Nuclear Physics B235[FS11] (1984) 395–416

SUPERSYMMETRIC DERIVATION OF THE ATIYAH-SINGER INDEX AND THE CHIRAL ANOMALY

D. FRIEDAN†

Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, IL 60637, USA

P. WINDEY‡

CERN, Geneva, Switzerland

Received 17 January 1984

We present an elementary derivation of the Atiyah-Singer formula for the index of the Dirac operator. This index is the space-time integral of the trace of the chiral anomaly. We calculate the full chiral anomaly using the supersymmetric path integral for a spinning particle moving through space-time.

1. Introduction

The Atiyah-Singer index theorem [1] equates the index of a partial differential operator on a manifold to a topological invariant, relating local information about the solutions of a partial differential equation to global properties of the manifold. All of the indices of operators which arise in geometry (and physics) are specializations of the index of the Dirac operator [1].

The index of the Dirac operator plays a role in quantum field theory because it can be interpreted as the space-time integral of the U(1) anomaly [2] for a fermion interacting with classical gauge and gravitational fields. The U(1) anomaly is a chiral anomaly: the anomalous divergence of a classically conserved chiral current in the presence of gauge and gravitational fields. The Adler-Bardeen anomaly, which can block the gauging of chiral fermions, is also a chiral anomaly [3].

In this paper we derive the general chiral anomaly, and thus the index formula, for arbitrary gauge and gravitational fields. The calculation is based on a proper time representation of the Dirac propagator in terms of the supersymmetric quantum mechanics of a spinning particle moving through space-time.

† Research supported in part by the Sloan Foundation and by the US Department of Energy contract DE-AC02-81ER-10957.
‡ Work done while at CEN Saclay.

The supersymmetry generator Q of the spinning particle is the Dirac operator $\not{D}$ in space-time. The wave functions which $Q=\not{D}$ act on are the spinor fields on space-time. The hamiltonian is $H=Q^{\dagger}Q$. The particle moves along paths in space-time which are parametrized by proper time τ and its anticommuting partner $\hat{\tau}$. The quantum-mechanical amplitude for the spinning particle to move between the space-time points x and y in "euclidean" super-time τ, $\hat{\tau}$ is

$$K_{\tau,\hat{\tau}}(x,y)=\mathrm{e}^{-\tau H-\hat{\tau}Q}\delta(x,y). \tag{1.1}$$

We will call this amplitude the super heat kernel. The Dirac propagator is the integral of the super heat kernel over the super proper time:

$$S_{\mathrm{F}}(x,y)=\int_0^{\infty}\mathrm{d}\tau\int\mathrm{d}\hat{\tau}\,K_{\tau,\hat{\tau}}(x,y), \tag{1.2}$$

exactly as in the proper time representation for the bosonic propagator. The anomaly will be determined by the standard calculation from the short distance properties of the Dirac propagator.

The strategy is as follows. In sect. 2 we point out that the Dirac operator $\not{D}$, acting on the Hilbert space of spinor fields, can be thought of abstractly as the generator Q of a quantum supersymmetry. The analytic index of $\not{D}$ then becomes the index $\mathrm{Tr}(-1)^F$ of Witten [4] for the supersymmetric quantum system with $Q=\not{D}$. In sect. 3 we write down the chiral anomaly, recall the relation between the U(1) anomaly and the index of the Dirac operator, and express the Adler-Bardeen anomaly in terms of the chiral anomaly.

In sect. 4 we realize the abstract supersymmetric system $Q=\not{D}$ as a spinning particle in space-time. The super heat kernel $K_{\tau,\hat{\tau}}(x,y)$ is given by a supersymmetric path integral whose action contains the background gauge and gravitational fields. In sect. 5, we calculate the short time expansion of $K_{\tau,0}(x,x)$ which gives the formula for the anomaly and the index.

In sect. 6 we sketch how the general index formula is specialized to give the Euler number and the Hirzebruch signature [5]. In sect. 7, we consider the situation in which the global topology of space-time does not allow a consistent definition of spinors [6], making it impossible to construct the Hilbert space for the spinning particle. We point out that this obstruction is equivalent to an inconsistency in the supersymmetric path integral, precisely analogous to Witten's SU(2) anomaly in four-dimensional gauge theory [7].

This work was reported by one of us (PW) at the XXIII Cracow School of Theoretical Physics in June, 19983 [8]. Similar methods have been used independently by Alvarez-Gaumé [9] to calculate the index and by Alvarez-Gaumé and Witten [10] in the calculation of the gravitational anomaly.

2. Supersymmetry and the index

In this section we will review some general facts about Witten's index [4] for a supersymmetric theory and then draw the parallel between it and the analytic index of the Dirac operator.

Consider a theory with supersymmetry charge Q, hamiltonian H and fermion parity $(-1)^F$:

$$H = Q^\dagger Q,$$

$$Q^\dagger = -Q,$$

$$Q(-1)^F + (-1)^F Q = 0. \tag{2.1}$$

It follows immediately from these defining properties that: (1) the energy is always zero or positive; (2) the zero energy states are exactly the supersymmetric states, i.e. the states annihilated by Q; and (3) each energy eigenvalue $E_i \neq 0$ is associated with a pair of eigenstates, one $|i, B\rangle$ bosonic and one $|i, F\rangle$ fermionic, which satisfy:

$$(-1)^F|i, B\rangle = |i, B\rangle, \qquad Q|i, B\rangle = \sqrt{E_i}|i, F\rangle,$$

$$(-1)^F|i, F\rangle = -|i, F\rangle, \qquad Q|i, F\rangle = -\sqrt{E_i}|i, B\rangle. \tag{2.2}$$

Witten pointed out that the number of bosonic zero energy states minus the number of fermionic zero energy states,

$$I = \operatorname*{Tr}_{H=0} (-1)^F, \tag{2.3}$$

is topologically invariant. States can only reach or leave zero energy in pairs, one fermionic state for each bosonic one, making no change in the index, because states at any nonzero energy, however small, are always paired. There must be at least $|I|$ zero energy states to produce the index, so a nonzero index implies that the supersymmetry cannot be broken. A useful formula for the index is

$$I = \mathrm{Tr}(-1)^F \mathrm{e}^{-\tau H}, \tag{2.4}$$

which holds for any $\tau > 0$, since only zero energy states contribute to the trace, the contribution of nonzero energy states cancelling because of the pairing (2.2).

The analytic index of the Dirac operator is developed in exactly parallel fashion. Let $\not{D}$ be the covariant Dirac operator on some even dimensional, oriented, compact manifold without boundary. On even dimensional oriented manifolds, the spinor fields can be divided into spaces of positive and negative chirality, eigenspaces of the chirality operator γ_5. The Dirac operator anticommutes with γ_5, so we can write the two operators in block form:

$$\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \not{D} = \begin{bmatrix} 0 & \not{D}_- \\ \not{D}_+ & 0 \end{bmatrix}, \tag{2.5}$$

with $\not{D}_- = -D_+^\dagger$. Note that $\not{D}_+$ takes positive chirality spinor fields to negative chirality fields, and $\not{D}_-$ does the reverse.

The analytic index [1] is defined to be

$$\mathrm{Ind}(\not{D}) = \dim\ker \not{D}_+ - \dim\ker \not{D}_+^\dagger\,, \tag{2.6}$$

i.e. the number of positive chirality spinor fields annihilated by $\not{D}$ minus the number of negative chirality spinor fields annihilated by $\not{D}$. The null space of $\not{D}_+$ is the same as the null space of the laplacian $\not{D}_+^\dagger \not{D}_+$, and the null space of $\not{D}_-$ is the same as that of $\not{D}_+ \not{D}_+^\dagger$. All nonzero eigenvalues of the two laplacians are exactly the same, because if $\not{D}_+^\dagger \not{D}_+ u = \lambda u$ then $\not{D}_+ \not{D}_+^\dagger(\not{D}_+ u) = \lambda(\not{D}_+ u)$. This allows us to write

$$\mathrm{Ind}(\not{D}) = \mathrm{Tr}\,\gamma_5 e^{-\tau \not{D}^\dagger \not{D}}\,. \tag{2.7}$$

The contributions from nonzero eigenvalues of $\not{D}^\dagger \not{D}$ cancel in the trace.

