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Entropy Flow Through Near-Critical Quantum Junctions

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Abstract This is the continuation of Friedan (J Stat Phys, 2017. doi: 10.1007/s10955-017-1752-8). Elementary formulas are derived for the flow of entropy through a circuit junction in a near-critical quantum circuit close to equilibrium, based on the structure of the energy–momentum tensor at the junction. The entropic admittance of a near-critical junction in a bulk-critical circuit is expressed in terms of commutators of the chiral entropy currents. The entropic admittance at low frequency, divided by the frequency, gives the change of the junction entropy with temperature—the entropic "capacitance". As an example, and as a check on the formalism, the entropic admittance is calculated explicitly for junctions in bulk-critical quantum Ising circuits (free fermions, massless in the bulk), in terms of the reflection matrix of the junction. The half-bit of information capacity per end of critical Ising wire is re-derived by integrating the entropic "capacitance" with respect to temperature, from $T = 0$ to $T = \infty$.

Keywords Quantum computers · Quantum statistical mechanics · Quantum transport · Entropy transport

1 Summary

This paper continues the elementary investigation of entropy flow in near-critical quantum circuits close to equilibrium begun in [7]. The entropy current operator, which is just the

In memory of Leo Kadanoff.

This is the second part of a two-part work originally published in 2005 as [5] and [6], here revised for clarity following helpful suggestions of the referee. The first part is [7] in this volume.

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energy current divided by the temperature, $j_S(x,t) = k\beta T_t^x(x,t)$, is used to analyze the flow of entropy through the general junction in a near-critical quantum circuit, especially when the quantum wires are critical in the bulk so that the only departure from criticality is in the junction. Entropy flows in and out of the junction in response to changes in the entropic potentials in the wires, which are just the temperature drops in the wires. The linear response coefficient for the entropy current is the entropic admittance of the junction. The junction entropy, $s(T)$, is the "charge" in the junction. The entropic "capacitance"—the temperature derivative, $\mathrm{d}s/\mathrm{d}T$—is extracted from the low frequency limit of the entropic admittance. The information capacity of the junction, $s(\infty) - s(0)$ is found by integrating with respect to T. The junction entropy itself needs a global calculation, but changes in the junction entropy can be determined locally, by studying the entropy currents near the junction. A trivial example—the general junction in a bulk-critical Ising circuit (i.e., free fermions, massless in the bulk)—is done explicitly, to illustrate the formalism and check that it works. Definitions, notation and motivation are carried over from [7]. As explained there, the near-critical one-dimensional quantum systems under consideration are those that are described by 1+1 dimensional relativistic quantum field theories, and they are assumed to be close to equilibrium.

A junction consists of N quantum wires joined at a single point. The wires are labelled by indices $A, B = 1 \ldots N$. Each wire is parametrized by a spatial coordinate $x \geq 0$. The endpoints, $x = 0$, are all identified to form a single point, the junction. The entropy flow through the junction is analyzed in terms of the entropy currents in the N external wires attached to the junction. The junction is treated as a black box, from the point of view of the larger circuit containing it. The single point $x = 0$ might stand for a complicated sub-circuit, but the internal structure of the junction shows itself only by its effects on the flow of entropy in and out of the junction.

Suppose that a small variation is made in the entropic potential in each wire, $\Delta V_S(t)^B = \mathrm{e}^{-i\omega t}\Delta V_S(0)^B$, alternating at frequency ω. The change in the entropic potential is the local temperature drop, $\Delta V_S(t)^B = -\Delta T(t)^B$, on wire B outside the junction. The changing entropic potentials in the wires cause entropy currents to flow through the wires, in and out of the junction, given by a linear response formula:

$$\Delta I_S(t)_A = \sum_{B=1}^{N} Y_S(\omega)_{AB}\,\Delta V_S(t)^B. \tag{1}$$

The entropic admittance matrix, $Y_S(\omega)_{AB}$, describes the entropy flow characteristics of the junction. Entropic admittance has fundamental units $k^2/\hbar$.

