Nuclear Physics B262 (1985) 593–609

STRINGS IN BACKGROUND FIELDS

C.G. CALLAN

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08540, USA

D. FRIEDAN*

Enrico Fermi Institute and Dept. of Physics, University of Chicago, Chicago, IL 60637, USA

E.J. MARTINEC** and M.J. PERRY***

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Received 19 June 1985

We formulate the propagation of strings in background fields, including the effects of metric, antisymmetric tensor, and dilaton expectation values, as well as gauge field backgrounds in the case of heterotic strings. The inclusion of background fermion fields is sketched. The equations of motion of all these fields are shown to be the consequence of (super) conformal invariance of the string.

1. Introduction

Supersymmetric string theories are promising candidates for a unified theory of the physical world. They offer the possibility of explaining the low-energy spectrum of chiral matter fields interacting through gauge and gravitational forces [1] in a framework free of perturbative ultraviolet infinities [2]. The high-energy spectrum is almost uniquely determined by requiring a consistent interplay between quantum mechanics and the reparametrization invariance of the string. Nevertheless, our understanding of the string theory is inadequate: at the moment we only know how to calculate in perturbation theory, and procedures for multiloop calculations are only now being developed. Since local gauge and coordinate symmetries are not manifest in the Feynman rules, their origin and meaning in string theory are still mysterious. A full understanding will have to await the discovery of the gauge-invariant, second-quantized string action. In order to help bridge this gap in our

* Supported in part by DOE grant DE-FG02-84ER-45144 and the Alfred P. Sloan Foundation.
** Supported in part by DOE grant DE-AC02-76ER-03072.
*** Supported in part by NSF grant PHY-80-19754 and the Alfred P. Sloan Foundation.

knowledge it would be useful to formulate string perturbation theory in a general background field. We would then derive insight from the gauge-covariance properties of the background. There are two perspectives on this approach—one can either consider it as providing a background on which the full string evolves, or think of integrating out the string modes in a background in order to derive an effective action at low energies. Both viewpoints are valid provided the string theory is weakly coupled. In this work we outline the background field formulation, presenting details in a future communication [3].

The consistency of string dynamics determines the background field configuration: by background fields, we mean the spacetime manifold $\mathcal{M}$ together with the background matter fields in it. Here consistency requires that the quantum string theory maintains its classical conformal invariance [4], since conformal transformations are part of the two-dimensional reparametrization group [5]. We will show that conformal invariance in the presence of background fields determines the background-field equations of motion. We do the calculations in some detail for bosonic background fields: the metric $G_{\mu\nu}$, antisymmetric tensor gauge potential $B_{\mu\nu}$, and dilaton Φ, as well as the vector potential A^a_μ in the case of heterotic strings [6]. We will find that *all* of the equations of motion of these fields are consequences of conformal invariance in the string theory. We also sketch the inclusion of backgrounds for the fermionic partners of these fields; their equations of motion follow from superconformal invariance (i.e. world-sheet conformal supersymmetry) of the strong fields.

2. The bosonic string

The bosonic string propagating in a non-trivial background is described by a generalized nonlinear sigma model defined on a two-dimensional surface with intrinsic metric γ^{mn}. For consistency, the model should be conformally invariant, so that the longitudinal modes of the string decouple from physical amplitudes [7]. Therefore we can only admit renormalizable interactions and, barring the discovery of a consistent quantization of the Liouville theory [8], the sigma model must be invariant under Weyl rescalings ($\gamma^{mn} \to \Lambda(\xi)\gamma^{mn}$) of the two-dimensional metric.

The most general classical action satisfying these criteria is

$$S_{nlsm} = \frac{1}{2\pi\alpha'}\int d^2\xi \left\{ \tfrac{1}{2}\sqrt{\gamma}\,\gamma^{mn}G_{\mu\nu}(X)\,\partial_m X^\mu \partial_n X^\nu + \tfrac{1}{2}\varepsilon^{mn}B_{\mu\nu}(X)\,\partial_m X^\mu \partial_n X^\nu \right\}$$

where $X^\mu(\xi)$, $\mu = 1, \ldots, D$ maps the string into a D-dimensional spacetime $\mathcal{M}$ and the dimensional coupling constant α' turns out to be the inverse string tension. The "coupling constant" functions $G_{\mu\nu}$ and $B_{\mu\nu}$ can be identified as the background spacetime graviton and antisymmetric tensor fields in which the string is propagating. Since these fields are massless in the closed string theory, it is reasonable to allow them to have background expectation values. It is no accident that the background fields couple to the operators which are, in the string theory, the vertex operators for emitting precisely those fields.

The closed string has one other massless excitation, namely the dilaton, and we should be able to give it a background expectation value as well. How to do this is a bit mysterious since all the renormalizable and Weyl-invariant sigma model terms have been used up! Fradkin and Tseytlin [9] have suggested that one should add to S_{nlsm} the renormalizable, but not Weyl invariant, term

$$S_{\mathrm{dil}} = \frac{1}{4\pi} \int \mathrm{d}^2\xi \sqrt{\gamma}\, R^{(2)} \Phi(X),$$

where $R^{(2)}$ is the scalar curvature of the two-dimensional manifold and $\Phi(X)$ is the background dilaton field in the spacetime $\mathfrak{M}$. Since Weyl invariance is so crucial to the consistency of string theory, it seems mad to introduce terms which explicitly break it. Nevertheless, we shall show that, properly treated, S_{dil} does the right thing.