When we make the following indentifications, we see that the two indices are exactly equivalent:

$$\begin{aligned} Q &= \not{D}\,, \\ (-1)^F &= \gamma_5\,, \\ H &= \not{D}^\dagger \not{D}\,. \end{aligned} \tag{2.8}$$

3. The chiral anomaly and the index

In this section we write the definition of the chiral anomaly for fermions in background gauge and gravitational fields. to make it clear that this is the general chiral anomaly we point out that it includes as special cases the U(1) anomaly [2] and also the Adler-Bardeen anomaly which can block the gauging of chiral fermions [3]. We recall that the index of the Dirac operator is the space-time integral of the U(1) anomaly. Then we explain the representation of the chiral anomaly as a matrix element in supersymmetric quantum mechanics.

We will be considering a compact n-dimensional manifold M, which could be the (compactified) space-time of a euclidean quantum field theory. This space-time is equipped with a gravitational field $g_{\mu\nu}(x)$ and a gauge field $A_{\mu b}^{a}(x)$. The gauge field is a matrix in a definite representation space. The representation is arbitrary. In particular, it might be reducible. For example, the gauge field might be the SU(3) color field, with representation space the fundamental representation of color tensored with a flavor space. Another example would have the representation space being the space-time tangent vectors themselves, so that the Dirac operator would be acting on fermions of spin higher than $\frac{1}{2}$. In that case A_μ would be the metric connection for $g_{\mu\nu}$ in the appropriate representation. We will call the representation space of the gauge field the internal space, and indices $a, b, \ldots$ will be called internal indices, even though the last example shows that this language is not always appropriate.

We assume that M is even dimensional and orientable, which means that we can choose a completely antisymmetric tensor $\varepsilon_{\mu_1 \ldots \mu_n}$, compatible with the metric: $n! = \varepsilon_{\mu_1 \ldots \mu_n} g^{\mu_1 \nu_1} \ldots g^{\mu_n \nu_n} \varepsilon_{\nu_1 \ldots \nu_n}$. We assume there is a spin structure, so that at each point there are Dirac matrices $\gamma_\mu = \gamma_\mu^\dagger$, $\nabla_\nu \gamma_\mu = 0$ which satisfy the standard anti-commutation relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} . \tag{3.1}$$

Using the orientation of M, we define

$$\gamma_5 = i^{-n/2} \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \gamma^{\mu_1} \ldots \gamma^{\mu_n} , \tag{3.2}$$

which, since M is even dimensional, satisfies

$$\gamma_5 = \gamma_5^\dagger , \qquad \gamma_5^2 = 1 . \tag{3.3}$$

The Dirac operator on spinors with internal indices is

$$\begin{aligned} \not{D} &= \gamma^\mu (\partial_\mu + \omega_\mu + A_\mu) , \\ \omega_\mu &= \tfrac{1}{8} (\partial_\sigma g_{\mu\rho} + e^r_\sigma \partial_\mu e^r_\rho) [\gamma^\rho , \gamma^\sigma] , \end{aligned} \tag{3.4}$$

where the vielbein e^r_μ satisfies $e^r_\mu e^r_\nu = g_{\mu\nu}$.

Now suppose $\Psi^{\alpha a}(x)$ is a free massless fermion field, $\overline{\Psi}_{\beta b}$ the conjugate field. The chiral current is

$$j^{\mu a}_{5 b}(x) = \overline{\Psi}_b \gamma_5 \gamma^\mu \Psi^a (x) . \tag{3.5}$$

We suppress indices when the meaning is clear. The covariant divergence of the chiral current is

$$\begin{aligned} \nabla_\mu j^\mu_5 &= \partial_\mu j^\mu_5 (x) + [A_\mu , j^\mu_5] \\ &= \overline{\Psi} \gamma_5 \not{D} \Psi - \not{D} \overline{\Psi} \gamma_5 \Psi . \end{aligned} \tag{3.6}$$

By the equation of motion $\not{D}\Psi = 0$, only the vacuum expectation value contributes to the divergence. This is the chiral anomaly. Using

$$S_{\mathrm{F}}(x, y) = \langle \overline{\Psi}(x)\Psi(y)\rangle,$$

$$\not{D}S_{\mathrm{F}}(x, y) = \delta(x, y), \tag{3.7}$$

we have

$$\nabla_\mu j_5^\mu(x) = 2 \operatorname*{tr}_{\text{spin}} [\gamma_5 \not{D} S_{\mathrm{F}}(x, y)]|_{x=y}$$

$$= 2 \operatorname*{tr}_{\text{spin}} [\gamma_5 \delta(x, x)] . \tag{3.8}$$

This needs to be regulated, but as the regularization is removed there will be a finite limit independent of the method of regularization, as long as locality and gauge invariance are respected. One standard choice of regularization is the heat kernel method

$$\nabla_\mu j_5^\mu(x) = \lim_{\tau\to 0} 2 \operatorname*{tr}_{\text{spin}} \left[\gamma_5 e^{-\tau \not{D}^\dagger \not{D}} \delta(x, y)\right]\Big|_{x=y} . \tag{3.9}$$

The relation between the U(1) anomaly and the index is now straightforward. The trace $j_{5a}^{\mu a}$ is the U(1) current. The right hand side of (3.9), traced over internal indices, is the U(1) anomaly. Integrating over space-time and comparing with formula (2.7) for the index, we get

$$I = \int \mathrm{d}^n x \, \tfrac{1}{2} \partial_\mu j_{5a}^{\mu a}(x) . \tag{3.10}$$

The relation between the general chiral anomaly and the gauging of chiral fermions is more delicate. Suppose we have some collection of fermion fields to be coupled to a gauge field, consisting of positive chirality fields $\Psi_+^a(x)$ and negative chirality fields $\Psi_-^a(x)$. The current to be coupled to a gauge field is

$$j_k^\mu = j_{k+}^\mu + j_{k-}^\mu ,$$

$$j_{k+}^\mu = \overline{\Psi}_+ \gamma^\mu \lambda_k^+ \Psi_+$$

$$j_{k-}^\mu = \overline{\Psi}_- \gamma^\mu \lambda_k^- \Psi_- , \tag{3.11}$$

where the λ_k^+, λ_k^- are the generators of the gauge group in the positive and negative chirality sectors respectively. It is essential, for a gauge invariant coupling, that j_k^μ be covariantly conserved.

In order to find out if the covariant divergence of j_k^μ is in fact zero we employ a device. We double the fermion content, adding for each fermion field a mirror image of the opposite chirality, but the same gauge representation. The result is a left-right symmetric theory of Dirac fermions Ψ^a. In terms of the Dirac field Ψ^a, the original currents are

$$j_{k+}^\mu = \overline{\Psi}\tfrac{1}{2}(1+\gamma_5)\gamma^\mu\lambda_k^+\Psi, \qquad j_{k-}^\mu = \overline{\Psi}\tfrac{1}{2}(1-\gamma_5)\gamma^\mu\lambda_k^-\Psi. \tag{3.12}$$

We can calculate the divergence of the original current in terms of the divergence of the chiral current:

$$\nabla_\mu j_k^\mu = \mathrm{tr}\left[(\lambda_k^+ - \lambda_k^-)\tfrac{1}{2}\nabla_\mu j_5^\mu\right], \tag{3.13}$$

since the vector currents in (3.12) are conserved by gauge invariance of the left-right symmetric theory. Therefore we can tell from the divergence of the chiral current when a chiral anomaly prevents the gauging of fermions.