When the wires are bulk-critical, the entropy current is a sum of chiral currents, $j_S(x,t)_A = j_R(x,t)_A - j_L(x,t)_A$, where $j_R(x,t)_A = j_R(x - vt)_A$ is the right-moving entropy current in wire A and $j_L(x,t)_A = j_L(x + vt)_A$ is the left-moving entropy current, v being the speed of "light" in the relativistic quantum field theory of the one-dimensional bulk-critical system. These are just the chiral energy currents of the conformally invariant bulk-critical system, divided by the temperature, T. The Kubo formula for the entropic admittance is re-written in terms of the chiral entropy currents:

$$Y_S(\omega)_{AB} = \frac{1}{i\omega}\int_{-\infty}^{\infty} \mathrm{d}t_1\ \mathrm{e}^{-i\omega(t_1-t_2)}\left\langle \frac{i}{\hbar}\big[-j_L(0,t_1)_B,\, j_R(0,t_2)_A - j_L(0,t_2)_A\big]\right\rangle_{eq}. \tag{2}$$

The time integral does not have the restriction $t_1 \leq t_2$ which expresses causality in the general Kubo formula: the fact that the response happens after the perturbation. Causality is automatically enforced in (2) by chirality, since a right-moving current cannot affect the junction, and a perturbation at the junction cannot produce any left-moving current. Equation (2) exhibits $Y_S(\omega)_{AB}$ as a well-behaved analytic function of ω, free from the possibility of a short-time divergence, which can arise in the general Kubo formula in quantum field theory because of the restriction $t_1 \le t_2$.

The Kubo formula is also re-written, again for bulk-critical wires, in the form

$$\begin{aligned}\frac{1}{i\omega}\int_{-\infty}^{t_2} \mathrm{d}t_1\, \mathrm{e}^{-i\omega(t_1-t_2)} &\left\langle \frac{i}{\hbar}\big[j_S(x_1,t_1)_B, j_S(x_2,t_2)_A\big]\right\rangle_{eq} \\ &= \mathrm{e}^{i\omega(x_1+x_2)/v} Y_S(\omega)_{AB} - \left(\mathrm{e}^{i\omega|x_1-x_2|/v} - \mathrm{e}^{i\omega(x_1+x_2)/v}\right) \\ &\quad \times \delta_{AB}\frac{c}{12}\frac{2\pi k^2}{\hbar}\left[1+\left(\frac{\hbar\beta}{2\pi}\right)^2\omega^2\right] \qquad (3)\end{aligned}$$

where $x_1, x_2 > 0$ are arbitrary points on the wires and c is the bulk conformal central charge. This formula gives a way to extract $Y_S(\omega)_{AB}$ from a measurement performed at a distance from the junction, by generating entropy current at an arbitrary point $x_1 > 0$ in wire A, then detecting the induced entropy current at an arbitrary point $x_2 > 0$ in wire B. These two elementary formulas for the entropic admittance are the basic results of this paper.

The net entropy current leaving the junction is $\sum_{A=1}^{N} \Delta I_S(t)_A$, so the change in the junction entropy, $\Delta s(t)$, is given by $\partial_t \Delta s(t) = -i\omega\Delta s(t) = \sum_{A=1}^{N} \Delta I_S(t)_A$, so

$$\Delta s(t) = \frac{1}{i\omega}\sum_{A,B=1}^{N} Y_S(\omega)_{AB}\,\Delta V_S(t)^B. \qquad (4)$$

The change in the junction entropy with the junction temperature is

$$\frac{\mathrm{d}s}{\mathrm{d}T} = \lim_{\omega\to 0}\frac{1}{i\omega}\sum_{A,B=1}^{N} Y_S(\omega)_{AB} \qquad (5)$$

because raising the temperature of the junction by ΔT is equivalent to dropping the temperature on all of the wires outside the junction by the same amount, $\Delta V_S(t)^B = -\Delta T(t)^B = \Delta T$. Equation (5) also gives the specific heat of the junction, $T\mathrm{d}s/\mathrm{d}T$. Integrating $\mathrm{d}s/\mathrm{d}T$ with respect to T gives the junction entropy, $s(T)$, as a function of temperature, up to an additive constant. Integrating $\mathrm{d}s/\mathrm{d}T$ from $T = 0$ to $T = \infty$ gives the total information capacity of the junction, $s(\infty) - s(0)$. This does not give the junction entropy itself. That requires a global calculation. But *changes* in $s(T)$ can be calculated locally, near the junction, using the entropy current.

