then, that d simply converts a K to a C and δ simply converts a B to an M.

As a simple illustration of the use of these rules, let us discuss the counting of gauge parameters for states with $\mathscr{N}=1$ and 2. For states with $\mathscr{N}=1$, the only gauge parameters are of the form $B|0)$. These suffice to gauge away all states in $|\phi)$ of the form $M|0)$. The remaining states in $|\phi)$ which we need to eliminate are those of the form $K|0)$. These states appear together with the states $M|0)$ in the first term of (3.8), but this term has been removed by our choice of gauge. The only remaining place that the states $K|0)$ appear is in the cross terms of (3.8); since d converts a K to a C, this state can overlap with states $B|0)$ in $|\eta)$. This matrix element uses only the term (3.20) in d, which contains no p^-. Thus, the states $K|0)$ act as Lagrange multipliers to eliminate the states $B|0)$ in $|\eta)$. Thus, we have exactly the gauge freedom we require to eliminate all states with $\mathscr{N}=1$.

The analogous argument for $\mathscr{N}=2$ illustrates some complications found at higher levels. The states in $|\phi)$ and $|\eta)$ which must be eliminated have the form

$$K^2|0),\ KM|0),\ M^2|0),\ KB|0),\ MB|0),\ BC|0)\,. \tag{3.22}$$

The gauge parameters in $|\varepsilon)$ and $|\theta)$ have the form

$$KB|0),\ MB|0),\ B^2|0)\,. \tag{3.23}$$

In addition, we must consider the gauge parameters of the gauge parameters, which characterize the redundancies in (3.23). These are states in $|G)$, of the form

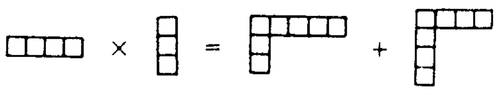
$$B^2|0)\,. \tag{3.24}$$

It is useful to think of these multiplets of states as components of tensors whose indices run over all positive integers. The commutation relations of these operators place restrictions on these tensors: $B^2|0\rangle$ is antisymmetric in its indices, and $M^2|0\rangle$ is symmetric. Thus, we can use (3.24) to gauge away the antisymmetric part of $MB|0\rangle$ in (3.23); the remaining symmetric part of this multiplet can gauge away the states $M^2|0\rangle$ is (3.22). $KB|0\rangle$ in (3.23) can gauge away $KM|0\rangle$, and $B^2|0\rangle$ in (3.23) can gauge away the antisymmetric part of $MB|0\rangle$ is (3.22). The remaining states in (3.22) are either Lagrange multipliers or are eliminated by Lagrange multipliers: $K^2|0\rangle$ eliminates the symmetric part of $MB|0\rangle$, and $KB|0\rangle$ eliminates $BC|0\rangle$.

Let us now generalize this counting argument to all levels. As a first step, we must reduce the full set of gauge parameters in $|E\rangle$ to those parameters which cannot be gauged away by higher-level gauge transformations. Consider, for example, the components of $|E\rangle$ of the form

$$K^pC^qM^{r-1}B^{s+1}|0\rangle . \tag{3.25}$$

Some of these components can be removed by acting with δ on components of $|G\rangle$ of the form $K^pC^qM^{r-2}B^{s+2}|0\rangle$. These components have their own redundancies, corresponding to the states $K^pC^qM^{r-3}B^{s+3}|0\rangle$, and so forth. The nonredundant components of $|E\rangle$ can be identified as follows: Operators M^r form an r-index symmetric tensor with indices in the set of values of N ($N>0$). Similarly, operators B^s form an s-index antisymmetric tensor. It is convenient to project states with both M's and B's onto states of definite (mixed) permutation symmetry, labeled by Young tableaux. For example, $M^4B^3|0\rangle$ belongs to

 (3.26)

Since we will be seeing many products of this form, let us refer to a Young tableau or r symmetrized boxes as $\{r\}$, a tableau of s antisymmetrized boxes as $[s]$, and a tableau with a row of r boxes above a column of s boxes as (r/s). In this language, (3.26) reads

$$\{4\}\times[3]=(5/2)+(4/3) . \tag{3.27}$$

One can then see that states (3.25) in $|E\rangle$ contain M's and B's in the representation $(r/s)+((r-1)/(s+1))$. Their redundancies belong to $((r-1)/(s+1))+((r-2)/(s+2))$. The redundancies of the redundancies belong to $((r-2)/(s+2))+((r-3)/(s+3))$. Continuing until one runs out of M's, and then resolving the net effect of these parameters, one finds that the nonredundant component of the gauge parameters in (3.25) have M's and B's combined to the symmetry (r/s).