It is essential for string consistency that, as a quantum field theory, the sigma model be locally scale invariant. This is equivalent to the requirement that the two-dimensional world-sheet stress-energy tensor of the theory be traceless. In our model, local scale invariance is broken explicitly by S_{dil} and implicitly by anomalies. The general structure of the trace is

$$2\pi T_m^m = \beta^{\Phi} \sqrt{\gamma}\, R^{(2)} + \beta^G_{\mu\nu} \sqrt{\gamma}\, \gamma^{mn} \partial_m X^\mu \partial_n X^\nu + \beta^B_{\mu\nu} \varepsilon^{mn} \partial_m X^\mu \partial_n X^\nu,$$

where β^{Φ}, β^G and β^B are local functionals of the coupling functions $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ.

The quantities of interest can be calculated in perturbation theory by a variety of methods. We choose the conformal gauge in which $\gamma_{mn} = \mathrm{e}^{2\sigma}\delta_{mn}$ on the world sheet, and work in complex coordinates z, $\bar{z}$($\delta_{z\bar{z}} = \delta_{\bar{z}z} = 1$, $\delta_{zz} = \delta_{\bar{z}\bar{z}} = 0$). The dimensional continuation of the action is

$$S_d = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2 z\, \mathrm{e}^{(d-2)\sigma} \left\{ G_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + B_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + \alpha'(-4\partial\bar{\partial}\sigma)\Phi \right\}.$$

As usual, we perform all the index algebra in two dimensions and continue the volume element to d dimensions. The trace of the stress-energy tensor is the same thing as the variation of the effective action with respect to σ. Using (two-dimensional) background-field perturbation theory, we have calculated the σ-dependence of the effective action and have found the following results:

$$\frac{\beta^{\Phi}}{\alpha'} = \frac{1}{\alpha'} \frac{D-26}{48\pi^2} + \frac{1}{16\pi^2} \left\{ 4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \tfrac{1}{12} H^2 \right\} + \mathrm{O}(\alpha'),$$

$$\beta^G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4} H_\mu^{\lambda\sigma} H_{\nu\lambda\sigma} + 2\nabla_\mu \nabla_\nu \Phi + \mathrm{O}(\alpha'),$$

$$\beta^B_{\mu\nu} = \nabla_\lambda H^\lambda_{\mu\nu} - 2(\nabla_\lambda \Phi) H^\lambda_{\mu\nu} + \mathrm{O}(\alpha').$$

Here $H_{\mu\nu\lambda} = 3\nabla_{[\mu}B_{\nu\lambda]}$ is the antisymmetric tensor field strength, and $R_{\mu\nu}$ is the Ricci tensor. The leading term in β^{Φ} was discovered by Polyakov [8] (the 26 arises from the conformal gauge Faddeev-Popov determinant), the $R_{\mu\nu}$ term was discussed by Friedan and others [4], and the inclusion of the H-field torsion has been considered by Witten [10] and Curtright and Zachos [11]. It is important to note that since the coefficient of $R^{(2)}\Phi$ is smaller by a factor α' (the loop-expansion parameter) than the other couplings, its *classical* contribution is of the same order as the one-loop *quantum contributions* of the $G_{\mu\nu}$ and $B_{\mu\nu}$ couplings. This is because $R^{(2)}\Phi$ is scale non-invariant at the classical level while the other couplings only lose scale invariance at the quantum level.

The vanishing of β^G and β^B in $T_{z\bar{z}}$ is sufficient to guarantee the existence of a Virasoro algebra [12] generated by the trace-free parts, T_{zz} and $T_{\bar{z}\bar{z}}$, of the stress tensor:

$$[T_{zz}, T_{ww}] = \tfrac{1}{2}(T_{zz} + T_{ww})\delta'(z-w) + \tfrac{1}{12}c\delta'''(z-w),$$

$$[T_{\bar{z}\bar{z}}, T_{\bar{w}\bar{w}}] = \tfrac{1}{2}(T_{\bar{z}\bar{z}} + T_{\bar{w}\bar{w}})\delta'(z-w) + \tfrac{1}{12}c\delta'''(z-w),$$

$$[T_{zz}, T_{\bar{w}\bar{w}}] = 0.$$

The Schwinger term, c, in this algebra is identical to our function β^{Φ} and the theory is fully conformally invariant only when β^{Φ} vanishes as well as β^G and β^B. The condition that all three "beta functions" vanish amounts to a set of equations of motion for the background fields. The rest of this section will be devoted to showing that these equations are a sensible generalization of the classical equations for the graviton, dilaton and antisymmetric tensor fields.

First we must solve a little puzzle: When $\beta^G = \beta^B = 0$ we have a conformal algebra with Schwinger term $c = \beta^{\Phi}$. The algebra certainly implies that c is a c-number, but β^{Φ} appears to be operator-valued starting with its $O(\alpha'^0)$ term. However, the Bianchi identities applied to β^G and β^B show that

$$\begin{aligned} 0 &= \nabla^{\mu}\left(R_{\mu\nu} - \tfrac{1}{4}H^2_{\mu\nu} + 2\nabla_{\mu}\nabla_{\nu}\Phi\right) \\ &= \nabla_{\nu}\left(-2(\nabla\Phi)^2 + 2\nabla^2\Phi + \tfrac{1}{2}R - \tfrac{1}{24}H^2\right), \end{aligned}$$

which is to say that β^{Φ} is constant in $\mathcal{M}$, or a c-number! Once β^{Φ} is known to be a c-number, it can be set equal to zero. We expect that this result, namely that the Φ equation is a consequence of the B and G equations plus the Bianchi identities, holds to all orders in α'.