$Y_S(\omega)_{AB}$ is a response function, therefore analytic in the upper half-plane. Equation (2) exhibits $Y_S(\omega)_{AB}$ as the Fourier transform in time of the equilibrium expectation value of a commutator of local operators, so the only singularities in $Y_S(\omega)_{AB}$ at temperature $T > 0$ should be poles in the lower half-plane. The gap between the real axis and the nearest pole of $Y_S(\omega)_{AB}$ is the inverse of the characteristic response time of the junction in the channel AB. The zero temperature limit of these gaps can serve as analogs of the mass gap in a bulk quantum field theory.

The conformal invariance of the bulk-critical wires implies a vanishing formula:

$$Y_S(2\pi i/\hbar\beta)_{AB} = 0. \qquad (6)$$

In a separate paper [9], this vanishing formula is used to re-write the proof of the gradient formula for the junction beta-function in terms of the real time statistical mechanics of the entropy current. The gradient formula, $\partial s/\partial\lambda^a = -g_{ab}(\lambda)\beta^b$, was originally proved in the less physical language of euclidean quantum field theory [8].

As an example, and as a check on the formalism, (2) is used to derive an explicit formula for $Y_S(\omega)_{AB}$ for the general junction in a bulk-critical quantum Ising circuit. The 1+1 dimensional quantum field theory of such a circuit is a theory of free Majorana fermions, massless in the bulk [2,4,10,11]. An Ising junction is characterized by its reflection matrix, $R(\omega)^A_B$, which gives the amplitude for left-moving fermions of energy $\hbar\omega$ to enter the junction through wire B, then leave the junction as right-moving fermions through wire A. Equation (2) is used to express the entropic admittance in terms of the reflection matrix:

$$Y_S(\omega)_{AB} = \frac{k^2\hbar\beta^2}{2\pi\omega}\int\int \mathrm{d}\omega_1\mathrm{d}\omega_2\,\delta(\omega_1+\omega_2-\omega)\left[R(\omega_1)^A_B R(\omega_2)^A_B - \delta^A_B\right] \times\frac{1}{8}(\omega_1-\omega_2)^2\left(\frac{1}{1+\mathrm{e}^{-\beta\hbar\omega_2}}-\frac{1}{\mathrm{e}^{\beta\hbar\omega_1}+1}\right). \tag{7}$$

Substituting in (5) gives the specific heat of the junction,

$$T\frac{\mathrm{d}s}{\mathrm{d}T} = \frac{k}{2\pi i}\int \mathrm{d}\eta\ \mathrm{Tr}\left[R^{-1}R'(\eta)\right]\ \frac{(\beta\hbar\eta)^2\mathrm{e}^{\beta\hbar\eta}}{2\left(\mathrm{e}^{\beta\hbar\eta}+1\right)^2}. \tag{8}$$

Integrating with respect to T gives $s(T)$ up to a constant. These formulas apply to the general Ising junction. $R(\omega)$ can depend on the temperature, as when the junction has internal structure. When $R(\omega)$ does *not* depend on the temperature, the integral over T can be performed explicitly, giving

$$s(T)-s(0) = \frac{k}{2\pi i}\int \mathrm{d}\eta\ \mathrm{Tr}\left[R^{-1}R'(\eta)\right]\ \frac{1}{2}\left[\frac{\beta\hbar\eta}{\mathrm{e}^{\beta\hbar\eta}+1}+\ln\left(1+\mathrm{e}^{-\beta\hbar\eta}\right)\right]. \tag{9}$$

The information capacity is then

$$s(\infty)-s(0) = \frac{k}{2\pi i}\int \mathrm{d}\eta\ \mathrm{Tr}\left[R^{-1}R'(\eta)\right]\ \frac{1}{2}\ln 2. \tag{10}$$

The information capacity consists of one half-bit for each pole of $R(\omega)^A_B$ in the lower half-plane (or each zero in the upper half-plane), when $R(\omega)$ is independent of temperature.