We will act on $|\Phi\rangle$ with these symmetry motions in a different way depending on whether or not $r\geq p$. If $r\geq p$, act δ on the nonredundant components of (3.25) to remove states of the form (3.19). The piece of (3.19) which remains has M's and

B's symmetrized according to $((r+1)/(s-1))$, so that the full set of operators displayed has the character

$$\{p\} \times [q] \times ((r+1)/(s-1)). \tag{3.28}$$

If $r < p$, decompose $\{p\} \times [q] \to ((p+1)/(q-1)) + (p/q)$. Act δ on the $((p+1)/(q-1))$ component to remove states of the form $K^p C^q M^r B^s|0\rangle$. Act d on the (p/q) component, to remove states of the form $K^{p-1}C^{q+1}M^{r-1}B^{s+1}|0\rangle$. The effect of this transformation is to reduce each group of states $K^p C^q M^r B^s|0\rangle$ with $r < p$ to the structure:

$$(p/q) \times ((r+1)/(s-1)) + (p/q) \times (r/s) = (p/q) \times \{r\} \times [s]. \tag{3.29}$$

Now let us examine the form of (3.4) that we have obtained. We have gauged away all states with $q = s = 0$, $r \geq p$. Thus, the states with $q = s = 0$, $r < p$ cannot appear in the first, diagonal term of (3.8). They can only appear in the off-diagonal terms involving $(d + \delta)$, using a d to convert it to the structure $K^{p-1}C^1 M^r|0\rangle$, which has a nonzero matrix element with states of the form $K^r M^{p-1} B^1|0\rangle$. As in our simple examples above, the terms with $q = s = 0$ act as Lagrange multipliers which eliminate terms with $s = 1$. After the gauge transformations described in the previous paragraph, both sets of states have been reduced to the multiplet $(p/0) \times \{r\}$, so all of the remaining states of the form (3.19) with $q = 0$, $s = 1$, and $r \geq p$ are eliminated. Now the states with $q = 0$, $s = 1$, $r < p$ appear only as Lagrange multipliers for the states with $q = 1$, $s = 1$, $r \geq p$. Comparing the representations into which these have been projected, we see that all of these states are eliminated. The pattern continues until all components of $|\Phi\rangle$ have either been removed or have acted as Lagrange multipliers to remove others.

In comparing this argument to that of ref. [12], the reader should note that here we find no nondynamical component fields in addition to the transverse fields. All unwanted components of $|\Phi\rangle$ disappear. It is never necessary to use the fact that the $|\eta\rangle$ components are purely auxiliary. This last feature is essential for generalizing this argument to the theories we will consider in sect. 5.

4. More bosonic strings

Now that we have discussed the simple example of the open bosonic string in a very thorough fashion, we are ready to construct the free field actions corresponding to the other known string theories. We will see that all of these actions can be written as expressions of the same structure as the open string action (3.4). In general, the operator Q will be replaced by another operator $\tilde{Q}$ which is not the BRST charge but which does satisfy $\tilde{Q}^2 = 0$ by virtue of the operator identities of sect. 2. The correspondence among these actions will be sufficiently strong that the proofs that each action leads to the correct covariant and light-cone gauge-fixed theory will be essentially identical to those given in sect. 3 for the open bosonic string.

We begin with the closed bosonic string. In this case, the Hilbert space of first-quantized string states factors into a product of two spaces, one containing left-moving and one containing right-moving string modes, each isomorphic to the open-string Hilbert space (excluding zero modes) and each possessing its own Virasoro algebra. We will denote operators acting on these spaces as unbarred and barred, respectively. The BRST charge on the full space is given by $(Q+\bar{Q})$, where Q has the structure of an open-string BRST charge (eq. (2.10)).