With this in mind, we can recast the string consistency equations in a form which makes clear that they are just conventional field equations in $\mathcal{M}$, provided that

$D=26$:

$$0=\beta^G_{\mu\nu}+8\pi^2 G_{\mu\nu}\frac{\beta^\Phi}{\alpha'}=\left(R_{\mu\nu}-\tfrac{1}{2}G_{\mu\nu}R\right)-T^{\text{matter}}_{\mu\nu},$$

$$0=8\pi^2\frac{\beta^\Phi}{\alpha'}+\tfrac{1}{2}G^{\mu\nu}\beta^G_{\mu\nu}=2(\nabla\Phi)^2-\nabla^2\Phi-\tfrac{1}{12}H^2,$$

$$0=\beta^B=\nabla_\lambda H^\lambda_{\mu\nu}-2(\nabla_\lambda\Phi)H^\lambda_{\mu\nu},$$

where

$$T^{\text{matter}}_{\mu\nu}=\tfrac{1}{4}\left[H^2_{\mu\nu}-\tfrac{1}{6}G_{\mu\nu}H^2\right]-2\nabla_\mu\nabla_\nu\Phi+2G_{\mu\nu}\nabla^2\Phi-2G_{\mu\nu}(\nabla\Phi)^2.$$

The first equation is recognizable as the Einstein equation and the second two are the matter equations of motion which guarantee conservation of the (spacetime) matter stress-energy tensor. These equations can be derived by varying the action [9,13]

$$\int \mathrm{d}^D x\sqrt{G}\,\mathrm{e}^{-2\Phi}\left\{R+4(\nabla\Phi)^2-\tfrac{1}{12}H^2\right\}$$

with respect to its three field variables. The peculiar Φ-dependence of this action exposes the fact that $\mathrm{e}^{2\Phi}$ is the string loop-expansion parameter: The constant mode of the dilaton field multiplies the Euler characteristic, $(1/4\pi)\int \mathrm{d}^2\xi\sqrt{\gamma}\,R^{(2)}$, of the world sheet in the sigma-model action. Thus an n-loop contribution to the functional integral, which comes from a world sheet with n handles, is proportional to $\mathrm{e}^{-2(1-n)\Phi}$.

After a conformal rescaling of G by

$$G_{\mu\nu}\to \mathrm{e}^{4\Phi/(D-2)}G_{\mu\nu},$$

(where D is the dimension of $\mathcal{M}$) the equations take on the more familiar form

$$0=\left(R_{\mu\nu}-\tfrac{1}{2}G_{\mu\nu}R\right)-T^{\text{matter}}_{\mu\nu},$$

$$0=\nabla^2\Phi+\tfrac{1}{6}\mathrm{e}^{-8\Phi/(D-2)}H^2,$$

$$0=\nabla_\lambda\left(\mathrm{e}^{-8\Phi/(D-2)}H^\lambda_{\mu\nu}\right),$$

where

$$T^{\text{matter}}_{\mu\nu}=\tfrac{1}{4}\left[H^2_{\mu\nu}-\tfrac{1}{6}G_{\mu\nu}H^2\right]\mathrm{e}^{-8\Phi/(D-2)}+\frac{4}{D-2}\left[(\nabla_\mu\Phi)(\nabla_\nu\Phi)-\tfrac{1}{2}G_{\mu\nu}(\nabla\Phi)^2\right].$$

These equations can be derived by varying an action [9]

$$\int \mathrm{d}^D x \sqrt{G} \left\{ R - \frac{4}{D-2} (\nabla\Phi)^2 - \tfrac{1}{12} \mathrm{e}^{-8\Phi/(D-2)} H^2 \right\},$$

which is essentially identical to the bosonic part of the Chapline-Manton [14] supergravity action. In particular, this makes clear that the constant mode of the dilaton field has the effect of rescaling the gravitational coupling constant.

So, to one-loop order, the string consistency equations are nothing but the classical field equations for the massless modes of the string, including the elusive dilaton field! Acceptable backgrounds for string physics are then just solutions of the classical field equations. At two-loop order, we will obtain contributions to the equations of motion and effective action which are of order $\alpha' R^2$ etc. and which represent short-distance corrections to the classical equations. We will have more to say about higher-loop corrections later on. For the moment let us note that the effective action, calculated to all orders, simply summarizes the expansion in powers of energy of all the no-string-loop S-matrix elements involving the massless string excitations. This puts some non-trivial restrictions (having to do with the fact that the string S-matrix has no negative metric or other unphysical contributions [15]) on the terms which can actually appear in the action. Whether a no-string-loop computation is accurate enough for any given application depends on the size of the string loop-coupling constant, e^{Φ}. Obviously the dynamics of the dilaton field will play a crucial role in determining the physics of the full string theory.

Conformal symmetry provides another means of establishing the link between the string and the background, considering the sigma model couplings, G, B and Φ, to be a background for the full string theory. The effective action is just the generating functional for S-matrix elements. Consider for example the metric coupling: an infinitesimal variation of the background metric produces the correlation functions

$$A_N = \langle V_1 \ldots V_N \rangle,$$

$$V_i = \int \delta G_{\mu\nu}^{(i)} \partial X^\mu \bar{\partial} X^\nu,$$

which are just the curved space generalizations of string graviton amplitudes. The Virasoro algebra generated by T_{zz} and $T_{\bar{z}\bar{z}}$ serve to distinguish physical from unphysical graviton vertex operators just as in flat space (recall that in flat space, $\delta G_{\mu\nu}^{(i)} = \zeta_{\mu\nu}^{(i)}(k)\mathrm{e}^{ik\cdot X}$ with physical vertices satisfying $k^2 = 0 = k^\mu_{\zeta\mu\nu}$) [7]. Thus, although one might have thought that there were two gravitons in the theory, namely fluctuations of the background and fluctuations of the string, they really are one and the same. One can therefore think of the metric coupling as a "string condensate". Of course similar remarks apply to the antisymmetric tensor and the dilaton. It is in fact an amusing exercise to find the dilaton operator in a non-trivial background, since it mixes with the graviton coupling even in flat space.