An elementary Ising junction, without substructure, has a reflection matrix that is independent of temperature, of the form

$$R(\omega) = \frac{\omega - i\lambda^T\lambda}{\omega + i\lambda^T\lambda} \tag{11}$$

where the matrix λ^A_B parametrizes the junction couplings (generalizing the boundary magnetic field in a 1-wire junction). Equation (10) then gives $s(\infty)-s(0) = (N/2)k\ln 2$. The information capacity of an elementary Ising junction is one half-bit per end of critical Ising wire. This reproduces a well-known result, at least for $N=1$. For the case $N=1$, the formula for $s(T)-s(0)$, obtained from (9) and (11), agrees with the formula for the boundary free energy and entropy of a single bulk-critical Ising wire, calculated directly [4,11]. The case $N=2$ is equivalent to a pair of free Majorana fermion fields on a single wire with boundary. When the reflection matrix is $U(1)$-invariant, this is equivalent to a single free massless Dirac fermion field on a wire with boundary, which is equivalent to the Toulouse limit of the spin-1/2 Kondo model. In this case, Eq. (9) for the boundary entropy is identical to the formula for the boundary entropy in the spin-1/2 Kondo model, specialized to the Toulouse limit, as discussed in [12], for example. Equation (8) for the junction specific heat is somewhat more general, allowing for the reflection matrix $R(\omega)$ to be temperature-dependent. Still, the field theory of a bulk-critical Ising circuit is a theory of free fermions, so there is no difficulty to calculate the junction specific heat. It is done here as an illustration and check of the method of using the entropy current to describe the flow of entropy through the junction. The direct calculation of the junction entropy is global. First, the entropy of the whole system is calculated, then the bulk entropy of the wires is subtracted. What remains is the junction entropy. This is a subtle calculation, because boundary conditions are needed at the far ends of the wires, whose contributions to the total entropy must also be subtracted. Using the entropy current to calculate *changes* in the junction entropy is a local method that can be carried out in an arbitrarily small neighborhood of the junction.

2 Entropy Flow Through a Junction

The energy–momentum tensor of a quantum circuit consists of a bulk part, located in the wires, and a contribution located in the junction (see Appendix 1). The bulk energy–momentum tensor in wire A is $T^{\mu}_{\nu}(x,t)_A$. The junction contributes only to the energy density. Its contribution is $\delta(x)T^{t}_{t}(t)_{junct}$, where $T^{t}_{t}(t)_{junct}$ is the junction energy. The junction energy is conventionally written $T^{t}_{t}(t)_{junct} = -\theta(t)$. Energy and momentum are locally conserved in each bulk wire:

$$0 = \partial_\mu T^{\mu}_{\nu}(x,t)_A \,. \tag{12}$$

Energy is conserved at the junction:

$$0 = -\partial_t\theta(t) + \sum_{A=1}^{N} T^{x}_{t}(0,t)_A. \tag{13}$$

The hamiltonian is

$$H_0 = -\theta(t) + \sum_{A=1}^{N}\int_0^\infty \mathrm{d}x\; T^{t}_{t}(x,t)_A. \tag{14}$$

