A new kind of quantum field theory
of $(n{-}1)$-dimensional defects in $2n$ dimensions

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I describe a project to open a new territory of quantum field theory where the fields live not on a space-time manifold but on certain complete metric spaces of $(n-1)$-dimensional objects (defects) in a $2n$-dimensional space-time. These metric spaces are "quasi Riemann surfaces"; they have properties formally analogous to Riemann surfaces. Every construction of a 2d CFT is to give an analogous construction of a CFT on the quasi Riemann surfaces, and thereby a new CFT on M. When $2n \geq 4$, the global symmetry of the 2d CFT will become a local gauge symmetry. Ordinary local quantum fields in space-time will be constructed by restricting to small objects. This note is a summary of the main points of [1]. References can be found there. Additional materials are collected at [2].

Let M be the euclidean space-time, an oriented conformal $2n$-manifold, compact and without boundary. When $n = 1$, M is a Riemann surface. The basic examples are $M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$. The Hodge $*$-operator acting on n-forms is conformally invariant, $(*\omega)_{\nu_1\cdots\nu_n}(x) = \omega_{\mu_1\cdots\mu_n}(x)\,\frac{1}{n!}\,\epsilon^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_n}(x)$ with $*^2 = (-1)^n$. Nothing else is used of the conformal structure.

The physical objects are represented mathematically as the integral $(n-1)$-currents in M, as constructed in Geometric Measure Theory (GMT). A *k-current* ξ in M is a distribution on k-forms, $\omega \in \Omega_k(M) \mapsto \int_\xi \omega \in \mathbb{R}$. The boundary operator ∂ on currents is dual to the exterior derivative, $\int_{\partial\xi} \omega = \int_\xi d\omega$, $\partial^2 = 0$. A k-simplex σ in M, $\sigma\colon \Delta^k \to M$ is represented by the k-current $\int_{[\sigma]} \omega = \int_{\Delta^j} \sigma^*\omega$ which is the delta-function concentrated on $\sigma(\Delta^j) \subset M$. The singular chains are the integer-linear combinations of simplices. The singular chain $\sigma = \sum_i m_i\sigma_i$ is represented by the current $[\sigma] = \sum_i m_i[\sigma_i]$. The *singular k-currents* $\mathcal{D}_k^{\mathrm{sing}}(M)$ are the k-currents representing the singular k-chains in M. Examples are the k-submanifolds. The current $[\sigma]$ represents the physical object in M, independent of how it is expressed as a sum of simplices. The physical difference between singular k-currents is measured by the *flat metric* $\|\xi_1 - \xi_2\|_{\mathrm{flat}}$, $\|\xi\|_{\mathrm{flat}} = \inf\{\mathrm{vol}(\xi - \partial\xi') + \mathrm{vol}(\xi')\colon \xi' \in \mathcal{D}_{k+1}^{\mathrm{sing}}(M)\}$. The metric completion of $\mathcal{D}_k^{\mathrm{sing}}(M)$ is the complete metric space $\mathcal{D}_k^{\mathrm{int}}(M)$ of *integral k-currents*, $\mathcal{D}_k^{\mathrm{sing}}(M) \subset \mathcal{D}_k^{\mathrm{int}}(M) \subset \mathcal{D}_k^{\mathrm{distr}}(M)$. The boundary of an integral current is an integral current, $\mathcal{D}_k^{\mathrm{int}}(M) \xrightarrow{\partial} \mathcal{D}_{k-1}^{\mathrm{int}}(M)$. $\mathcal{D}_k^{\mathrm{int}}(M)$ is a metric abelian group — an abelian group and, compatibly, a complete metric space.

Recall the 2d gaussian model, the free 1-form in 2d. $j(x)$ is a 1-form on a Riemann surface satisfying $dj = d(*j) = 0$. The integrals $d\phi = j$, $d\phi^* = *j$ are 0-forms ϕ, ϕ^* which take values in dual circles, $\phi(x) \in \mathbb{R}/2\pi R\mathbb{Z}$, $\phi^*(x) \in \mathbb{R}/2\pi R^*\mathbb{Z}$, $RR^* = 1$. and which are determined up to $U(1) \times U(1)$ global symmetries $\phi(x) \to \phi(x) + a$, $\phi^*(x) \to \phi^*(x) + a^*$. The vertex operators $V_{p,p^*}(x) = e^{ip\phi(x) + ip^*\phi^*(x)}$ describe point defects. They carry charges $(p, p^*) \in \mathbb{Z}/R \times \mathbb{Z}/R^*$, transforming by $V_{p,p^*} \to V_{p,p^*}\, e^{ipa + ip^*a^*}$.