4. The supersymmetric spinning particle

In this section we will build path integral representations for the Dirac propagator $S_F(x, y)$, the chiral anomaly and the index. We will proceed in the following steps:

(i) review the supersymmetric quantum mechanical theory of the spinning particle in flat space [11];

(ii) go to a superfield formulation in which it is easy to implement general covariance and gauge invariance;

(iii) express the proper time representation of the Dirac propagator in terms of the path integral for the spinning particle.

The dynamical variables which describe the spinning particle are the position operator $x^\mu(t)$ and its super-partner $\psi^\mu(t)$. The lagrangian is

$$L = \tfrac{1}{4}\dot{x}^\mu\dot{x}^\mu + \tfrac{1}{4}\psi^\mu\dot{\psi}^\mu. \tag{4.1}$$

The coefficient $\frac{1}{4}$ is introduced to simplify subsequent formulae. The lagrangian (4.1) is invariant under the supersymmetry transformation

$$\delta x^\mu = \varepsilon\psi^\mu, \qquad \delta\psi^\mu = -\varepsilon\dot{x}^\mu. \tag{4.2}$$

The canonical (anti-)commutation relations are

$$\begin{aligned} \left[ip_\mu, x^\nu\right] &= \delta_\mu^\nu, \\ [\psi^\mu, \psi^\nu]_+ &= 2\delta^{\mu\nu}. \end{aligned} \tag{4.3}$$

In "euclidean" time,

$$ip_\mu = -\tfrac{1}{2}\dot{x}_\mu . \tag{4.4}$$

The supersymmetry generator is

$$Q = \psi^\mu(ip_\mu) . \tag{4.5}$$

These (anti-)commutation relations are represented on wave functions which are space-time spinor fields $u^\alpha(x)$. ψ^μ is represented by γ^μ and ip_μ by ∂_μ. The generator is $Q = \gamma^\mu\partial_\mu = \not{D}$.

Our object is to construct the super heat kernel $e^{-\tau H - \hat{\tau} Q}$. The path integral with lagrangian (4.1) will give the ordinary heat kernel $e^{-\tau H}$. But Q commutes with H, so we need only add to the action a term proportional to $\hat{\tau}Q$:

$$S = \int_0^\tau \mathrm{d}t\left(\tfrac{1}{4}\dot{x}^\mu\dot{x}^\mu + \tfrac{1}{4}\psi^\mu\dot{\psi}^\mu - \frac{1}{2}\frac{\hat{\tau}}{\tau}\psi^\mu\dot{x}^\mu\right). \tag{4.6}$$

Next, we want to introduce internal degrees of freedom, but for the moment without coupling to a gauge field. The wave functions are to acquire internal indices without any effect on the dynamics. Let $\bar{\eta}_b$ and η^a be canonically conjugate fermion operators:

$$[\bar{\eta}_b, \eta^a]_+ = \delta^a_b . \tag{4.7}$$

If the lagrangian is modified by addition of the term $\bar{\eta}\dot{\eta}$, then the new variables are constant in time. The modified lagrangian remains supersymmetric. The internal operators are represented in a Fock space: $\bar{\eta}$ as creation operator and η as destruction operator. The wave functions can be written as functions $u^\alpha(x, \bar{\eta})$ of the creation operators:

$$u^\alpha(x,\bar{\eta}) = u^\alpha_0(x) + u^{\alpha a}_1(x)\bar{\eta}_a + u^{\alpha ab}_2\bar{\eta}_a\bar{\eta}_b + \cdots . \tag{4.8}$$

These wave functions contain antisymmetric tensors of all ranks in the internal space, but we will eventually want amplitudes between states with only a single internal index. A systematic way to isolate these states would be to use the number operator $N_\eta = \bar{\eta}\eta$. The k-eigenspace of N_η contains the spinor fields with k internal indices. N_η commutes with Q and H, so we can introduce a term $(i\alpha/\tau)N_\eta$ in the lagrangian to give a modified heat kernel

$$K = e^{-\tau H - \hat{\tau} Q - i\alpha N_\eta} . \tag{4.9}$$

The index would be a generating function

$$I(\alpha) = \sum_k I_k e^{-i\alpha k} , \tag{4.10}$$

where I_k is the index of the Dirac operator acting on antisymmetric k-tensors in the internal space. We will actually proceed without this complication and directly compute I_1, which was the index we were originally interested in, and only quote the result for $I(\alpha)$.

It will be easiest to introduce background fields into the superfield version of this theory. The super-field associated with x^μ is

$$X^\mu = x^\mu + \theta\psi^\mu . \tag{4.11}$$

The variables η and $\bar{\eta}$ are contained in superfields

$$N^a = \eta^a + \theta\phi^a , \qquad \bar{N}_b = \bar{\eta}_b + \theta\phi_b , \tag{4.12}$$

where ϕ and $\bar{\phi}$ will play the role of auxiliary fields. Note that N and $\bar{N}$ are fermionic superfields. The superlagrangian is

$$L = \tfrac{1}{4}\left(1 + 2\theta\frac{\hat{\tau}}{\tau}\right) DX^\mu \partial_t X^\mu - \bar{N}DN , \tag{4.13}$$

where $D = \theta\partial_t - \partial_\theta$, $D^2 = -\partial_t$. The action is

$$S = \int_0^\tau \mathrm{d}t \int \mathrm{d}\theta L . \tag{4.14}$$

The component lagrangian is reproduced by using the equations of motion $\phi = \bar{\phi} = 0$ to eliminate the auxiliary fields. The supersymmetry transformation of a field is its commutator with $\varepsilon Q = \varepsilon(\theta\partial_t + \partial_\theta)$.

If we rescale $t \to \tau t$, $\theta \to \tau^{1/2}\theta$, $\mathrm{d}\theta \to \tau^{-1/2}\mathrm{d}\theta$, $D \to \tau^{-1/2}D$, $\psi \to \tau^{-1/2}\psi$, $\phi \to \tau^{-1/2}\phi$, $\bar{\phi} \to \tau^{-1/2}\bar{\phi}$, and define

$$g = \tau^{1/2} - \theta\hat{\tau} , \tag{4.15}$$

the action takes the compact form

$$S = -\int_0^1 \mathrm{d}t \int \mathrm{d}\theta \left[\frac{1}{4g^2}(DX)D(DX) + \bar{N}DN \right] . \tag{4.16}$$

The parameter g plays the role of a metric in one-dimensional superspace. The action (4.16) can be thought of as a gauge-fixed version of a more general reparametrization invariant action.

To take account of a background metric and gauge field we contract space-time indices with $g_{\mu\nu}(X) = g_{\mu\nu}(x) + \theta\psi^\sigma\partial_\sigma g_{\mu\nu}(x)$ and we replace DN with the corresponding covariant superderivative

$$D_A N = \left(D + DX^\mu A_\mu(X)\right) N , \tag{4.17}$$

giving

$$S=\int_0^1 \mathrm{d}t \int \mathrm{d}\theta \left[\frac{1}{4g^2} g_{\mu\nu}(X) DX^\mu \partial_t X^\nu - \overline{N} D_{\mathrm{A}} N\right]. \tag{4.18}$$

DX and $\partial_t X$ are already covariant.

We now have a manifestly covariant, manifestly supersymmetric lagrangian whose supersymmetry generator in the flat space limit is the Dirac operator. We want to show that in the covariant theory the supersymmetry generator is the covariant Dirac operator

$$\not{D} = \not{D}_0 + \gamma^\mu \bar{\eta} A_\mu \eta, \tag{4.19}$$

where $\not{D}_0$ is the Dirac operator on ordinary spinors. Recall that $\bar{\eta}$ and η are respectively creation and destruction operators acting on the wave functions (4.8).