The departure from criticality is expressed by the trace of the energy–momentum tensor, written $\Theta(x,t) = -T^{\mu}_{\mu}(x,t)$. It consists of the bulk contribution in each wire, $\Theta(x,t)_A = -T^{\mu}_{\mu}(x,t)_A$, and the contribution located in the junction, $\delta(x)\theta(t)$. When the wires are bulk-critical, the $\Theta(x,t)_A$ all vanish. The departure from criticality is then entirely in the junction, expressed by

$$\theta(t) = \beta^a(\lambda)\phi_a(t) \tag{15}$$

where the $\phi_a(t)$ are the relevant and marginal operators located in the junction, and $\beta^a(\lambda)$ is the junction beta-function. The junction is critical when a change of scale has no effect, which is equivalent to $\theta(t)$ being a multiple of the identity, which is equivalent to

$$0 = \partial_t\theta(t) = \sum_{A=1}^{N} T^{x}_{t}(0,t)_A \tag{16}$$

which is also the condition that the boundary energy be stationary.

The entropy current operators in the wires are

$$j_S(x,t)_A = k\beta T^{x}_{t}(x,t)_A. \tag{17}$$

The entropy density operators in the wires are

$$\rho_S(x,t)_A = k\beta T^t_t(x,t)_A - \langle k\beta T^t_t(x,t)_A\rangle_{eq}. \tag{18}$$

In addition, the junction makes a contribution, $\delta(x)q_S(t)$, to the entropy density operator, where

$$q_S(t) = -k\beta\theta(t) + \langle k\beta\theta(t)\rangle_{eq}. \tag{19}$$

$q_S(t)$ is the junction entropy operator. The entropy density operator measures the variation of the entropy density away from its equilibrium value. $q_S(t)$ measures the variation of the junction entropy away from its equilibrium value:

$$\Delta s(t) = \langle q_S(t)\rangle. \tag{20}$$

Conservation of energy at the junction implies conservation of entropy:

$$0 = \partial_t q_S(t) + \sum_{A=1}^{N} j_S(0,t)_A. \tag{21}$$

Any change in the junction entropy is equal to the net flow of entropy into the junction.

The entropy flow characteristics of the junction are its responses to small changes in the entropic potentials on the wires. First consider a 1-wire junction, which is simply the boundary of a wire. Put a small alternating entropic potential, $\Delta V_S(t)$, outside the junction, constant along the wire:

$$\Delta\Phi_S(x,t) = \Delta V_S(t) = \mathrm{e}^{-i\omega t}\Delta V_S(0) \qquad \text{for } x > x_1 \tag{22}$$

where x_1 is some fixed point very close to 0. The hamiltonian is perturbed by

$$\Delta H = \int_0^\infty \mathrm{d}x\ \rho_S(x,t)\Delta\Phi_S(x,t) = \Delta V_S(t)\int_{x_1}^\infty \mathrm{d}x\ \rho_S(x,t). \tag{23}$$

Entropy current will flow in response to the perturbation, in and out of the boundary, at frequency ω. In the linear response approximation, the induced current is

$$\Delta I_S(t) = \Delta\langle j_S(0,t)\rangle = Y_S(\omega)\Delta V_S(t), \tag{24}$$

where the linear response coefficient, $Y_S(\omega)$, is the entropic admittance of the boundary. $Y_S(0) = 0$ for a 1-wire junction, because the entropy flowing into the boundary encounters the same entropic potential as the entropy flowing out, when $\omega = 0$.

A uniform small change in the entropic potential, constant over the whole system, has no effect, because the hamiltonian is perturbed by a multiple of itself. Therefore, raising the entropic potential by $\Delta V_S(t)$ everywhere outside the boundary is equivalent to lowering the potential on the boundary by $\Delta V_S(t)$. The perturbation can just as well be written $\Delta H(t) = -\Delta V_S(t)q_S(t)$. Now measure the entropy current at a point $x_2 > 0$ very near the boundary. The Kubo formula for the entropy current is