At the string theory tree level (i.e. spherical world-sheet topology), conformal invariance also implies that the sigma model is a solution of the *string* equations of motion. The global conformal group SL(2, C) is a symmetry of the correlation functions even before integrating over the location of the vertices on the world sheet. This means that the one-point function

$$\left\langle \int \delta G_{\mu\nu} \partial X^{\mu} \bar{\partial} X^{\nu} \right\rangle$$

must be scale invariant since scale transformations are part of the global conformal group. However only the identity operator is scale invariant, so the one-point function must vanish—i.e. the equations of motion are satisfied (this argument was first presented by Candelas, Horowitz, Strominger, and Witten in ref. [4]). This need not be true on surfaces with handles, where dilations are not a symmetry; in other words, there can be quantum corrections to the equations of motion. One must find some other argument, perhaps using spacetime supersymmetry (see below), that the tree-level vacuum remains stable.

Since our discussion has implicitly been restricted to closed strings, we have had nothing to say about gauge bosons. If we consider open strings as well, then the massless excitations of the string will include gauge bosons and couplings to background gauge fields can be introduced by adding to the sigma-model action line integrals about the string boundary of the type

$$\oint A_{\mu} \frac{\mathrm{d} X^{\mu}}{\mathrm{d} s} \mathrm{d} s .$$

There are severe technical obstacles to calculating the effect of such couplings on the trace of the world-sheet stress-energy tensor. They have to do with the fact that the contribution of A_{μ} to the metric equation of motion comes from vanishingly small boundaries which are, in effect, joining an open onto a closed string topology. Since it is difficult to regulate the associated singularities, we prefer to restrict our attention to closed strings and eventually introduce gauge fields through the heterotic string mechanism.

3. Supersymmetry and higher-loop corrections

We now consider sigma models with $N=1$ world-sheet supersymmetry. These theories describe type II superstrings [16] in curved backgrounds, since a projection onto even world-sheet fermion number renders them spacetime supersymmetric [17, 18]. As we shall see, in these models the higher-loop corrections to the beta functions are particularly simple. Since the use of complex superspace coordinates facilitates the analysis, we introduce the superpartners $\theta, \bar{\theta}$ of the commuting complex coordinates z, $\bar{z}$. In this basis, the only non-vanishing γ-matrix elements are $\gamma^{z}_{\bar{\theta}\theta} = \gamma^{\bar{z}}_{\bar{\theta}\bar{\theta}} = 1$; the superspace covariant derivatives simplify to

$$D = \partial_\theta + \theta\partial_z \qquad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$$

and the supersymmetry algebra is just $D^2 = \partial_z$ and $\bar{D}^2 = \partial_{\bar{z}}$.

The supersymmetric version of the dimensionally continued bosonic sigma model is obtained through the correspondence rules

$$X^\mu \to X^\mu(z, \bar{z}, \theta, \bar{\theta}),$$

$$\sigma(z, \bar{z}) \to \sigma(z, \bar{z}, \theta, \bar{\theta}),$$

$$\partial_z, \partial_{\bar{z}} \to D, \bar{D},$$

$$d^2z \to d^2z\, d^2\theta .$$

The result is

$$S_{N=1} = \frac{1}{4\pi\alpha'} \int d^2z\, d^2\theta\, e^{(d-2)\sigma}\{ G_{\mu\nu} DX^\mu \bar{D} X^\nu + B_{\mu\nu} DX^\mu \bar{D} X^\nu + \alpha'(-4D\bar{D}\sigma)\Phi \} .$$

Perturbative background field calculations may be done with this action by performing a normal coordinate expansion and then using superspace Feynman rules [19].

The requirement to be met in order to build a string theory is that the effective action be sigma-independent, or that the trace of the super stress-energy tensor vanish. By the supersymmetric generalization of the arguments of the previous section, we know that the general form of this trace will be

$$2\pi T_{\theta\bar{\theta}} = \beta^\Phi(D\bar{D}\sigma) + \beta^G_{\mu\nu} DX^\mu \bar{D} X^\nu + \beta^B_{\mu\nu} DX^\mu \bar{D} X^\nu ,$$

where the distinction between β^G and β^B is that one is symmetric and the other is antisymmetric in the indices. The various beta functions can be computed, as in the previous section, as perturbative expansions in powers of α'. Setting the beta functions equal to zero produces the equations of motion for the background fields.

There are some simplifications in the beta functions which occur on passing to the supersymmetric model and which will be important to us. In the first place, the presence of fermions does not affect the one-loop beta functions. Further, in the sigma model with only $G_{\mu\nu}$ couplings, it is known that the two-loop [19], and strongly believed that all higher-loop [20] corrections to β^G vanish, so that a one-loop solution to the equations of motion is probably a solution to all orders! Our calculations indicate that the two-loop corrections to β^G and β^B continue to vanish even when the dilaton and the antisymmetric tensor fields are included* [21]. However, we know by examining a special case, namely the supersymmetric nonlinear sigma model on a group manifold [22], that β^{Φ} continues to receive corrections to all orders when $H_{\mu\nu\lambda} \neq 0$. It is tempting to conjecture that the strongest possible result is true, namely that in the supersymmetric theory β^{G} and β^{B} continue to vanish to all orders, with the corrections to β^{Φ} summing up so that D_{crit} of the resulting conformal field theory is ten.