Taking any point as origin, we can choose coordinates and a basis in the internal space, so that $A_\mu(0)=0$ and $\partial_\sigma g_{\mu\nu}(0)=0$. Then, from the flat space result, both $\not{D}$ and Q are equal to $\gamma^\mu \partial_\mu$ at the origin of coordinates. Since the origin of coordinates is arbitrary, it follows that $Q=\not{D}$ everywhere. The effectiveness of this argument depends on the fact that Q and $\not{D}$ are first order in ip_μ and ∂_μ respectively.

To see in a more concrete way that $Q=\not{D}$ for the covariant theory, we can re-express the action in component notation and eliminate the auxiliary fields using their equations of motion. The result is

$$S=\int_0^\tau \mathrm{d}t\, \tfrac{1}{4} g_{\mu\nu}(x)\left(\dot{x}^\mu \dot{x}^\nu + \psi^\mu \nabla_t \psi^\nu - 2\frac{\hat{\tau}}{\tau}\dot{x}^\mu \psi^\nu\right) + \bar{\eta}\, \nabla_t \eta - \tfrac{1}{2}\bar{\eta} F_{\mu\nu} \psi^\mu \psi^\nu \eta, \tag{4.20}$$

where

$$\nabla_t \psi^\mu = \partial_t \psi^\mu + \dot{x}^\sigma \Gamma^\mu_{\rho\sigma} \psi^\rho,$$

$$\nabla_t \eta = \left(\partial_t + \dot{x}^\mu A_\mu\right)\eta,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right],$$

$$\Gamma^\mu_{\rho\sigma} = \tfrac{1}{2} g^{\mu\nu}\left(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}\right). \tag{4.21}$$

The verification that $Q=\not{D}$ is now a straightforward application of the canonical formalism. The only subtle point is that $e^r_\mu \psi^\mu$ is the correct canonical variable. The term involving the vielbein in eq. (3.4) for the covariant Dirac operator comes from rewriting the kinetic term for ψ: $g_{\mu\nu}\psi^\mu \dot{\psi}^\nu = (e^r_\mu \psi^\mu)\partial_t(e^r_\nu \psi^\nu) + \psi^\mu \psi^\nu \dot{x}^\rho (e^r_\nu \partial_\rho e^r_\mu)$.

The super heat kernel $K(x,y) = \mathrm{e}^{-\tau H - \hat{\tau} Q}\delta(x,y)$ satisfies the differential equation

$$(\hat{\tau}\partial_\tau - \partial_{\hat{\tau}})K(x,y) = \not{D}K(x,y). \tag{4.22}$$

The Heisenberg operator $X(\tau, \hat{\tau})$ is

$$X(\tau, \hat{\tau}) = x(\tau) + \hat{\tau}\psi(\tau)$$
$$= e^{\tau H + \hat{\tau} Q} X(0,0) e^{-\tau H - \hat{\tau} Q}, \tag{4.23}$$

where $X(0,0) = x$. The supertime dependence of N and $\overline{N}$ is expressed similarly. The diagonal of the super heat kernel is given by

$$\langle X, N(\tau, \hat{\tau}) | X, \overline{N}(0,0) \rangle = \langle X, N | e^{-\tau H - \hat{\tau} Q} | X, \overline{N} \rangle, \tag{4.24}$$

where the $|X, \overline{N}\rangle$ form a Dirac basis of states. In terms of the functional integral,

$$\langle X_0, N_0(\tau, \hat{\tau}) | X_0, \overline{N}_0 \rangle = \int D[X] D[\overline{N}] D[N] e^{-S}$$
$$\times \delta(X_{t=0} - X_0) \delta(\overline{N}_{t=0} - \overline{N}_0) \delta(N_{t=0} - N_0), \tag{4.25}$$

where the path integral is over fields periodic in time, and the delta function of a superfield is

$$\delta(X) = \delta^n(x) \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \psi^{\mu_1} \ldots \psi^{\mu_n}. \tag{4.26}$$

The expression (4.25) will be interpreted explicitly in the next section as the diagonal part of the super heat kernel.

The Dirac states $|X\rangle$ deserve some comment. They reflect Cartan's treatment of spinors [12]. Each point x^μ in space-time is associated with a Dirac state $|X(0,0)\rangle = |x\rangle$. This represents a particular spinor at the space-time point x^μ. Certain linear combinations of the operators ψ^μ (which are just the γ matrices) will annihilate the state $|x\rangle$. These operators we write ξ^r. They must all anti-commute since the square of a γ matrix is a pure number, and $(\xi^r)^2|x\rangle = 0 = (\xi^r + \xi^s)^2|x\rangle$. There must be $n/2$ operators, ξ^r, and we can always make a change of basis such that

$$\xi^r = \tfrac{1}{2}(\psi^r + i\psi^{r+n/2}). \tag{4.27}$$

The complementary collection of operators

$$\bar{\xi}_r = \tfrac{1}{2}(\psi_r - i\psi_{r+n/2}) \tag{4.28}$$

also anti-commute among themselves. $\bar{\xi}$ and ξ have canonical anti-commutation relations

$$[\bar{\xi}_r, \xi^s]_+ = \delta_r^s. \tag{4.29}$$

The term $\frac{1}{4}\psi\dot{\psi}$ in the lagrangian can be rewritten as $\bar{\xi}\dot{\xi}$. It has the same form as the term $\bar{\eta}\dot{\eta}$ for the internal degrees of freedom.

The space of spinors at x, which has dimension $2^{n/2}$, is the Fock space generated by the $\frac{1}{2}n$ $\bar{\xi}$-operators acting as creation operators on the state $|x\rangle$. A spinor wave function $u^{\alpha}(x)$ can be regarded as a function $u(x, \bar{\xi})$ of the creation operators $\bar{\xi}$. The wave function $u^{\alpha}(x)$ corresponds to the state $\int d^n x\, d^{n/2}\bar{\xi}\, u(x, \bar{\xi})|x\rangle$.

The states $|X\rangle$ form a Dirac basis in the sense that $\langle X|Y\rangle = \delta(X-Y)$. The projections $|X_0\rangle\langle X_0|$, which we use to extract the diagonal part of the super heat kernel, satisfy

$$|X_0\rangle\langle X_0||Y_0\rangle\langle Y_0| = \delta(X_0 - Y_0)|X_0\rangle\langle X_0|. \tag{4.30}$$

As operators they are written

$$|X_0\rangle\langle X_0| = \delta(X - X_0). \tag{4.31}$$

5. Calculation of the heat kernel

The most straightforward way to calculate the anomaly and the index from the short time behaviour of the heat kernel would be to go back to the component field formulation, expand around the minima of the action (4.20), and keep only the leading term in the limit $\tau \to 0$. The calculation becomes a simple evaluation of three determinants. This is described in detail in [8].

We prefer to adopt a different and more formal route using superfields, because we believe it sheds some light on how supersymmetry singles out only the relevant terms in the full expansion of the super heat kernel.

