

Cosmology from the two-dimensional renormalization group acting as the Ricci flow

Daniel Friedan*

*New High Energy Theory Center and Department of Physics and Astronomy,
Rutgers, The State University of New Jersey,
Piscataway, New Jersey 08854-8019 U.S.A. and*

Science Institute, The University of Iceland, Reykjavik, Iceland

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The two-dimensional renormalization group acting as the Ricci flow $\Lambda \frac{\partial}{\partial \Lambda} g_{\mu\nu} = R_{\mu\nu}$ produces a specific 1+3 dimensional space-time metric which describes an expanding universe that starts with a big bang $a \sim t^{1/\sqrt{3}}$ then decelerates until $z = 0.2$ then accelerates until ending at $t_{\max} = 1.6 t_H$ with a big blowup $a \sim (t_{\max} - t)^{-1/\sqrt{3}}$. The only free parameters are the overall time scale and the value of the present time t_0 . These are fixed by the Hubble constant $H_0 = t_H^{-1}$ and the present deceleration parameter $q(t_0)$. This crude calculation of cosmology omits all but the gravitational field. The only energy-momentum is purely gravitational dark matter and energy. This is a preliminary exploration towards a specific, comprehensive, testable calculation of cosmology from a fundamental theory in which physics is produced by a quantum version of the two-dimensional renormalization group.

The renormalization group of the two-dimensional general nonlinear model acts at leading order as the Ricci flow [1–3]

$$\Lambda \frac{\partial}{\partial \Lambda} g_{\mu\nu} = R_{\mu\nu} \quad (1)$$

The field of the nonlinear model takes its values in a Riemannian manifold M . The Riemannian metric $g_{\mu\nu}$ encodes the couplings of the model. The beta-function at leading order is the Ricci tensor $R_{\mu\nu}$. The renormalization group drives $g_{\mu\nu}$ to a fixed point (modulo at most a finite number of relevant parameters). The fixed points are the solutions of

$$R_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu \quad (2)$$

where $v^\mu(x)\partial_\mu$ is a vector field on M . A vector field is an infinitesimal reparametrization $\dot{x}^\mu = v^\mu(x)$ of M . This is a field redefinition in the nonlinear model equivalent to the redundant perturbation $\dot{g}_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$. Thus the fixed point equation (2) expresses physical 2-d scale invariance¹.

The 2-d quantum field theory is manifestly well defined when $g_{\mu\nu}$ has euclidean signature. Assume that M is a four-dimensional Riemannian manifold of the form $I \times S^3$ where I is a real interval and S^3 is the unit 3-sphere. Assume $\text{SO}(4)$ invariance. Parametrize $I \times S^3$ as a spherical shell in \mathbb{R}^4 . The flat euclidean metric is $\delta_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2 ds_{S^3}^2$ where $ds_{S^3}^2$ is the round metric on the unit 3-sphere. The general $\text{SO}(4)$ -invariant metric has the form

¹ The general nonlinear model (also called the *nonlinear sigma model*) was constructed in [1–3] in $2+\epsilon$ dimensions with fixed point equation $R_{\mu\nu} - \epsilon g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$. The solutions were called quasi-Einstein metrics. The Ricci flow was introduced in Mathematics in [4]. The solutions of $R_{\mu\nu} - \epsilon g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$ have been called *Ricci solitons* or *Ricci flow solitons* in the Mathematics literature.

$g_{\mu\nu}dx^\mu dx^\nu = F_1(r)^2 dr^2 + F_2(r)^2 ds_{S^3}^2$. The metric can be made conformally flat

$$g_{\mu\nu}dx^\mu dx^\nu = a^2 (d\tau^2 + ds_{S^3}^2) \quad a = e^{f(\tau)} \quad (3)$$

by reparametrizing $r \rightarrow \tau(r)$ with $dr/d\tau = F_2/F_1$. The general SO(4)-invariant vector field is $v^\tau(\tau)\partial_\tau$. Then [5]

$$\begin{aligned} R_{\mu\nu}dx^\mu dx^\nu &= (-3\partial_\tau f_\tau) d\tau^2 + (-\partial_\tau f_\tau + 2 - 2f_\tau^2) ds_{S^3}^2 \\ (\nabla_\mu v_\nu + \nabla_\nu v_\mu)dx^\mu dx^\nu &= (2\partial_\tau v_\tau - 2v_\tau f_\tau) d\tau^2 + (2v_\tau f_\tau) ds_{S^3}^2 \\ f_\tau &= \partial_\tau f \quad v_\tau = g_{\tau\tau}v^\tau = a^2 v^\tau \end{aligned} \quad (4)$$