We will write the action for the closed string in terms of string forms satisfying the condition:

$$(K-\bar{K})|\phi) = 0. \tag{4.1}$$

Correspondingly, we will reduce the space of ghost zero modes from that spanned by the operators $b_0, c^0, \bar{b}_0, \bar{c}^0$ to that spanned by two formal operators satisfying

$$(\tilde{b}_0)^2 = (\tilde{c}^0)^2 = 0, \qquad \{\tilde{b}_0, \tilde{c}^0\} = 1. \tag{4.2}$$

In principle, a gauge-invariant action might enforce the condition (4.1). In the gauge-fixed action of ref. [1], the coefficients of the extra ghost zero modes are Lagrange multiplier fields which impose (4.1). In this paper, however, we content ourselves with imposing (4.1) from outside and use only the operators (4.2).

To construct an action, begin by defining $|\tilde{\omega})$ to be a state such that

$$\tilde{b}_0|\tilde{\omega}) = 0, \qquad (\tilde{\omega}|\tilde{c}^0|\tilde{\omega}) = 1, \tag{4.3}$$

following the properties of the ghost zero mode subspace of the open string, described at the beginning of sect. 3. Define

$$\tilde{Q} = \tilde{c}^0 K + d + \delta + \bar{d} + \bar{\delta} - 2\tilde{b}_0(\Downarrow + \bar{\Downarrow}). \tag{4.4}$$

This equation is symmetric between left- and right-movers and satisfies $\tilde{Q}^2 = 0$ on states satisfying (4.1). Finally, let

$$|\Phi) = \big(|\phi) + \tilde{c}^0|\eta)\big)|\tilde{\omega}), \tag{4.5}$$

which ϕ a 0-form and η a (-1)-form. The closed-string action then can be written as:

$$S = -\tfrac{1}{2}(\Phi|\tilde{Q}|\Phi). \tag{4.6}$$

This action has the gauge invariance $\delta_E|\Phi) = \tilde{Q}|E)$, which, in particular, includes the transformations $\delta_E|\phi) = (d+\delta)\varepsilon$ which correspond to the gauge symmetries of the closed string actions discussed in refs. [7] and [8]. The gauge-fixing can clearly be done exactly as for the open string, after enlarging the index space of N to run over all unbarred and barred integers.

The Neveu-Schwarz open string action can be constructed in a similar fashion. Here we may use precisely the ghost zero mode operators c^0 and b_0; the appropriate ghost vacuum again obeys (3.2). The index N runs over bosonic integer and fermionic half-integer values. However, we have already made clear in sect. 2 that our fundamental operator relations (2.14) and (2.15) are left unchanged by this modification. Thus, we may take over the formalism of the bosonic open string directly. Defining $|\Phi)$ as in (3.3), the free Neveu-Schwarz string action is

$$S = -\tfrac{1}{2}(\Phi|Q|\Phi), \tag{4.7}$$

where Q is now the BRST charge of the Neveu-Schwarz model. The gauge-fixing can again be done just as for the open string, the only change being that we must utilize, in the descent to the light-cone gauge, graded Young symmetrization of tensors.

It is important to note that the Neveu-Schwarz string forms, as we have described them so far, contain as expansion coefficients tensor fields of different statistics. For example, the general form $|\phi)$ has the expansion

$$\begin{aligned} |\phi) = \big\{ &\chi(x) - i\psi^{\mu}_{-1/2}A_\mu(x) - i\beta_{-1/2}\bar{c}(x) - i\gamma^{-1/2}c(x) \\ &- i\alpha^{\mu}_{-1}V_\mu(x) - \tfrac{1}{2}\psi^{\mu}_{-1/2}\psi^{\mu}_{-1/2}T_{\mu\nu}(x) + \cdots \big\}|0) . \end{aligned} \tag{4.8}$$