We want to evaluate the diagonal matrix element (4.25). Let's first see what the structure of the answer is in absence of internal degrees of freedom. We have to evaluate

$$\begin{aligned} K(\tau, \hat{\tau}, X_0) &= \langle X_0|e^{-\tau H - \hat{\tau} Q}|X_0\rangle \\ &= \int D[X] e^{-S} \delta(X_{t=0} - X_0), \end{aligned} \tag{5.1}$$

where the path integral is over periodic paths $X_{t=1} = X_{t=0}$. We perform the path integral by writing $X(t) = X_0 + \delta X(t)$ and integrating over the fluctuations which obey $\delta X(0) = \delta X(1) = 0$. The result will be expanded in the form

$$K(\tau, \hat{\tau}, X_0) = (4\pi\tau)^{-n/2} \sum_{p=0}^{n} K_{\mu_1 \ldots \mu_p}(\tau, \hat{\tau}, x_0)\psi_0^{\mu_1} \ldots \psi_0^{\mu_p}, \tag{5.2}$$

where the $K_{\mu_1\ldots\mu_p}$ are anti-symmetric tensors. The normalization factor $(4\pi\tau)^{-n/2}$ is introduced for later convenience. The perturbation series for the path integral (5.1) begins at order g^0, ignoring the overall normalization. Therefore the $K_{\mu_1\ldots\mu_p}(\tau, \hat{\tau}, x_0)$ have expansions in τ which begin at order τ^0.

Recall that the diagonal part of the super heat kernel, $K_{\tau,\hat{\tau}}(x, x)$, is a matrix on spinors at each point in space-time, and that any matrix on spinors can be decomposed into a linear combination of antisymmetric products of γ matrices. Eq. (5.2) provides this decomposition for $K_{\tau,\hat{\tau}}(x, x)$. Each antisymmetric product of the grassmann variables ψ_0^μ in (5.2) is interpreted as the corresponding antisymmetric product of γ matrices (scaled by $\tau^{1/2}$):

$$K_{\tau,\hat{\tau}}(x_0, x_0) = (4\pi)^{-n/2} \sum_{p=0}^{n} \tau^{(p-n)/2} K_{\mu_1\ldots\mu_p}(\tau, \hat{\tau}, x_0)\gamma^{\mu_1}\ldots\gamma^{\mu_p}. \tag{5.3}$$

We can fix the normalization of the super heat kernel by reference to its flat space value $K_{\tau,\hat{\tau}}(x, x) = (4\pi\tau)^{-n/2}$. Because of the overall normalization in (5.2–5.3), the scalar $p = 0$ term $K(\tau, \hat{\tau}, x_0) \to 1$ in the limit $g \to 0$.

Each term $K_{\mu_1\ldots\mu_p}(\tau, \hat{\tau}, x_0)$ has an expansion in powers of τ beginning at order τ^0 so the leading term of the rank p tensor will go as $\tau^{(p-n)/2}$. Now it is easy to see how the term of $O(\tau^0)$ is singled out in the computation of the anomaly. Remember we wanted to calculate $\mathrm{tr}_{\mathrm{spin}}\gamma_5 e^{-\tau H}\delta(x, x)$. But the trace of γ_5 with every antisymmetric tensor of rank p in the γ-matrices is zero but for the highest one $p = n$, which contains the product of all of the γ-matrices. Only the rank n term will contribute to the trace over spin and its leading behavior as $\tau \to 0$ is $O(\tau^0)$. So this leading part is all we need to calculate to get the formula for the anomaly and the index.

The last result will not be modified by the introduction of internal degrees of freedom. The coefficients of the expansion of the super heat kernel simply become matrices in the internal space.

The actual calculation is done as follows. For small g, the X field fluctuates around minima X_0 of the action. The fluctuations δX are of order $g^{1/2}$. The minima satisfy the equation of motion

$$gD_\Gamma(g^{-1}\partial_t X^\mu) = 0, \tag{5.4}$$

where D_Γ is the covariant superderivative

$$D_\Gamma Y^\mu = DY^\mu + DX^\sigma \Gamma^\mu_{\nu\sigma}(X)Y^\nu. \tag{5.5}$$

Eq. (5.4) is the super geodesic equation. For $g \sim 0$ only the absolute minima of the action contribute. These are the constants $X_0^\mu = x_0^\mu + \theta\psi_0^\mu$, $\partial_t X_0^\mu = 0$. Note that $DX_0^\mu = -\psi_0^\mu$. Since we have no present need of the $\hat{\tau}$ dependence we now set $\hat{\tau} = 0$, and treat g as an ordinary number.

The contribution from the η integral can be determined to lowest order in g directly from the Heisenberg equation of motion

$$\dot{\eta} + [\bar{\eta}\underline{F}\eta, \eta] = 0, \tag{5.6}$$

where

$$\underline{F} = \tfrac{1}{2}F_{\mu\nu}\psi_0^\mu\psi_0^\nu. \tag{5.7}$$

This gives

$$\bar{\eta}(1) = e^{-\bar{\eta}\underline{F}\eta}\bar{\eta}(0)e^{\bar{\eta}\underline{F}\eta},$$

$$\langle\eta(1)|\bar{\eta}\rangle = \langle\eta|e^{\bar{\eta}\underline{F}\eta}|\bar{\eta}\rangle. \tag{5.8}$$

Thus, in the leading order, the diagonal of the super heat kernel acts as the matrix $e^{\bar{\eta}\underline{F}\eta}$ on the internal space. On wave functions with exactly one internal index this is simply $e^{\underline{F}}$.

For the remaining spin dependence it is necessary to do the gaussian integral over the fluctuations in $X(t)$. We expand the action to second order in the fluctuations, using $g_{\mu\nu}(X) = g_{\mu\nu}(X_0) + \delta X^\sigma\partial_\sigma g_{\mu\nu}(X_0)$, and writing $\Gamma_0 = \Gamma(X_0)$:

$$S = \int_0^1 \mathrm{d}t \int \mathrm{d}\theta \frac{1}{4g^2} g_{\mu\nu}(X_0)\delta X^\mu(-D_{\Gamma_0}\partial_t)\delta X^\nu. \tag{5.9}$$

The integral is over fluctuations δX which are periodic and vanish at $t=0$. It is possible to express the result in terms of the superdeterminant of $(D_{\Gamma_0})^2$, but it is just as simple to calculate from the component form of (5.9), or directly from (4.20), if we simplify the calculation by choosing special coordinates with x_0 as origin such that $\Gamma(0) = 0$ and

$$\Gamma^\tau_{\sigma\nu}(x) \sim \tfrac{1}{2}R^\tau_{\sigma\alpha\nu}x^\alpha. \tag{5.10}$$

The quadratic part of the action is

$$S = \int_0^1 \mathrm{d}t\, \delta x(-\partial_t + \underline{R})\partial_t\delta x + \psi\partial_t\psi, \tag{5.11}$$

where

$$\underline{R}^\mu_\nu = \tfrac{1}{2}\psi_0^\alpha\psi_0^\beta R^\mu_{\nu\alpha\beta}.$$

$$R^\mu_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu_{\nu\beta} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - (\alpha \leftrightarrow \beta). \tag{5.12}$$

This gives, up to a normalization,

$$\int D[\delta X]\mathrm{e}^{-S} = \det\nolimits_0^{-1/2}(\partial_t - \underline{R}), \tag{5.13}$$

where $\det_0$ is the determinant leaving out the constant modes.

Up to a normalization,

$$\det_0^{-1/2}(\partial_t - \underline{R}) = \det^{-1/2}\left[\prod_{k\neq 0}(\underline{R} - 2\pi i k)\right]$$
$$= \det^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\tfrac{1}{2}\underline{R}\right]. \tag{5.14}$$

Putting together (5.8), (5.14) and the flat space normalization we get the leading part of the diagonal of the super heat kernel

$$K_{\tau,0}(x_0, x_0) \sim (4\pi\tau)^{-n/2} e^{\underline{F}} \det^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\tfrac{1}{2}\underline{R}\right]. \tag{5.15}$$

$\underline{R}$ and $\underline{F}$ are given by (5.7) and (5.12). Antisymmetric products of the ψ_0^μ are to be replaced in (5.15) by the corresponding antisymmetric products of $\tau^{1/2}\gamma^\mu$.