The fixed point equation (2) becomes the ordinary differential equation

$$\partial_\tau f_\tau = -2f_\tau v_\tau - 2f_\tau^2 + 2 \quad \partial_\tau v_\tau = 4f_\tau v_\tau + 3f_\tau^2 - 3 \quad (5)$$

We shall see below that there is an essentially unique solution of (5) which analytically continues in τ to real time $T = i^{-1}\tau$

$$ds^2 = a^2 (-dT^2 + ds_{S^3}^2) \quad (6)$$

The real-time ode is

$$\begin{aligned} \partial_T f_T &= -2f_T v_T - 2f_T^2 - 2 \quad \partial_T v_T = 4f_T v_T + 3f_T^2 + 3 \\ \partial_T &= i\partial_\tau \quad f_T = if_\tau = \partial_T a/a \quad v_T = iv_\tau \end{aligned} \quad (7)$$

The solution is

$$f_T = \frac{\cos 2T + \sqrt{3}}{\sin 2T} \quad v_T = \frac{-\sqrt{3}}{\sin 2T} \quad a = t'_0 \sin^{1+\nu} T \cos^{-\nu} T \quad T \in (0, \pi/2) \quad (8)$$

with $\nu = \sqrt{3}/2 - 1/2 = 0.3660\dots$. The only free parameter is the overall time scale t'_0 . In co-moving time t

$$\begin{aligned} ds^2 &= -dt^2 + a^2 ds_{S^3}^2 \quad dt = adT \\ \frac{t}{t'_0} &= \int_0^T \sin^{1+\nu} T' \cos^{-\nu} T' dT' = \frac{1}{2} B_{\sin^2 T} \left(\frac{2+\nu}{2}, \frac{1-\nu}{2} \right) \\ t \in (0, t_{\max}) \quad \frac{t_{\max}}{t'_0} &= \frac{1}{2} B \left(\frac{2+\nu}{2}, \frac{1-\nu}{2} \right) = 1.470\dots \end{aligned} \quad (9)$$

$B(p, q)$ is the Euler beta function, $B_x(p, q)$ the incomplete beta function. At the limits

$$\begin{aligned} T \rightarrow 0 \quad \frac{t}{t'_0} &\rightarrow \frac{T^{2+\nu}}{2+\nu} \quad a \rightarrow t'_0 (2+\nu)^{1/\sqrt{3}} \left(\frac{t}{t'_0} \right)^{1/\sqrt{3}} \\ T \rightarrow \frac{\pi}{2} \quad \frac{t_{\max} - t}{t'_0} &\rightarrow \frac{\left(\frac{\pi}{2} - T \right)^{1-\nu}}{1-\nu} \quad a \rightarrow t'_0 (1-\nu)^{-1/\sqrt{3}} \left(\frac{t_{\max} - t}{t'_0} \right)^{-1/\sqrt{3}} \end{aligned} \quad (10)$$

This is an expanding universe which begins with a big bang $a \sim t^{0.577\dots}$ and ends with a big blowup $a \sim (t_{\max} - t)^{-0.577\dots}$. The Hubble parameter $H = \partial_t a/a = f_T/a$ is

$$H = \frac{1}{t'_0} (\cos^2 T + \nu) \sin^{-2-\nu} T \cos^{-1+\nu} T \quad (11)$$

The deceleration parameter $q = -a\partial_t^2 a/(\partial_t a)^2 = -\partial_T f_T/f_T^2$ is

$$q = \frac{2(\sqrt{3}\cos 2T + 1)}{(\cos 2T + \sqrt{3})^2} \quad T_{q=0} = \frac{1}{2} \arccos(-1/\sqrt{3}) = (0.6959\dots)\frac{\pi}{2} \quad (12)$$

The expansion decelerates ($q > 0$) until $T = T_{q=0}$ then accelerates ($q < 0$) until the end.