The field $A_\mu(x)$ is the vector gauge field of the Neveu-Schwarz theory; we would like to make this a Bose field. Then $|\phi)$ must be a Grassmann-valued form. This makes $c(x)$ and $\bar{c}(x)$ Grassmann fields, as is correct for the ghost and antighost of A_μ. But the integer-spin physical fields $\chi(x)$, $V_\mu(x)$, $T_{\mu\nu}(x)$ are also assigned Grassmann values. In general, the fields with the wrong statistics are those whose coefficients contain an even number of the fermionic creation operators ψ^{μ}_{-n}, β_{-n}, γ^{-n}. These fields may be removed by projecting all string forms onto their components with

$$G = (-1)^{\mathcal{N}_F + 1} = 1, \tag{4.9}$$

where $\mathcal{N}_F$ is the total number of fermion creation operators included in that state. This G is of course just the projection operator of Gliozzi, Scherk, and Olive [36] needed to define the supersymmetric string theory! The remarkable correlation between two-dimensional and space-time statistics first appeared in Siegel's papers [1] on the gauge-fixed bosonic string theory. The observation that the GSO projection must be made in order to preserve the correct statistics of fields in the Neveu-Schwarz-Ramond theory has also been made by LeClair [23].

The Neveu-Schwarz-Ramond theory contains three types of closed strings, those with Neveu-Schwarz boundary conditions for both left- and right-movers, those with Ramond boundary conditions for one set of modes, and those with Ramond boundary conditions for both left- and right-moving modes. The sectors of the first and third type lead to bosonic string states; however, it is convenient to treat the third type together with the fermionic strings. We are ready, though, to write the action for the first sector. In fact, this action is exactly (4.6), with the bosonic string operators d, δ, $\boldsymbol{K}$, $\Downarrow$ replaced by their Neveu-Schwarz counterparts and with a GSO projection applied independently to the left-and right-moving components of each form. This projection does not affect the proof of gauge invariance or the process of gauge-fixing, both of which proceed exactly as above. By replacing only the left-moving operators by Neveu-Schwarz operators, while keeping the right-moving operators those of the bosonic string, we find a free field action for the bosonic states of the heterotic string [37].

5. Fermionic strings

When we attempt to extend the analysis of the previous section to the Ramond string, we find two complications. The first is that the string action must be converted from a second-order form involving the kinetic operator $\boldsymbol{K} \sim (\partial^2 + \mathcal{M}^2)$ to a first-order lagrangian involving $\boldsymbol{F} \sim (i\not{\partial} + \mathcal{M})$. The second is that the superconformal ghosts β and γ now have zero modes which must be taken into account. We will see that these two problems can be dealt with in a simple way by using the same trick that we applied to the bosonic closed string, that of ignoring completely the space of the additional zero modes and defining an appropriate auxiliary charge $\tilde{Q}$, which satisfies $\tilde{Q}^2 = 0$ by virtue of the identities of sect. 2. In the case of the fermionic string, however it is known that the space of superconformal ghost zero modes is very large, including states with all possible values of the "Bose-sea" charge of ref. [31]. It is likely, then, that the Ramond theory we present here is simply a projection down from a much richer formal structure.

We can build the $\tilde{Q}$ of the Ramond string by using abstract operators $\tilde{b}_0$ and $\tilde{c}^0$ with the properties (4.2), together with a vacuum state $|\tilde{\omega}\rangle$ satisfying (4.3). Let us define

$$\tilde{Q} = \tilde{c}^0 \boldsymbol{F} + (-1)^{\mathcal{N}_{\mathrm{F}}} (d + \delta) - \tilde{b}_0 (\boldsymbol{F} \Downarrow + \Downarrow \boldsymbol{F}), \tag{5.1}$$

where $\mathcal{N}_{\mathrm{F}}$ is the fermion counting operator defined below (4.9). With this factor included, $\tilde{Q}$ is a Grassmann-valued differential operators. $\tilde{Q}^2 = 0$ follows from (2.24), (2.25).