The vanishing of higher-loop corrections to $T_{z\bar{z}}$ in the case where $B_{\mu\nu} = \Phi' = 0$ is reminiscent of the Adler-Bardeen theorem [23]. Indeed, when the background admits $N = 2$ world-sheet supersymmetry, $T_{z\bar{z}}$ is in the same supermultiplet as the divergence of the axial current which rotates the two supersymmetries and a version of the Adler-Bardeen theorem might well hold [24].** Unfortunately, the dilaton couples only to the $N = 1$ world-sheet Einstein supermultiplet and there is no way of implementing the above scheme unless $\Phi' = 0$. Consequently the methods which have been used to prove the no-renormalization theorem for the purely metric sigma models [20] will probably have to be rethought carefully.

At this point it is appropriate to raise the question of string loop corrections to the equations of motion. This is a subtle matter which we do not fully understand. On the one hand, the conformal anomalies of the sigma model are short-distance effects and ought, barring some disaster, to be the same for all world-sheet topologies. On the other hand, in the supersymmetric theory, this would mean that the classical equations of motion are perturbatively exact, that scale invariance is an exact symmetry and that the mass scale of the vacuum is not determined in perturbation theory. This may be the case in backgrounds admitting spacetime supersymmetry, where it might be possible to prove a nonrenormalization theorem. The alternative to this peculiar situation is that higher world-sheet topologies modify the consistency conditions in some way, yet this would imply different consistency conditions on different topologies. It is difficult to see how these different demands on the background are reconciled. We expect that detailed studies of how our background field calculations actually work on higher world-sheet topologies will shed light on these perplexing questions.

4. The heterotic string

The heterotic string [6] is a promising starting point for deriving realistic models of particle physics. It has an $N = \frac{1}{2}$ Majorana-Weyl world-sheet supersymmetry: only the left-moving bosons have fermionic partners and the supersymmetry of opposite chirality is absent. Right-moving fermions do exist, but they are used to form a chiral $E_8 \otimes E_8$ or SO(32) current algebra. The massless bosons of the string

* The vanishing of the two-loop beta function with general $H \neq 0$ has been shown in ref. [21].

** This idea has occurred independently to the second set of authors in [20].

theory include the gauge bosons of this current algebra as well as the usual graviton, dilaton and antisymmetric tensor field.

The sigma-model action corresponding to background values of all these massless fields can be written in chiral superspace as

$$S_{N=1/2} = \frac{2}{4\pi\alpha'} \int \mathrm{d}^2 z\, \mathrm{d}\theta\, \mathrm{e}^{(d-2)\sigma} \{ G_{\mu\nu} DX^\mu \bar{\partial} X^\nu + B_{\mu\nu} DX^\mu \bar{\partial} X^\nu + \alpha'(-4D\bar{\partial}\sigma)\Phi$$

$$+ A_{\mu a}(X) DX^\mu j^a + \theta \psi^i \partial \psi^i \} .$$

Here $X^\mu(z, \bar{z}, \theta)$ and $\sigma(z, \bar{z}, \theta)$, are the string coordinates and conformal factor, $A^a_\mu(x)$ is the background spacetime gauge field and ψ^i are the right-handed fermions on which the algebra of currents $j^a = \psi^i T^a_{ij} \psi^j$ is realized, with T^a the generators of the gauge algebra. Because the gauge fields couple over the entire string rather than on the boundary, we shall see that, contrary to the open string case, they are no harder to deal with than the graviton and antisymmetric tensor field. We emphasize that the use of the fermionic representation of the gauge current algebra is inessential: the calculations may be phrased in an invariant way, but the use of the fermionic formulation is convenient and conceptually familiar. Similar remarks apply to the spacetime fermions. There seems to be some confusion as to whether the world-sheet supersymmetric sigma model always describes the superstring, since the variables of the superstring are spacetime spinors, not vectors like ψ^μ. However the world-sheet supersymmetric sigma-model can have a subsector which is spacetime supersymmetric [16, 17]. Moreover, the model may be phrased entirely in terms of SO(9, 1) current algebra [28] —the representation in terms of ψ^μ is just a convenience (indeed, other representations are more suited to the treatment of spacetime fermion backgrounds [18, 3]).

Once again, we obtain equations of motion by imposing the condition that $T_{z\bar{z}}$, the trace of the energy-momentum tensor, vanish. The one-loop results for β^Φ, β^G and β^G are unchanged. In addition, there is a now a beta function for A^a_μ which turns out to have the value (ignoring contributions from the dilaton and antisymmetric tensor fields)

$$\beta^A = \nabla^\nu F^a_{\mu\nu} + \mathrm{O}(\alpha') ,$$

where $F_{\mu\nu} = T^a F^a_{\mu\nu}$ is the Yang-Mills field strength appropriate to the gauge group of the model. Thus β^A gives the proper Yang-Mills field equation to this order.

The general two-loop beta-function calculation is quite complicated because of the many possible mixings between different background fields. We shall focus on some aspects which are of particular importance to phenomenological applications: the appearance of gauge-variant Chern-Simons terms in the antisymmetric tensor field strength, and the appearance of a gauge field-strength-squared term as well as curvature-squared terms in the Einstein equations.