The chiral anomaly is

$$\tfrac{1}{2}\nabla_\mu j_5^\mu = \operatorname*{tr}_{\text{spin}} \gamma_5 K_{\tau,0}(x,x). \tag{5.16}$$

The trace with γ_5 selects the coefficient of $\gamma^1 \ldots \gamma^n$ in $K_{\tau,0}(x,x)$, times a factor $i^{n/2}$ from the definition (3.2) of γ_5 and a factor $2^{n/2}$ which comes from the trace of the identity matrix on spinors. A simple way to write the result is

$$\mathrm{d}^n x\, \tfrac{1}{2}\nabla_\mu j_5^\mu = \left(\frac{i}{2\pi}\right)^{n/2} e^{\underline{F}} \det^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\tfrac{1}{2}\underline{R}\right]\Big|_{\psi^\mu = \mathrm{d}x^\mu}. \tag{5.17}$$

On the right-hand side of (5.17) only the term proportional to $\mathrm{d}^n x$ should be kept. The formula for the Atiyah-Singer index is obtained by taking the trace in the internal indices and integrating over x:

$$I = \int \left(\frac{i}{2\pi}\right)^{n/2} \operatorname{tr}(e^{\underline{F}}) \det^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\tfrac{1}{2}\underline{R}\right], \tag{5.18}$$

where now

$$\underline{F} = \tfrac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu,$$
$$\underline{R}^\mu_\nu = \tfrac{1}{2}R^\mu_{\nu\alpha\beta}\,\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{5.19}$$

For the sake of completeness we record the index formula for all the antisymmetric tensor products of the internal space:

$$I(\alpha) = \sum_k I_k e^{-i\alpha k}$$
$$= \int \left(\frac{i}{2\pi}\right)^{n/2} \det(1 + e^{\underline{F} - i\alpha}) \det^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\left(\tfrac{1}{2}\underline{R}\right)\right]. \tag{5.20}$$

This follows from the formula

$$\operatorname{tr} \mathrm{e}^{\bar{\eta}(\underline{F}-i\alpha)\eta} = \det\left(1 + \mathrm{e}^{\underline{F}-i\alpha}\right), \tag{5.21}$$

which in turn follows from the fact that both sides of (5.21) are equal to the periodic fermionic path integral which gives $\det(\partial_t - \underline{F} + i\alpha)$.

We illustrate the index formula (5.18) by two familiar examples. The winding number of a two-dimensional abelian gauge field is [13]

$$I = \int \frac{i}{2\pi} \frac{1}{2} F_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} = \frac{i}{4\pi} \int \mathrm{d}^2 x\, \varepsilon^{\mu\nu} F_{\mu\nu}. \tag{5.22}$$

The Pontryagin number of a 4-dimensional non-abelian gauge field is [13]

$$\begin{aligned} I &= \int \left(\frac{i}{2\pi}\right)^2 \frac{1}{2!} \operatorname{tr}\left(\tfrac{1}{4} F_{\mu\nu} F_{\alpha\beta}\right) \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta} \\ &= \frac{-1}{32\pi^2} \int \mathrm{d}^4 x\, \varepsilon^{\mu\nu\alpha\beta} \operatorname{tr}\left(F_{\mu\nu} F_{\alpha\beta}\right). \end{aligned} \tag{5.23}$$

As an example of the chiral anomaly, let us take the non-abelian gauge field in $n = 4$ dimensions, with gauge group generators λ_k^+ on the positive chirality spinors and λ_k^- on the negative chirality spinors, and $F_{\mu\nu} = F_{\mu\nu}^k(\lambda_k^+ + \lambda_k^-)$. Following the discussion of sect. 3,

$$\begin{aligned} \nabla_{\mu} j_k^{\mu} &= \operatorname{tr}\left[(\lambda_k^+ - \lambda_k^-) \tfrac{1}{2} \partial_{\mu} j_5^{\mu}\right] \\ &= \frac{-1}{32\pi^2} \operatorname{tr}(\lambda_k^+ \lambda_l^+ \lambda_m^+ - \lambda_k^- \lambda_l^- \lambda_m^-) \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^l F_{\alpha\beta}^m. \end{aligned} \tag{5.24}$$

The condition that the current j_k^{μ} be covariantly conserved is the well known condition that the sums of the cubes of the charges should be the same for each chirality.

6. Euler number and Hirzebruch signature

We will sketch in this section how to specialize the general index formula to get the Euler number and the Hirzebruch signature. A more complete version of this material can be found in ref. [5]. The point we wish to emphasize is that the Dirac operator becomes an operator on fields with spin different from $\frac{1}{2}$ when the internal space itself has nonzero spin.

The Euler number of a space-time manifold is the index of the exterior derivative d on differential forms. When the space-time dimension is even there is a one-to-one correspondence $\omega \leftrightarrow \hat{\omega}$ between the differential form $\omega = \omega_{\mu_1 \ldots \mu_k} \mathrm{d}x^{\mu_1} \ldots \mathrm{d}x^{\mu_k}$ and the matrice $\hat{\omega} = \omega_{\mu_1 \ldots \mu_k} \gamma^{\mu_1} \ldots \gamma^{\mu_k}$. The $\hat{\omega}$ span all the matrices on spinors. The space of matrices on spinors we write $S \otimes \bar{S}$, where S is the space of spinors and $\bar{S}$ is its dual space.

A Dirac operator can be defined with $\bar{S}$ as internal space. The correspondence $\omega \leftrightarrow \hat{\omega}$ allows us to consider this $\not{D}$ as a first order differential operator on forms. It is an exercise with γ matrices to show that this operator is $d - d^*$. The gauge field A_μ is simply the spin connection for $g_{\mu\nu}$.

Let Λ^p $p = 0, 1, \ldots n$ be the space of p-forms on the tangent space of a given n-dimensional manifold. One defines as usual the exterior derivative d_p (here the index p is kept for clarity)

$$d_p : \Lambda^p \to \Lambda^{p+1} . \tag{6.1}$$

For example, on the one-form $\omega_\mu dx^\mu \in \Lambda^1$,

$$(d_1 \omega)_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu . \tag{6.2}$$

The adjoints are the divergence operators

$$d_p^* : \Lambda^{p+1} \to \Lambda^p , \qquad d_0^* \omega = - \nabla^\mu \omega_\mu . \tag{6.3}$$

This defines the de Rham complex

$$0 \to \Lambda^0 \xrightarrow{d_0} \Lambda^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n \to 0 , \tag{6.4}$$

in which the composition of two successive operators is zero. The laplacian operators are

$$\Delta_p = d_{p-1} d_{p-1}^* + d_p^* d_p : \Lambda_p \to \Lambda_p . \tag{6.5}$$

The Euler number is

$$\chi = \sum_{p=0}^{n} (-1)^p \dim H_p = \sum_{p=0}^{n} (-1)^p \dim \ker(\Delta_p) , \tag{6.6}$$

where $\ker(\Delta_p)$ is the space of harmonic p-forms ($d\omega = d^*\omega = 0$) and is identified with the cohomology class H_p (the space of closed p-forms $d\omega_p = 0$ modulo the exact ones $\omega_p = d\omega_{p-1}$) whose dimension is the pth Betti number.

It is now easy to recast the Euler number as an index of the type described in sect. 2. First split the space of forms into even and odd subspaces

$$\Lambda^+ = \bigoplus_{p \text{ even}} \Lambda^p , \qquad \Lambda^- = \bigoplus_{p \text{ odd}} \Lambda^p . \tag{6.7}$$

Then define

$$Q = d - d^* ,$$

$$(-1)^F = (-1)^p . \tag{6.8}$$

A comparison of the definitions gives $\chi = I$.