The basic fields of the Ramond theory are string forms $|\lambda)$ carrying 10-dimensional fermion indices. The Dirac matrices which act on these indices are the zero modes of $\psi^{\mu}(\sigma)$:

$$\psi_0^{\mu}=\sqrt{\tfrac{1}{2}}\,i\Gamma^{\mu}. \tag{5.2}$$

Since application of Γ^{μ} flips chirality, it is appropriate to consider the fermion parity $(-1)^{\mathcal{N}_F+1}$ of a 10-dimensional spinor to be given by its chirality Γ^{11}. This assignment becomes explicit if the spinor representation of O(9, 1) is represented by states formed as products of ψ_0^{μ}'s acting on a vacuum state. We must also require the zero mode operators $\tilde{b}_0, \tilde{c}^0$ to have odd GSO parity. Thus

$$G=\left[\tilde{b}_0,\tilde{c}^0\right]\cdot\Gamma^{11}\cdot(-1)^{\hat{\mathcal{N}}_F}, \tag{5.3}$$

where $\hat{\mathcal{N}}_F$ refers to nonzero modes only. With this definition, $[G,\tilde{Q}]=0$. As in the Neveu-Schwarz case, a GSO projection must be made to insure that all component fields have the correct statistics. The Ramond string fields will have Grassmann character if they satisfy (4.9).

The properties of our basic operators under hermitian conjugation are the following: $\Gamma^{\mu\dagger}=-\Gamma^0\Gamma^{\mu}\Gamma^0$; $F^{\dagger}=-\Gamma^0F\Gamma^0$; $d^{\dagger}=\Gamma^0\delta\Gamma^0$; $\tilde{Q}^{\dagger}=-\Gamma^0\tilde{Q}\Gamma^0$. It is useful to define $(\bar{\lambda}|=(\lambda|\Gamma^0$; the Dirac matrix will compensate the Grassmann character of $\tilde{Q}$ in the Ramond theory action.

To complete our construction, we introduce

$$|\Lambda)=\left(|\lambda)+\tilde{c}^0|\xi)\right)|\tilde{\omega}), \tag{5.4}$$

with λ a 0-form of chirality $\Gamma^{11}=+1$ and ξ a $G=+1$ (-1)-form. Then the Ramond action can be written:

$$S=-\sqrt{\tfrac{1}{2}}\,i(\bar{\Lambda}|\tilde{Q}|\Lambda). \tag{5.5}$$

This action has the gauge invariance $\delta_E|\Lambda)=\tilde{Q}|E)$, where E is now a spinor-valued (-1)-form. By directly applying the steps leading from (3.11) to (3.14) this action can be gauge-fixed to the Feynman-Siegel gauge

$$S_{FS}=-\sqrt{\tfrac{1}{2}}\,i(\bar{\lambda}|F|\lambda), \tag{5.6}$$

where now $|\lambda)$ is a general $G=+1$ form in the space of nonzero modes. Using the arguments given for the descent to the light-cone gauge, (5.6) can also be gauge-fixed to the form

$$S=-\sqrt{\tfrac{1}{2}}\,i(\bar{\lambda}_T|F|\lambda_T), \tag{5.7}$$

where $|\lambda_T)$ is a transverse state. From this formula, one can then easily reach the light-cone gauge action by integrating out the components of $|\lambda_T)$ satisfying $\Gamma^+|\lambda_T)=0$.

Written in components (after performing the zero mode algebra), the action (5.7) takes the form

$$S = -\sqrt{\tfrac{1}{2}}\, i\{(\bar\lambda|F|\lambda) + (\bar\xi|F \Downarrow + \Downarrow F|\bar\xi) - (\bar\lambda|d+\delta|\xi) - (\bar\xi|d+\delta|\lambda)\}\,. \tag{5.8}$$

Note that in this expression, unlike the bosonic string actions, the auxiliary field $|\xi)$ has become dynamical. This turns out to have no effect on the gauge-fixing of the action by the methods of sect. 3; our arguments there did not make use of the explicit form of the term quadratic in the auxiliary field. However, it is interesting to note that this fact does play a role in more conventional, component-by-component covariant gauge-fixing. As a concrete example, let us consider the first excited mass level. The components of $|\lambda)$ at this level are the vector-spinor coefficients of the states $\alpha^{\mu}_{-1}|0)$ and $\psi^{\mu}_{-1}|0)$; these have opposite chirality and thus can form a massive spin-$\frac{3}{2}$ field. The conventional covariant quantization of this field would bring in 3 massive spin-$\frac{1}{2}$ ghosts, the third being the Nielsen-Kallosh ghost [38, 39]. The Feynman-Siegel gauge action for the Ramond string contains only two massive ghosts. But (5.8) also contains, at this level, two dynamical components of $|\xi)$, corresponding to the states $b_{-1}|0)$ and $\beta_{-1}|0)$; these have opposite chirality and combine to form a massive spin-$\frac{1}{2}$ fermion with normal statistics. This fermion precisely compensates the Nielsen-Kallosh ghost.