To compute effects of the external gauge field, we need the normal coordinate expansion of the corresponding term in the sigma-model lagrangian. To second order in the quantum field ξ, the result is

$$4\pi\alpha' \mathcal{L}_A = A^a_\mu D X^\mu j_a + F^a_{\mu\nu}(DX^\mu)\xi^\nu j_a - \tfrac{1}{2}F^a_{\mu\nu}\xi^\mu(D\xi^\nu)\, j_a - \xi^\nu A^a_\nu \tilde{D} j_a$$

In the last term, which is obtained by integration by parts, the spacetime derivative has been completed to a gauge-covariant derivative $\tilde{D}_z$ in order to complete all the $F_{\mu\nu}$ to Yang-Mills field strengths. It is tempting to use the classical equation of motion for the fermion fields to argue that this term vanishes. However, the current is a chiral gauge current and has an anomaly which can be expressed in terms of the projection of the background gauge field onto the string:

$$\tilde{D}_z j^a_{\bar{z}} = -\frac{1}{8\pi} F^a_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu .$$

Taking this anomaly into account, and carrying out a further normal coordinate expansion, we can rewrite the last term of the previous equation as

$$\mathcal{L}'_A = \frac{1}{4\pi} F^a_{\mu\nu} A^a_\lambda \xi^\lambda (\partial\xi^\mu \bar{\partial} X^\nu - \bar{\partial}\xi^\mu \partial X^\nu).$$

We will use these vertices to calculate the gauge field contribution to the various beta functions.

As a first application, consider the two-loop diagram of fig. 1, where the vertices come from the term $F^a_{\mu\nu} D X^\mu \xi^\nu j_a$, the dashed line is the ξ propagator and the solid lines are the fermi propagators. A modest amount of calculation shows that this graph contributes $-\frac{1}{2}\alpha' \operatorname{tr}(F_{\mu\lambda}F_\nu^{\ \lambda})$ to β^G. This is the contribution of the gauge-field energy-momentum tensor to our version of the Einstein equation. No other graphs give a similar contribution. It is perhaps disturbing that the gauge field enters at two-loop order, while the similar contribution from the antisymmetric tensor field enters at one-loop order. However, the dimensionalities of the fields and the fact that the only dimensional parameter in the game, α', is also the loop-counting parameter make this inevitable.

The same sort of analysis can be carried out for the vertices involving a coupling to the background connection and curvature. Once again it is important to note that

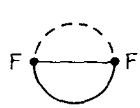


Fig. 1. The two-loop graph which generates the Yang-Mills field contribution to the energy-momentum tensor.

there is an anomaly, this time in the generating current of local Lorentz transformations. The two-loop contribution to β^G will contain field strength-squared terms which are in general rather unpleasant to work out. When the background is Ricci-flat, however, these contributions simplify to

$$\tfrac{1}{2}\alpha' R_{\mu}^{\lambda\rho\sigma} R_{\nu\lambda\rho\sigma} .$$

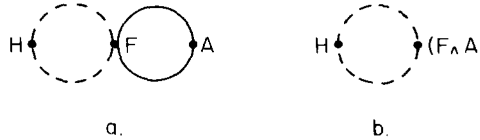
The key point is that $N=\frac{1}{2}$ supersymmetry, unlike $N=1$, is not powerful enough to remove higher-loop contributions to the beta-function. We shall also see that the sign of this contribution is quite significant.

The term in the sigma-model lagrangian containing the background gauge field is formally invariant to gauge transformations, indicating that the physics we extract from the theory will not depend on the gauge chosen for the background field. Unfortunately, the proof of gauge invariance makes use of covariant conservation of the anomalous gauge current, j_z^a. The equations of motion for the background fields are therefore not actually gauge-invariant! If all goes well, the gauge non-invariance will be of a very special kind, organizing itself into the Chern-Simons completion of the antisymmetric tensor field strength familiar from ten-dimensional supergravity theories.

A complete proof of this is beyond the scope of our investigations, but we can see how it begins to work in two-loop order. Consider the graphs shown in fig. 2, where the dashed lines are the ξ propagators and the solid lines are the fermi propagators. The left-most vertex is the term

$$(\partial X^{\mu}\bar{\partial}\xi^{\nu} - \bar{\partial}X^{\mu}\partial\xi^{\nu})\xi^{\lambda}H_{\mu\nu\lambda}$$

arising from the normal coordinate expansion of the antisymmetric tensor term in the sigma-model lagrangian. The other vertices come from the expansion of the gauge-field interaction term and, in particular, the right-most vertex in fig. 2b is the term arising from the anomaly. Since the anomaly is a one-loop effect, the graph in fig. 2b is really a two-loop contribution despite having only one explicit loop. The



a. b.

Fig. 2. Two-loop graphs which generate the Chern-Simons piece of the antisymmetric tensor field strength.

contribution of these graphs to β^G turns out to be

$$\frac{\alpha'}{16\pi} H_{\mu\nu\lambda} \operatorname{tr}(F^{\mu\nu} \wedge A^{\lambda}).$$

There is another set of graphs which contributes the same sort of term, with the gauge Chern-Simons term, $\operatorname{tr}(F \wedge A)$ replaced by the Lorentz Chern-Simons term.

The net effect of all this is clearly to replace the antisymmetric tensor field strength, $H_{\mu\nu\lambda}$, which appeared in β^G at one-loop order, by the well-known Chern-Simons completion [25]

$$H \to \tilde{H} = H + \tfrac{1}{8}\alpha' \operatorname{tr}(F \wedge A) - \tfrac{1}{8}\alpha' \operatorname{tr}(R \wedge \omega).$$

A careful study of three-loop graphs would no doubt show the squares of the Chern-Simons terms appearing with the proper coefficients. Let us assume that all the gauge variance arising from the anomaly is absorbed in this redefinition of H—on the basis of our calculations this is plausible but by no means proven. Then, by the well-known properties of the Chern-Simons terms, gauge invariance of the full theory can be recovered by making the antisymmetric tensor field gauge-*variant* in the manner of Green-Schwarz and Chapline-Manton [25, 14]. In other words, the gauge non-invariance of the anomalous sigma model [26] is not a disaster (providing our assumption about what happens at higher orders is right) but essential to reproducing the correct ten-dimensional supergravity physics. It is important to note that the gauge-variance discussed here has nothing to do with the chiral anomalies of ten-dimensional field theory which, being one-string-loop effects, are invisible at the level we are working.