For a first estimate of the time scale t'_0 and the present time t_0 use the deceleration parameter at the present time $q_0 = q(T_0) \approx 0.6$ in equation (12) to obtain $T_0 = 0.77\pi/2$. Then equate the Hubble constant H_0 to $H(T_0)$ in (11) to obtain $t'_0 = 1.1t_H$ where $t_H = 1/H_0 \approx 1.4 \times 10^{10}y$. Then (9) gives $t_{\max} = 1.6t_H$, $t_0 = 0.73t_H$ and (8) gives $a_0 = a(t_0) = 1.5t_H$.

Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (13)$$

is satisfied with energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{8\pi G} (\nabla_\mu v_\nu + \nabla_\nu v_\mu - \nabla_\sigma v^\sigma g_{\mu\nu}) \quad (14)$$

which is that of a perfect fluid of density $\rho(t)$ and pressure $p(t)$

$$\begin{aligned} T_{\mu\nu} dx^\mu dx^\nu &= \rho(t) dt^2 + p(t) a^2 ds_{S^3}^2 \\ 8\pi G \rho(t) &= a^{-2} (3f_T^2 + 3) \quad 8\pi G p(t) = a^{-2} (4f_T v_T + 3f_T^2 + 3) \end{aligned} \quad (15)$$

The energy-momentum is purely gravitational, tautologically dark. The equation-of-state parameter $w(t) = p/\rho$ and the density parameter $\Omega(t) = 8\pi G \rho/3H^2 = 1 + 1/f_T^2$ are

$$w = \frac{\cos 2T}{3\cos 2T + 2\sqrt{3}} \quad \Omega = 1 + \left(\frac{\sin 2T}{\cos 2T + \sqrt{3}} \right)^2 \quad (16)$$

The estimate $T_0 = 0.77\pi/2$ gives present values $w_0 = -0.6$, $\Omega_0 = 1.5$. The cosmological parameters are graphed as functions of t in Figure 1. Past values for a selection of redshifts $z = a_0/a - 1$ are listed in Table I.

We analyze the real-time ode (7) by adapting some of the methods that were used in [7] to analyze the ode analogous to (5) for the euclidean-signature $SO(3)$ -invariant fixed point equation in three dimensions. The real-time ode (7) has time-reflection symmetry $(T, f_T, v_T) \rightarrow (-T, -f_T, -v_T)$ and constant of motion

$$C = \frac{1}{a^2} \left(\frac{2}{3} v_T^2 + 2f_T v_T + f_T^2 + 1 \right) \quad \partial_T C = 0 \quad (17)$$

Changing variables to $h_\pm = f_T + (1 \pm 1/\sqrt{3})v_T$ the constant of motion and ode become

$$h_+ h_- + 1 = C a^2 \quad \partial_T h_\pm = -(1 + h_\pm^2) + \lambda_\pm (h_+ h_- + 1) \quad \lambda_\pm = 2 \pm \sqrt{3} \quad (18)$$