The fermionic closed string theories can be constructed along the same basic lines. The closed strings with Ramond left-movers and Neveu-Schwarz right-movers can be written for string forms satisfying the constraint (4.1). Define

$$\tilde Q = \tilde c^{0} F + (-1)^{A'}(d+\delta+\bar d+\bar\delta) - \tilde b_0(F\Downarrow + \Downarrow F + F\overline{\Downarrow} + \overline{\Downarrow} F)\,. \tag{5.9}$$

Then the appropriate action is given by (5.7), where now $|A)$ is a string form built on the product space of left- and right-movers, GSO projected independently in each subspace. The fermionic heterotic string action is constructed in the same way, using the bosonic string operators to build the right-moving subspace.

Finally, we turn to the closed superstring theory corresponding to Ramond boundary conditions for both left- and right-movers. In this sector, our simplistic treatment of the zero modes breaks down. We have been able to construct a quantum-mechanically complete theory, but this theory has two defects. First, it requires a constraint which, in a general frame, is dynamical. Second, it requires that part of the GSO projection be done after quantization rather than before. Despite these defects, we are encouraged to present this formulation because it does generalize the formal structure we have set out for the other strings, and because it continues our formulation of the other closed superstrings in a suggestive pattern.

The basic fields in this sector will be string fields carrying two Dirac indices and satisfying the condition

$$(F-\bar{F})|\beta)=0. \tag{5.10}$$

Since $F^2=K$, this condition implies (4.1). However, while (4.1) is a purely algebraic condition, this condition contains time derivatives of $|\omega)$. Choose

$$\tilde{Q}=\tilde{c}^0+(d+\delta+\bar{d}+\bar{\delta})-\tilde{b}_0(F(F\Downarrow+\Downarrow F)+\bar{F}(\bar{F}\overline{\Downarrow}+\overline{\Downarrow}\bar{F})), \tag{5.11}$$

where $\tilde{b}_0, \tilde{c}^0$ satisfy (4.2) and have even GSO parity with respect to both the left-moving and the right-moving GSO operators G and $\bar{G}$. Define

$$|B)=(|\beta)+\tilde{c}^0|\rho))|\tilde{\omega}), \tag{5.12}$$

$$(\bar{\bar{B}}|=(B|\Gamma^0\overline{\Gamma^0}. \tag{5.13}$$

Then the gauge-invariant action for this sector may be written

$$S=\tfrac{1}{4}(\bar{\bar{B}}|\tilde{Q}|B). \tag{5.14}$$

$|B)$ should be restricted to have the correct statistics: $G\cdot\bar{G}=1$. However, if we apply at this point the separate conditions $G=\bar{G}=1$, the chirality conditions do not match and (5.14) vanishes. Note that the closed superstring charges that we have defined (eqs. (4.4), (5.9), (5.11)) fall into a simple pattern.

Despite the fact that the constraint (5.10) is dynamical in a general frame, we can quantize this system straightforwardly by observing that, in the light-cone frame, (5.10) becomes a set of nondynamical relations. To make this point clear, we will discuss in a very explicit way the quantization of the massless level of this string. This level contains antisymmetric tensor fields, and so one would suspect that it should have a gauge invariance. In our formulation, however, there is no gauge invariance; the required reduction of degrees of freedom is implemented by the dynamical constraint. (The constraint (5.10) looks suggestively like a gauge-fixing condition for a Duffin-Kemmer lagrangian.) The light-cone quantization of the remaining levels will then follow by analogous manipulations, after fixing of the light-cone gauge for the oscillators in the manner of sect. 3.