What we know so far about the two-loop beta function of the heterotic string sigma-model can be summarized as follows:

$$\beta^G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4}\tilde{H}^2_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi + \tfrac{1}{2}\alpha'\left(R_{\mu\lambda\sigma\tau}R_\nu^{\ \lambda\sigma\tau} - \operatorname{tr}F^2_{\mu\nu}\right) + \cdots$$

Although all the details are not worked out, we expect that, after a suitable conformal rescaling of the metric, the equations of motion can be derived from an action of the form

$$\int \mathrm{d}^D x \sqrt{G} \left\{ R - \frac{1}{D-2}(\nabla\Phi)^2 - \tfrac{1}{12}\mathrm{e}^{-8\Phi/(D-2)}H^2 \right.$$
$$\left. + \tfrac{1}{2}\alpha' \mathrm{e}^{-4\Phi/(D-2)}\left(R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} - \operatorname{tr}F^2\right)\right\}.$$

Since there are quite a few terms we have not computed (in particular, terms second-order in the curvature which vanish for Ricci-flat spaces), we are in no position to make general statements about background fields which might satisfy the equations to all orders. The exception is the case where the gauge and gravitational fields conspire so as to make a theory which is actually $N=1$ supersymmetric and gauge-invariant. For example, in the case where $B = \Phi' = R_{\mu\nu} = 0$, we may embed the spin connection in a subgroup of the gauge group via $\omega_\mu^{\alpha\beta} = A_\mu^a T_a^{\alpha\beta}$ so that, in effect, the curvature and gauge field strength are the same. Then the Lorentz and gauge Chern-Simons terms cancel against each other and the R^2 and F^2 terms in β^G do also. Thus the β^G, and presumably all other equations of motion, are satisfied to the order we have calculated. This is no accident as this choice of background has reinstated both $N=1$ supersymmetry ($N=2$ if $\mathcal{M}$ is Kähler) and gauge invariance in that the right-handed gauge fermions and the left-handed Lorentz vector fermions now undergo exactly the same interactions for the subset of the theory coupling to the background. The component term $F_{\mu\nu}^a \psi^\mu \psi^\nu T_{ij}^a \psi^i \psi^j$ reproduces the four-fermion interaction of the $N=1$ sigma model. Presumably, because the theory is now fully supersymmetric, such a background field satisfies the equations of motion to all orders and is a satisfactory vacuum solution.

5. Spacetime fermions

Finally, we note that background spacetime fermions may be incorporated into the sigma model using the recently discovered covariant fermion vertex V_F^α [18]. We have seen that bosonic background fields (except the dilaton) couple to the corresponding bosonic vertex operators of the string theory. This suggests that we add to the sigma-model action similarly constructed background fermion terms. For instance, in the heterotic sigma model,

$$S_{\text{fermion}} = \int \lambda_{a\alpha}(X) j^a V_F^\alpha + \xi_{\mu\alpha}(X) \bar{\partial} X^\mu V_F^\alpha$$

describes the effect of background gluino fields, $\lambda_{a\alpha}$, and gravitino fields, $\xi_{\mu\alpha}$. This is an unconventional sigma model, involving anticommuting coupling parameters, but it is renormalizable by power counting and should be perfectly well-defined. An extension of this kind is absolutely essential if we are to deal with issues of spacetime, as opposed to world-sheet, supersymmetry. Such couplings are also necessary in the type II theories to incorporate, e.g., all the massless antisymmetric tensor backgrounds.

In order to define a satisfactory string theory, the sigma model must be superconformal-invariant. This is by no means guaranteed because the fermion couplings are not written in terms of superfields [18], unlike the bosonic couplings which can be written in manifestly supersymmetric form. In the bosonic case, the vanishing of $T_{z\bar{z}}$ automatically ensured the vanishing of $T_{\bar{z}\theta}$ since both operators are in the same supermultiplet. This is no longer true when spacetime fermions are present. Instead, one finds that $T_{z\bar{z}} = 0$ is the generalized d'Alembertian equation of the fermion background while $T_{\bar{z}\theta} = 0$ is the generalized Dirac equation which contains more information. This may seem somewhat mysterious, but it is related to the fact that the Ramond operator [27] (the string generalization of the Dirac operator) is also the generator of conformal supersymmetry on a cylindrical closed string world sheet. We have verified that for vanishing background fields the requirement of superconformal invariance in $O(\alpha')$ forces the gluino and gravitino fields to satisfy the Dirac and Rarita-Schwinger equations, respectively. The extension to general background fields is tedious but straightforward. Details will be reported elsewhere [3].

6. Conclusions

Our work has shown that all the massless particles of the string can be incorporated in a background field approach. The equations of motion are consequences of superconformal invariance: conformal invariance supplies the dynamics for spacetime bosons, local world-sheet supersymmetry gives the spacetime fermion dynamics. All the interesting equations of physics are subsumed in this invariance principle. Viewed as an expansion of the low energy effective lagrangian in powers of α', the loop expansion of the sigma model is capable of reproducing all the terms of spacetime field theory—Chern-Simons terms, four fermion interactions, dilaton couplings, higher derivative interactions, etc.—in a very simple way. Our results also lend strong support to the idea that backgrounds which admit $N = 1$ world-sheet supersymmetry are exact solutions of the full (tree-level) string equations. We have not dealt with the effects of higher-loop *string* corrections, and anticipate problems consistently maintaining conformal invariance of the sigma model on world sheets of differing topologies. Perhaps these problems are mitigated by the intervention of spacetime supersymmetry. We hope that our investigations shed light on the important issues of string theory, particularly the problems of finding phenomenologically interesting compactifications to four dimensions and discovering the gauge-invariant, second-quantized string action.