Now we want to write I in terms of the Dirac operator $\not{D}$ on $S \otimes \bar{S}$. If ω is a p-form, then $(-1)^p \omega \leftrightarrow \gamma_5 \hat{\omega} \gamma_5$. Let us split the matrices on spinors into two subspaces

$$S \otimes \bar{S}_+ = \{\omega :\ \omega\gamma_5 = \omega\} , \qquad S \otimes \bar{S}_- = \{\omega :\ \omega\gamma_5 = -\omega\} . \tag{6.9}$$

The γ_5 which anticommutes with $\not{D}$ acts on the left on $S \otimes \bar{S}$, so multiplying by γ_5 on the right commutes with $\not{D}$. We write $\not{D} = \not{D}_+ \oplus \not{D}_-$, on the two subspaces (6.9). From (6.9),

$$(-1)^p \omega \leftrightarrow \gamma_5 \hat{\omega} \qquad \text{for} \qquad \hat{\omega} \in S \otimes S^*_+ ,$$

$$(-1)^{p+1} \omega \leftrightarrow \gamma_5 \hat{\omega} \qquad \text{for} \qquad \hat{\omega} \in S \otimes S^*_- . \tag{6.10}$$

It follows that

$$I_+ = \mathrm{Ind}(\not{D}_+) = \sum_p (-1)^p \dim H_p^- , \qquad I_- = \mathrm{Ind}(\not{D}_-) = \sum_p (-1)^{p+1} \dim H_p^+ , \tag{6.11}$$

where H_p^+ and H_p^- are the spaces of harmonic forms in $S \otimes S^*_+$ and $S \otimes S^*_-$ respectively. Then

$$\chi = \sum_p (-1)^p \dim H_p = I_+ - I_- . \tag{6.12}$$

The Hirzebruch signature is simply $\mathrm{sign}(M) = \mathrm{Ind}(\not{D}) = I_+ + I_-$. The definition is $\mathrm{sign}(M) = \mathrm{Ind}(d - d^*)$, where the index is taken with respect to the following self-adjoint operator which anti-commutes with $d - d^*$ and has square one:

$$\tau(\omega) = (-1)^{p(p-1)/2} * (\omega) , \tag{6.13}$$

where $* : \Lambda^p \to \Lambda^{n-p}$ is the Hodge duality operator, i.e. contraction with $\varepsilon_{\mu_1 \ldots \mu_n}$. But it can be checked that $\tau(\omega) \leftrightarrow \gamma_5 \hat{\omega}$, from which it follows immediately that $\mathrm{sign}(M) = I_+ + I_-$.

Finally, we need to calculate $I_\pm = \mathrm{Ind}(\not{D}_\pm)$. To use the index formula (5.18) we need to know $F^\pm_{\mu\nu}$, the curvature of the metric spin connection on $\bar{S}_\pm$. This is simply the ordinary curvature represented by infinitesimal rotations on the dual space of the spinors:

$$F^{\pm}_{\mu\nu} = -\tfrac{1}{2}(1 \pm \gamma_5)\tfrac{1}{4}R^{\alpha}_{\beta\mu\nu}\gamma_\alpha\gamma^\beta . \tag{6.14}$$

The Euler number and Hirzebruch signature are then given by

$$\chi = I_+ - I_- = \int \left(\frac{i}{2\pi}\right)^{n/2} \det{}^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\tfrac{1}{2}\underline{R}\right]\mathrm{tr}\left[\gamma_5 \mathrm{e}^{-R^{\alpha}_{\beta}\gamma_\alpha\gamma^\beta/4}\right]$$

$$\mathrm{sign}(M) = I_+ + I_- = \int \left(\frac{i}{2\pi}\right)^{n/2} \det{}^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\sinh\tfrac{1}{2}\underline{R}\right]\mathrm{tr}\left[\mathrm{e}^{-R^{\alpha}_{\beta}\gamma_\alpha\gamma^\beta/4}\right], \tag{6.15}$$

where the definition of $\underline{R}$ is given in (5.19). The traces can be calculated with the aid of the fermionic path integral with lagrangian $(\zeta^\mu \partial_t \zeta^\mu + \zeta^\nu \underline{R}^{\nu}_{\mu}\zeta^\mu)$, or by putting $\underline{R}^{\alpha}_{\beta}$ in two by two block diagonal form and calculating the following identity:

$$\mathrm{e}^{-\underline{R}^{\alpha}_{\beta}\gamma_\alpha\gamma^\beta/4} = \det{}^{1/2}\left(\cosh\left(\tfrac{1}{2}\underline{R}\right)e^{-\zeta_0 \tanh(\underline{R}/2)\zeta_0/2}\right., \tag{6.16}$$

where the ζ_0 behave just as the ψ_0, as anticommuting variables whose products represent antisymmetric products of γ matrices. Combining (6.15)–(6.16) we get

$$\chi = \int \frac{1}{(4\pi)^{n/2}(n/2)!}\varepsilon^{\mu_1 \cdots \mu_n}\underline{R}_{\mu_1\mu_2}\cdots\underline{R}_{\mu_{n-1}\mu_n}, \tag{6.17}$$

$$\mathrm{sign}(M) = \int \left(\frac{i}{\pi}\right)^{n/2} \det{}^{-1/2}\left[\left(\tfrac{1}{2}\underline{R}\right)^{-1}\tanh\tfrac{1}{2}\underline{R}\right]. \tag{6.18}$$

7. The global obstruction to spin

There are space-times in which it is impossible to consistently and covariantly define spinors [6]. The obstruction comes from global topology. We will describe the obstruction here as a Z_2 anomaly in the quantum mechanics of the spinning particle: the same sort of global anomaly that Witten found in SU(2) gauge theory [7]. The quantum mechanics turns out to be sensible if and only if the space-time carries spinors. It would be surprising otherwise, since the wave functions of the spinning particle *are* the spinors.

The argument is exactly the same as in [7]. To construct the spinning particle through the path integral with action (4.20) it is necessary to make sense of the integral over ψ^μ. The value of the path integral is $\sqrt{\det(\nabla_t)}$, which depends on the closed path $x^\mu(t)$ in space-time and on the space-time metric $g_{\mu\nu}$. The operator ∇_t acting on vector fields $\psi^\mu(t)$ along the loop is real and skew-symmetric, so its eigenvalues are grouped in complex conjugate pairs of imaginary numbers $i\lambda_k$, $i\lambda_{-k}$

$= -i\lambda_k$. For a typical path $x^\mu(t)$ there are no zero eigenvalues, so it makes sense to put the eigenvalues in order: $0 < \lambda_1 \leqslant \lambda_2 \ldots$ Then we can write, leaving out the normalization,

$$\sqrt{\det(\nabla_t)} = \prod_{k>0} \lambda_k . \tag{7.1}$$

This definition is extended to all the nearby closed paths by following the λ_k, $k > 0$, as the path $x^\mu(t)$ changes, keeping continuity in the corresponding eigenfunctions of ∇_t.

This procedure can be used to extend the definition of $\sqrt{\det(\nabla_t)}$ to any loop $x^\mu(t)$ by following any route from the initial loop. It only remains to check if the definition depends on the route taken. Since the definition of $\sqrt{\det(\nabla_t)}$ is only extended locally, over neighborhoods in loop space, two routes could produce different values. The product (7.1) will always contain one eigenvalue from each complex conjugate pair, so the only possible discrepancy is in the sign. Following a route in loop space, $\sqrt{\det(\nabla_t)}$ changes sign every time a pair of eigenvalues $i\lambda_k$, $-i\lambda_k$ crosses zero. The definition is consistent if and only if each route sees the same number of crossings mod_2. This is equivalent to the requirement there be an even number of crossings along any closed path in loop space.

This obstruction to constructing the spinning particle, i.e. the number of eigenvalue crossings mod_2 along closed paths in loop space, is a topological invariant. Any deformation of the route in loop space or of the space-time metric changes the number of crossings in pairs. A closed path in loop space describes a surface in space-time. So the obstruction to constructing the spinning particle is a Z_2 valued continuous function on surfaces.