Choose the following representation of the Γ^μ matrices:

$$\Gamma^+=\begin{pmatrix}0 & -i\sqrt{2}\\ 0 & 0\end{pmatrix},\qquad \tfrac{1}{2}\Gamma^-=\begin{pmatrix}0 & 0\\ i\sqrt{2} & 0\end{pmatrix},\qquad \tfrac{1}{2}\Gamma i=\begin{pmatrix}i\gamma_8^i & 0\\ 0 & -i\gamma_8^i\end{pmatrix}, \tag{5.15}$$

where γ_8^i are a set of real symmetric Dirac matrics of O(8). Express the massless level of $|\beta\rangle$ as $\boldsymbol{b} = \beta\Gamma^0$; $\boldsymbol{b}$ transforms under Lorentz transformations like a Dirac matrix. The action (5.14), restricted to this level, takes the form

$$S = \tfrac{1}{4}\mathrm{tr}\left[\Gamma^0 \boldsymbol{b}^{\mathrm{T}} \Gamma^0 \boldsymbol{b}\right]. \tag{5.16}$$

Decompose $\boldsymbol{b}$ in the basis of eq. (5.15), as follows:

$$\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_{\uparrow} & \boldsymbol{b}_{+} \\ \boldsymbol{b}_{-} & \boldsymbol{b}_{\downarrow} \end{pmatrix}. \tag{5.17}$$

On the massless level, $\boldsymbol{F} = -\sqrt{2}\,\partial\!\!\!/$, so (5.10) may be written:

$$p\!\!\!/\boldsymbol{b} = -\boldsymbol{b}p\!\!\!/\,. \tag{5.18}$$

Let $\hat{p} = p^i\gamma_8^i$. Then the full content of (5.18) is expressed by the relations:

$$\begin{aligned} \sqrt{2}\,p^{+}\boldsymbol{b}_{\downarrow} &= -\sqrt{2}\,p^{+}\boldsymbol{b}_{\uparrow} - [\hat{p}, \boldsymbol{b}_{-}], \\ \sqrt{2}\,p^{+}\boldsymbol{b}_{+} &= \sqrt{2}\,p^{-}\boldsymbol{b}_{-} - \{\hat{p}, \boldsymbol{b}_{\uparrow}\}. \end{aligned} \tag{5.19}$$

Use these equations to eliminate $\boldsymbol{b}_{\downarrow}$ and $\boldsymbol{b}_{+}$ in (5.16). Then $\boldsymbol{b}_{\uparrow}$ may be seen to be auxiliary and can be integrated out. This reduces (5.16) to the form

$$S = \tfrac{1}{2}\mathrm{tr}\left[\boldsymbol{b}^{\mathrm{T}}_{-}\,\frac{p^2}{2p^{+}}\,\boldsymbol{b}_{-}\right]. \tag{5.20}$$

If one now imposes the chirality conditions on $\boldsymbol{b}_{-}$ which follow from $G = \overline{G} = 1$, we are left with a theory of a double chiral O(8) bispinor. This is the correct physical content for the massless sector of the Ramond/Ramond closed string.

To generalize this discussion to higher mass levels of the string, we need two observations. First, the light-cone gauge-fixing procedure of sect. 3 still allows us to remove all states with longitudinal, time-like, and ghost excitations. Then the quantization procedure reduces to the treatment of the explicit Dirac indices. On higher levels, (5.10) equates two massive Dirac operators. The mass terms always couple two different field components which have opposite chirality but the same GSO parity. Thus, each massive Dirac operator may be written as the action on a pair of Dirac spinors of the operator

$$(i\partial\!\!\!/ + M\Gamma_M), \tag{5.21}$$

where Γ_M anticommutes with the Γ^{μ}. (If these massive equations follow by dimensional reduction, in the manner suggested by Siegel and Zwiebach [8]. $\Gamma_M = \Gamma^{11}$.) Then the analysis of the previous paragraph can be repeated for every massive level by treating the mass term in (5.21) as an extra component of the transverse momentum. This demonstration completes our formulation of free field theories, which can be explicitly gauge-fixed to the known physical spectra, for all of the known strings and superstrings.

We are grateful to Charles Thorn and Edward Witten for many enlightening remarks. Special thanks go to Luca Mezincescu for pointing out an error in a previous version of the Ramond/Ramond closed string theory. T.B. thanks Hermann Nicolai and Pierre Ramond for informative discussions of their work.

Note added in proof

After this paper was submitted, several preprints have come to our attention which describe work closely related to ours [41–49].

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