Recall the free n-form in $2n$ dimensions. $F(x)$ is an n-form on M satisfying $dF = d(*F) = 0$. The integrals $dA = F$, $dA^* = *F$ are $(n{-}1)$-forms A, A^* on M, determined up to local gauge symmetries $A \to A + df$, $A^* \to A^* + df^*$ given by $(n{-}2)$-forms f, f^*. A and A^* are $U(1)\times U(1)$ gauge potentials in the sense that, for $\xi \in \mathcal{D}^{\text{sing}}_{n-1}(M)$, $\int_\xi A \in \mathbb{R}/2\pi R\mathbb{Z}$ and $\int_\xi A^* \in \mathbb{R}/2\pi R^*\mathbb{Z}$, with $RR^* = 1$. Fields $V_{p,p^*}(\xi) = e^{ip\int_\xi A + ip^*\int_\xi A^*}$ describe $(n{-}1)$-dimensional defects, transforming by $V_{p,p^*}(\xi) \to V_{p,p^*}(\xi)\, e^{ip\int_{\partial\xi} f + ip^*\int_{\partial\xi} f^*}$.

Define scalar fields ϕ, ϕ^* on $\mathcal{D}^{\text{sing}}_{n-1}(M)$ by $\phi(\xi) = \int_\xi A$ and $\phi^*(\xi) = \int_\xi A^*$. For a gauge transformation f, f^*, define $a(\partial\xi) = \int_{\partial\xi} f$ and $a^*(\partial\xi) = \int_{\partial\xi} f^*$. Then $\phi(\xi) \to \phi(\xi) + a(\partial\xi)$ and $\phi^*(\xi) \to \phi^*(\xi) + a^*(\partial\xi)$ and $V_{p,p^*}(\xi) = e^{ip\phi(\xi)+ip^*\phi^*(\xi)} \to V_{p,p^*}(\xi)\, e^{ipa(\partial\xi)+ip^*a^*(\partial\xi)}$. Fix an $(n{-}2)$-boundary $\partial\xi_0$ and consider the abelian subgroup $\mathcal{D}^{\text{sing}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} = \{\xi\colon\ \partial\xi \in \mathbb{Z}\partial\xi_0\}$ on which the gauge symmetries act as a global $U(1)\times U(1)$ generated by the two numbers $a(\partial\xi_0)$ and $a^*(\partial\xi_0)$.

To continue the analogy with the 2d gaussian model on $\mathcal{D}^{\text{sing}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0}$, calculus is needed, thus the metric completion $Q = \mathcal{D}^{\text{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0}$. GMT provides a construction of currents in any such complete metric space, so there is the space $\mathcal{D}^{\text{int}}_j(Q)$ of integral currents in Q. Define the j-forms on Q to be the dual objects, $\Omega_j(Q) = \text{Hom}(\mathcal{D}^{\text{int}}_j(Q), \mathbb{R})$, with the exterior derivative defined as the dual of the boundary operator, $d\omega(\xi) = \omega(\partial\xi)$. The infinitesimal j-simplices generate $\mathcal{D}^{\text{int}}_j(Q)$, so the tangent bundle TQ can be defined as the set of infinitesimal 1-simplices in Q. Then the 1-forms are the sections of the dual cotangent bundle T^*Q.

There are natural maps $\Pi_{j,n-1}\colon\ \mathcal{D}^{\text{int}}_j(Q) \to \mathcal{D}^{\text{int}}_{j+n-1}(M)$ derived from the equivalence $\Delta^j \times \Delta^{n-1} \cong \Delta^{j+n-1}$. $\Pi_{1,n-1}$ identifies each tangent space $T_\xi Q$ with a subspace of $\mathcal{D}^{\text{distr}}_n(M)$ which is closed under $*$ (a crucial technical point that requires the flat metric completion). Therefore $*$ acts on each $T_\xi Q$. The n-form F on M pulls back to a 1-form j on Q, $j = \Pi^*_{1,n-1}F$. A and A^* pull back to the 0-forms ϕ and ϕ^*. They satisfy $d\phi = j$, $d\phi^* = *j$, so now there is a classical 2d gaussian model on each space Q, except that $*^2 = 1$ for n even. So define $J = \epsilon_n *$, choosing numbers ϵ_n such that $J^2 = -1$. For n even, the currents must be complexified, $Q = \mathcal{D}^{\text{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}^{\text{int}}_n(M)$, so that J acts on the tangent spaces $T_\xi Q$. Now, for all n, there are the chiral 1-forms $j_\pm = \frac{1}{2}(1 \pm i^{-1}J)j$ and the chiral scalars $d\phi_\pm = j_\pm$ of the 2d gaussian model on Q.

The spaces Q are the “quasi Riemann surfaces”. They are the fibers of a bundle $\mathcal{Q}(M) \to \mathcal{B}(M) = \{\mathbb{Z}\partial\xi_0\}$. There is a 2d gaussian model on each fiber. The 2d global symmetry groups $G = U(1)\times U(1)$ on the fibers comprise a local gauge symmetry over the base space $\mathcal{B}(M)$.

Quantization is expressed by the Schwinger-Dyson equation on the 2-pt functions. In the 2d gaussian model, the chiral fields are (anti-)holomorphic and the S-D equation is the Cauchy-Riemann equation $\frac{\partial}{\partial\bar z}\,\frac{1}{z-z'} = \pi\delta^2(z-z')$ which is the foundation for complex analysis on Riemann surfaces. The 2d gaussian model might have led to complex analysis on Riemann surfaces had it not already existed.