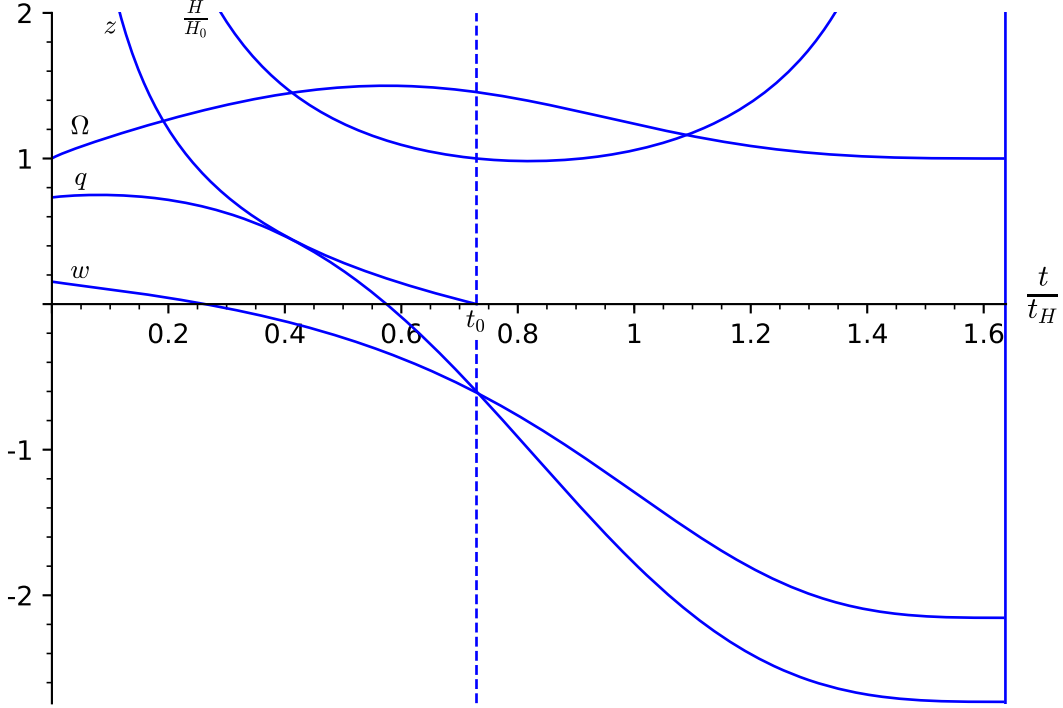


FIG. 1. The cosmological parameters as functions of the co-moving time t . The present is t_0 .

TABLE I. Cosmological parameters at selected values of z

	z	H/H_0	q	w	Ω
$z \gg 1$	$\gg 1$	$0.75 z^{1.7}$	0.73	0.16	1.0
	1000	1.2×10^5	0.73	0.16	1.0
	100	2.2×10^3	0.73	0.15	1.0
	10	48	0.74	0.15	1.0
$T = \frac{1}{2}T_0$	1.7	4.1	0.74	0.08	1.2
	1.0	2.4	0.69	0.02	1.3
$q = 0$	0.2	1.1	0	-0.3	1.5
$t = t_0$	0	1	-0.6	-0.6	1.5

The cosmological solution (8) is $C = 0$, $h_- = \cot T$, $h_+ = -\tan T$.

The phase portrait of the ode is shown in Figure 2. Every $C \neq 0$ trajectory asymptotes to a $C = 0$ trajectory. The separatrix S and its time-reflection S' each does so at one end; all the other $C \neq 0$ trajectories do so at both ends. The asymptotic behavior is one of

$$\begin{aligned}
 T \rightarrow 0^\pm \quad h_- &\rightarrow \frac{1}{T} & h_+ &\rightarrow -T - \frac{1}{3}T^3 + cT(T^2)^{1+\nu} & \frac{c}{C} &> 0 \\
 & & h_- &\rightarrow c'T(T^2)^{-\nu} & \frac{c'}{C} &> 0
 \end{aligned} \tag{19}$$

The presence of an irrational power of T , namely $(T^2)^{1+\nu}$ or $(T^2)^{-\nu}$, implies that such a solution cannot be the analytic continuation of a solution in imaginary time $\tau = iT$.