Let $x^\mu(s, t)$, as s varies from 0 to 1, be a closed path in loop space. Following Witten, we will calculate the number of eigenvalue crossings of ∇_t as s goes from 0 to 1 in terms of the zeros of a Dirac operator on the s, t surface. Define the operator ∇_s acting on vector valued functions $\psi^\mu(s, t)$ by analogy with ∇_t:

$$\nabla_s \psi^\mu = \partial_s \psi^\mu + \partial_s x^\alpha \Gamma^\mu_{\nu\alpha}(x) \psi^\nu . \tag{7.2}$$

Then define the two-dimensional Dirac operator

$$\not{\nabla} = \sigma^1 \nabla_t + \sigma^3 \nabla_s , \tag{7.3}$$

which acts on two-dimensional spinors tensored with space-time vectors. $\not{\nabla}$ has two zero modes for every eigenvalue crossing. The doubling occurs because ∇_t had to be turned into a 2×2 matrix of operators in order to make a covariant operator on the s, t surface.

The number of zero modes of $\not{\nabla}$ is a topological invariant (mod_4). To see this, define the following operators on the nonzero modes:

$$\tau_1 = |\not{\nabla}|^{-1} \not{\nabla}, \qquad \tau_2 = i\sigma_2 , \qquad \tau_3 = \tau_1 \tau_2 . \tag{7.4}$$

The real, skew-symmetric operators τ_k form the Clifford algebra

$$\tau_j \tau_k + \tau_k \tau_j = -2\delta_{jk} .$$

The only irreducible representation of this algebra is 4-dimensional, so eigenvalues of $\not{\nabla}$ can only approach or leave zero in quartets.

Thus the number of eigenvalue crossings of ∇_t along the path $x^\mu(s,t)$ equals (mod_2) half the number of zero modes of the two-dimensional Dirac operator $\not{\nabla}$.

The topological obstruction to making spinors is also associated with 2-surfaces in space-time. Making spinors means covering the orthogonal group at each point with the spin group, its double covering. The metric connection performs parallel transport of tangent vectors, so to every closed loop it associates, by transport around the loop, an orthogonal linear transformation of tangent vectors. A closed path in loop space thus determines a closed path in $\mathrm{O}(n)$, which is homotopically either the trivial closed path or the nontrivial one. Only the trivial one lifts to $\mathrm{spin}(n)$, so, in order for there to exist a spin structure on space-time, all of the closed paths in $\mathrm{O}(n)$ induced by closed paths in loop space must be trivial. Conversely, if every closed path in loop space gives a contractible closed path in $\mathrm{O}(n)$ then spinors do exist.

The construction of spinors comes down to a question about surfaces, in particular, about the bundle of space-time tangent vectors $\psi^\mu(s,t)$ over the s,t surface. Because we are dealing with homotopy invariants of paths in loop space, we can limit the discussion to spherical surfaces. Strictly speaking, this is only true if space-time is simply connected, since otherwise there is more than one connected component to loop space, and more than one spin structure is possible. We ignore this complication. On each hemisphere of a spherical surface the bundle of space-time tangents can be trivialized, so the only issue is what happens at the equator. The two trivializations are patched together by a function from the equator to $\mathrm{O}(n)$. The triviality or non-triviality of this loop in $\mathrm{O}(n)$ is exactly the same as the triviality or nontriviality of the loop of parallel transports in $\mathrm{O}(n)$. The patching loop in $\mathrm{O}(n)$ can be deformed so that it lies in an $\mathrm{O}(2)$ subgroup, say the upper left 2×2 block. The bundle of space-time tangents then splits into $n-2$ trivial one-dimensional bundles plus a possibly nontrivial 2-dimensional bundle. Any such nontrivial 2-dimensional bundle can be taken to be the bundle of majorana spinors on the sphere.

There are two possible decompositions of the operator $\not{\nabla}$. If spinors exist, the bundle of space-time tangents over the sphere being trivial, then $\not{\nabla}$ decomposes into n copies of the ordinary Dirac operator on the sphere. The ordinary Dirac operator has two zero modes, so $\not{\nabla}$ has $2n$, a multiple of four. On the other hand, if spinors do not exist, then $\not{\nabla}$ decomposes into $n-2$ copies of the ordinary Dirac operator on the sphere and one copy of the Dirac operator on spinors $\otimes$ spinors. Spinors $\otimes$ spinors are the differential forms. $\not{\nabla} = d - d^*$ has two zero modes among the forms on the two sphere: the constants and $\varepsilon_{\mu\nu}$. Thus, if spinors do not exist, there are $2(n-2)+$

$2 = 2n - 2$ zero modes, which is equal to 2 (mod_4). So we confirm that the global quantum anomaly is the same as the topological obstruction to having a spin structure.

The standard suggestion for coping with space-times which admit no spinors is to use internal degrees of freedom to compensate for the obstruction to spinors, so that neither set of degrees of freedom makes sense globally on space-time, but the combined set does. In the formalism we have been using for the spinning particle, this means making η a real field. The superfields DX and N would have separate identities only locally; globally they would have to be regarded as two components of a single object. It would be interesting to know if, for fermions obeying a Dirac equation of this twisted type, there are global gravitational-gauge anomalies, like Witten's SU(2) anomaly, which could limit the possibility of compensating with internal degrees of freedom for the non-existence of spinors.

We would like to thank I.M. Singer whose report of a discussion with E. Witten initiated this work. We also thank S. Shenker for lengthy, interesting discussions at the beginning of this work. PW would like to thank Alvarez-Gaumé for a detailed and interesting discussion on his treatment [9a] of part of the material covered in this paper and also C. Itzykson for discussion. D.F. expresses appreciation to CEN Saclay, and PW to the Enrico Fermi Institute, for hospitality during the course of the work.

References

[1] M.F. Atiyah and I.M. Singer, *Bull. AMS* 69 (1983) 422; Ann. of Math. 87 (1968) 484
[2] S. Adler, Phys. Rev. 177 (1969) 24;
J. Bell and R. Jackiw, Nuov. Cim. 60A (1969)
[3] S. Adler and W. Bardeen, Phys. Rev. 182 (1969) 15;
W. Bardeen, Phys. Rev. 184 (1969) 18;
J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95
[4] E. Witten, Nucl. Phys. B202 (1982) 253
[5] P. Gilkey, The index theorem and the heat equation, Math. Lect. Series 4 (Publish or Perish, 1974);
R.S. Palais, Seminar on the Atiyah-Singer index theorem, Ann. of Math. Study, vol. 57 (Princeton Univ. Press);
P. Shanahan, The Atiyah-Singer index theorem, Lect. Notes in Math., vol. 638 (Springer, 1978)
[6] A. Haefliger, C.R. Acad. Sc. Paris 243 (1956) 358;
S. Hawking and C.N. Pope, Phys. Lett. 73B (1978) 42
[7] E. Witten, Phys. Lett. 117B (1982) 324
[8] P. Windey, Acta Physica Polonica, to be published
[9] (a) L. Alvarez-Gaumé, Comm. Math. Phys. 90 (1983) 161;
(b) L. Alvarez-Gaumé, HUTP-83/A035: "A note on the Atiyah-Singer Index Theorem"
[10] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1984) 269
[11] F.A. Berezin and M.S. Marinov, JETP Lett. 21 (1975) 320;
L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P. Howe, Phys. Lett. 64B (1976) 435;
P. Di Vecchia, L. Brink and P. Howe, Nucl. Phys. 118B (1976) 77
[12] E. Cartan, *Leçons sur la thèorie des spineurs* (Hermann, Paris)
[13] S. Coleman, The uses of instantons, in Erice lectures 1975, ed. Zichichi (Academic Press)