The S-D equation of the free n-form has a form independent of n, $\langle\int_{\bar{\xi}_1}\bar{A}_\alpha\int_{\xi_2}dF_\beta\rangle = 2\pi i\epsilon_{\alpha\beta}I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$. The lhs is $\langle\bar{A}_\pm(x)dF_\pm(x')\rangle$ smeared against two currents. The rhs is a slight modification of the intersection number, $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle = \epsilon_{n,k_2-n}I_M(\bar{\xi}_1,\xi_2)$, $\epsilon_{n,m} = (-1)^{nm+m(m+1)/2}\,\epsilon_n^{-1}$. The intersection number is $I_M(\xi_1,\xi_2) = \int_M \frac{1}{k_1!\,k_2!}\xi_1^{\mu_1\cdots}\;\xi_2^{\nu_1\cdots}\epsilon_{\mu_1\cdots\nu_1\cdots}\;d^{2n}x$, nonzero only if $k_1+k_2=2n$. The $\epsilon_{n,m}$ are chosen so that $I_M\langle\,\bar{\xi}_1,\xi_2\,\rangle$ has properties independent of n: (1) it is skew-hermitian, (2) $I_M\langle\,\overline{\partial\xi_1},\xi_2\,\rangle = -I_M\langle\,\bar{\xi}_1,\partial\xi_2\,\rangle$, and (3) $I_M\langle\,\bar{\xi}_1,J\xi_2\,\rangle$ on n-currents is hermitian and positive definite.

Pulled back to $\mathcal{Q}$, the S-D equation of the n-form becomes that of the 2d gaussian model, $\langle\int_{\bar{\eta}_1}\bar{\phi}_\alpha\int_{\eta_2}dj_\beta\rangle = 2\pi i\epsilon_{\alpha\beta}I_Q(\bar{\eta}_1,\eta_2)$. The key point is that the skew-hermitian form on Q $I_Q(\bar{\eta}_1,\eta_2) = I_M\langle\,\Pi_{j_1,n-1}\eta_1,\Pi_{j_2,n-1}\eta_2\,\rangle$ is nonzero only if $j_1+j_2=2$, just like the intersection number in a 2-manifold.

Now the free *quantum* n-form on M has become the *quantum* 2d gaussian model on Q. Moreover, the metric space Q has the structure needed to write the Cauchy-Riemann equation. This is taken to be the defining structure of a *quasi Riemann surface.* All of the constructions of 2d CFT are based on the Cauchy-Riemann equation and the 2d gaussian model. So there is the prospect of carrying out those constructions on each of the fibers Q in the bundle of quasi Riemann surfaces, to obtain, for each 2d CFT, a new CFT of defects in M. Putting the 2d CFT on each fiber is ambiguous up to the global symmetry group, which becomes a local gauge symmetry in the bundle of quasi Riemann surfaces.

Conformal tensor analysis on quasi Riemann surfaces has to be developed in analogy with ordinary Riemann surfaces. Conjecturally, every quasi Riemann surface Q is isomorphic to $\mathcal{D}_0^{\mathrm{int}}(\Sigma)$ for Σ the 2d conformal space with the same Jacobian as Q. This would allow constructing a 2d CFT on all Q by lifting an ordinary 2d CFT from Σ to $\mathcal{D}_0^{\mathrm{int}}(\Sigma)$, a purely 2d problem. The conjecture would imply the automorphism group of a quasi Riemann surface is a very interesting univeral object, an extension of all the global symmetry groups of 2d CFTs by all the space-time conformal groups.

There should be a large collection of structure preserving maps from the complex disk $\mathbb{D}_1$ into Q, such that meromorphic functions pull back. The local structure of a CFT on Q will be expressed as the ordinary 2d CFT on each of this collection of *quasi holomorphic curves*, each with its the radial quantization, Virasoro algebras, and operator product expansion.

What is the space-time interpretation of the local gauge symmetry in the bundle of quasi Riemann surfaces? For example, there should be a local $SU(2)\times SU(2)$ symmetry corresponding to the global $SU(2)\times SU(2)$ at the self-dual point $R=1$ of the 2d gaussian model.

Tiny defects look like points in M, so fields $\Phi(\xi)$ on the small ξ in Q give ordinary local quantum fields on M. Will these form new local qfts in

$2n$-dimensions?

A cornucopeia of foundational problems present themselves — opportunities to leverage 2d qft to develop a new technology for qft in $2n$ dimensions.

[1] D. Friedan, *Quantum field theories of extended objects*, `arXiv:1605.03279 [hep-th]`.

[2] `http://www.physics.rutgers.edu/~friedan/#eqft`
slides of two talks in Israel, February 2017
Quasi Riemann surfaces, September, 2017
Quasi Riemann surfaces II. Questions, comments, and speculations, September, 2017
slides of a talk at Chicheley Hall, UK, September 2017