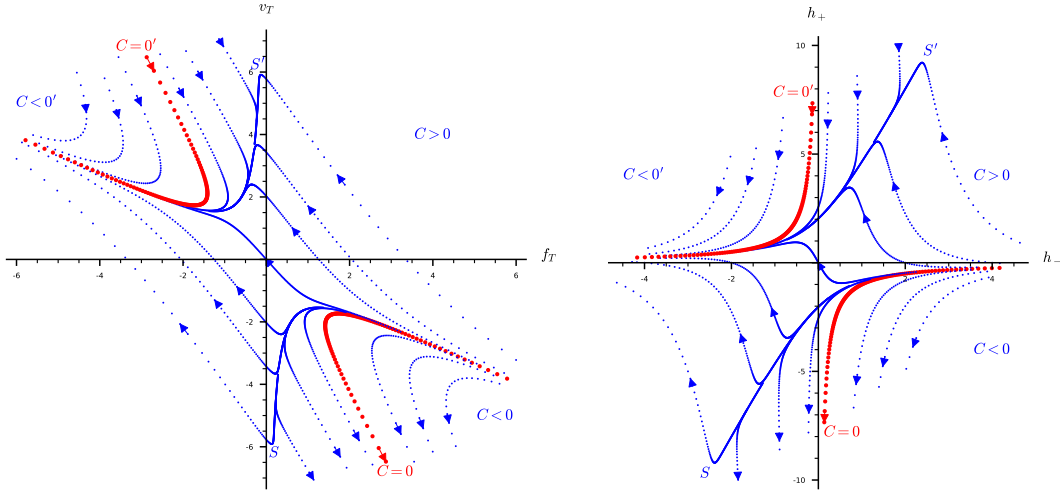


FIG. 2. Phase portrait of the real-time ode (7,18) in terms of (f_T, v_T) on the left, (h_-, h_+) on the right. The trajectories are the solutions. The arrows point in the direction of increasing T . The interval between points in the trajectories is $\Delta T = 0.01$. The universe is expanding when $f_T = \partial_t a$ is positive. The expansion is accelerating when f_T is increasing. The two red trajectories are the $C = 0$ solutions. The one in the lower right quadrant labeled $C=0$ is the cosmological solution (8). Its time reflection is labeled $C=0'$. The separatrix is labeled S . Its time-reflection is labeled S' .

The trajectories that start from $h_- = \infty$, $h_+ = 0$ at $T = 0$ are characterized by the numerical invariant

$$c = (h_+ h_- + 1)(h_-^2)^{1+\nu} e^{\int_0^T \frac{2 dT'}{h_-(T')}} = \lim_{T \rightarrow 0} (h_+ h_- + 1)(h_-^2)^{1+\nu} \quad (20)$$

The separatrix S has $c_S = 0.284\dots$. The $c < c_S$ trajectories — the trajectories below S — are all roughly realistic as cosmologies. They all start with a big bang $a \sim t^{1/\sqrt{3}}$ then decelerate then accelerate to a big blow-up $a \sim t^{-1/\sqrt{3}}$, all driven by purely gravitational dark energy-momentum. The gap between the $C=0$ cosmological solution and the separatrix S means that the cosmological solution (8) is stable against real-time perturbations. The zone of stability on the side $C > 0$ shrinks to zero as $T \rightarrow 0$.

The two-dimensional renormalization group was developed into a comprehensive fundamental theory of physics in which the 2-d renormalization group acts not on a space of classical space-time fields but rather on measures on that space, producing not a solution of the classical field equations but rather a measure — a functional integral — expressing a solution of the quantum field equations [8–10]. The present exploratory calculations of cosmology from the 2-d renormalization group will need to be connected to this fundamental theory. Assumptions such as $SO(4)$ space-time symmetry will need to be justified by properties of the 2-d renormalization group flow — by stability properties of the fixed point and/or as the dynamical result of a distinguished trajectory of the rg flow.

The analyticity assumption needs precise characterization and justification. The assumption here that $f_T(T)$ and $v_T(T)$ should be analytic in T singles out the $C = 0$ cosmological solution. Analyticity serves as a selection principle providing specificity. It should express a physical principle.

The next exploratory steps will add standard model fields to the calculation, then spatial fluctuations and quantum effects, hoping to find analytic solutions of 2-d renormalization group fixed point equations that capture more and more qualitative features of the cosmology of the real world, aiming to find a solution that can be tested in quantitative detail. This would provide support for the fundamental theory [8] in which a quantum version of the two-dimensional renormalization group acts mechanically to produce physics and would provide guidance for deriving real world cosmology from that fundamental theory.

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* dfriedan@gmail.com; www.physics.rutgers.edu/~friedan/

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- [5] Calculations are shown in the accompanying supplemental material which consists of a note *Calculations for "Cosmology ..."* and three SageMath [6] notebooks of numerical calculations. The supplemental material can be found at http://www.physics.rutgers.edu/pages/friedan/papers/2019/Cosmology_I_supplementary_material/ or at https://share.cocalc.com/share/6a7035ba-9879-4b05-b3d0-688b1309a21c/Cosmology_I_supplementary_material/?viewer=share/.
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