C.C., M.P. and E.J.M. wish to thank C. Lovelace for inspiring us to embark upon our investigations and E. Witten for valuable remarks. C.C. and M.P. would like to thank D.J. Gross for early discussions about the dilaton mystery and also for informing us of the existence of the work of ref. [9]. D.F. acknowledges an early conversation with D.Z. Freedman about the renormalization of c and extensive discussions with S. Shenker. E.J.M. would like to thank A. Abouelsaood, R. Rohm, and S. Shenker for helpful discussions and U. of Chicago for its hospitality during part of this work.

References

[1] P. Candelas, G. Horowitz, A. Strominger, and E. Witten, ITP preprint NSF-ITP-84-170, Nucl. Phys. B, 258 (1985) 46
E. Witten, Princeton preprints (1985);
M. Dine, V. Kaplunovsky, M. Mangano, C. Nappi, and N. Seiberg, Princeton preprint (1985);
J. Breit, B. Ovrut, and G. Segré, Penn preprint UPR-0279T
[2] M. Green and J. Schwarz, Nucl. Phys. B198 (1982) 441
[3] C. Callan, D. Friedan, E. Martinec, and M. Perry, in preparation
[4] This idea has a long history; it was conjectured by, among others, J. Scherk and J. Schwarz, Nucl. Phys. B81 (1974) 118;
T. Yoneya, Prog. Theor. Phys. 56 (1976) 1310;
The beta function and scale invariance in generalized sigma models was investigated by D. Friedan, Phys. Rev. Lett. 45 (1980) 1057, UC Berkeley thesis; Ann. of Phys., to be published
L. Alvarez-Gaumé and D.Z. Freedman, Phys. Lett. 94B (1980) 171;
L. Alvarez-Gaumé, D.Z. Freedman, and S. Mukhi, Ann. Phys. 134 (1981) 85;
more recent work includes C. Lovelace, Phys. Lett. 135B (1984) 75 and unpublished;
P. Candelas et al., ref. [1];
D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. 54 (1985) 620;
S. Jain, R. Shankar, and S. Wadia, Tata Inst. preprint TIFR/TH/85-3; See also ref. [9]
D. Friedan, Z. Qiu, and S. Shenker, in Proc. of APS meeting, Santa Fe (1984), ed. T. Goldman and M. Nieto (World Scientific, 1985)
[5] D. Friedan, in Proc. of the 1982 Les Houches summer school, ed. J. Zuber and R. Stora (North-Holland, 1984);
A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333;
D. Friedan, Z. Qiu, and S. Shenker, Vertex operators in mathematics and physics, ed., J. Lepowski (Springer, 1984)
[6] D. Gross, J. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. 54 (1985) 502, Princeton preprint (1985) to appear
[7] P. Goddard and C. Thorn, Phys. Lett. 40B (1972) 235;
J.L. Gervais and B. Sakita, Nucl. Phys. B34 (1971) 477
[8] A.M. Polyakov, Phys. Lett. 103B (1981) 207 211
[9] E.S. Fradkin and A.A. Tseytlin, Lebedev Inst. preprint N261 (1984)
[10] E. Witten, Comm. Math. Phys. 92 (1984) 455
[11] T. Curtright and C. Zachos, Phys. Rev. Lett. 53 (1984) 1799;
E. Braaten, T. Curtright, and C. Zachos, Florida preprint UFTP 85-01
[12] M.A. Virasoro, Phys. Rev. D1 (1970) 2933; see also ref. [5].
[13] M. Dine and N. Seiberg, Couplings and scales in superstring models, IAS preprint (May 1985)
[14] G. Chapline and N. Manton, Phys. Lett. 120B (1983) 105
[15] B. Zweibach, U.C. Berkeley preprint (1985)
[16] M. Green and J. Schwarz, Nucl. Phys. B181 (1981) 502;
Candelas et al. in ref. [4].
[17] F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122 (1977) 253;
M. Green and J. Schwarz in ref. [16];
D. Friedan, Z. Qiu, and S. Shenker, U. of Chicago preprint EFI 84-35
[18] D. Friedan, E. Martinec, and S. Shenker, U. of Chicago preprint EFI 85-32.
[19] L. Alvarez-Gaumé, D.Z. Freedman, and S. Mukhi in ref. [4]
[20] C. Hull, IAS preprints (1985);
L. Alvarez-Gaumé and P. Ginsparg, Harvard preprint HUTP 85/A016 (1985)
[21] B. Fridling and A. van de Ven, to appear
[22] R. Rohm, Princeton preprint (1984);
D. Friedan and S. Shenker, unpublished;
P. DiVecchia, V.G. Knizhnik, J.L. Petersen, and P. Rossi, Nucl. Phys. B253 (1985) 701; see also refs. [10,11]

[23] S.L. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517
[24] L. Alvarez-Gaumé, private communication
[25] M. Green and J. Schwarz, Phys. Lett. 149B (1984) 117
[26] G. Moore and P. Nelson, Phys. Rev. Lett. 53 (1984) 1519
[27] P. Ramond, Phys. Rev. D3 (1971) 2493
[28] J. Cohn, D. Friedan, Z. Qiu, and S. Shenker, to appear