$$\begin{aligned}\Delta\langle j_S(x_2,t_2)\rangle &= \textstyle\int_{-\infty}^{t_2} \mathrm{d}t_1 \left\langle \frac{i}{\hbar}\left[\Delta H(t_1), j_S(x_2,t_2)\right]\right\rangle_{eq} \\ &= \Delta V_S(0,t_2) \textstyle\int_{-\infty}^{t_2} \mathrm{d}t_1\ \mathrm{e}^{-i\omega(t_1-t_2)}\left\langle \frac{-i}{\hbar}\left[q_S(t_1), j_S(x_2,t_2)\right]\right\rangle_{eq}\end{aligned} \tag{25}$$

so the entropic admittance is

$$Y_S(\omega) = \lim_{x_2\to 0} \int_{-\infty}^{t_2} \mathrm{d}t_1\ \mathrm{e}^{-i\omega(t_1-t_2)} \left\langle \frac{-i}{\hbar}\big[q_S(t_1), j_S(x_2,t_2)\big]\right\rangle_{eq} \tag{26}$$

Conservation of entropy at the junction, $0 = \partial_t q_S(t) + j_S(0,t)$, allows the Kubo formula to be written

$$Y_S(\omega) = \lim_{x_2\to 0} \frac{1}{i\omega} \int_{-\infty}^{t_2} \mathrm{d}t_1\ \mathrm{e}^{-i\omega(t_1-t_2)} \left\langle \frac{i}{\hbar}\big[j_S(0,t_1), j_S(x_2,t_2)\big]\right\rangle_{eq}. \tag{27}$$

The change of the boundary entropy is then

$$\Delta s(t) = \langle q_S(t)\rangle = (i\omega)^{-1}\langle j_S(t)\rangle = (i\omega)^{-1} Y_S(\omega)\Delta V_S(t). \tag{28}$$

In the static limit,

$$\Delta s = C_S \Delta V_S \tag{29}$$

where

$$C_S = \lim_{\omega\to 0} (i\omega)^{-1} Y_S(\omega). \tag{30}$$

Change of entropic charge divided by change of entropic potential might be called the entropic "capacitance" of the boundary (abusing the electrical analogy). The entropic potential of the boundary has been *lowered* by ΔV_S, which is equivalent to *raising* the temperature of the boundary by $\Delta T = \Delta V_S$, so the entropic "capacitance" is

$$C_S = \frac{\mathrm{d}s}{\mathrm{d}T}. \tag{31}$$

The specific heat of the boundary is $T\mathrm{d}s/\mathrm{d}T = TC_S$.

For an N-wire junction, impose small alternating potentials, $\Delta V_S(t)^B$, $B = 1\ldots N$, on the wires outside the junction:

$$\Delta\Phi_S(x,t)^B = \Delta V_S(t)^B = \mathrm{e}^{-i\omega t}\Delta V_S(0)^B \qquad \text{for } x > x_1. \tag{32}$$

The induced entropy current flowing out of the junction through wire A is

$$\Delta I_S(t)_A = \lim_{x_2\to 0} \Delta\langle j_S(x_2,t)_A\rangle = \sum_{B=1}^{N} Y_S(\omega)_{AB}\Delta V_S(t)^B. \tag{33}$$

The linear response coefficients, $Y_S(\omega)_{AB}$, form the entropic admittance matrix of the junction. The entropic potential in the wires perturbs the hamiltonian by

$$\Delta H(t)_1 = \sum_{B=1}^{N}\int_{x_1}^{\infty} \mathrm{d}x\ \rho_S(x,t)_B \Delta V_S(t)^B \tag{34}$$

which is gauge equivalent to

$$\Delta H(t) = \sum_{B=1}^{N} j_S(x_1,t)_B \frac{1}{i\omega}\Delta V_S(t)^B. \tag{35}$$

That is, the difference is

$$\Delta H(t) - \Delta H(t)_1 = \partial_t\left[\sum_{B=1}^{N}\int_{x_1}^{\infty}\mathrm{d}x\ \rho_S(x,t)_B \frac{1}{i\omega}\Delta V_S(t)^B\right]. \tag{36}$$

The Kubo formula for the induced entropy current at a point, x_2, in wire A is

$$\begin{aligned}\Delta\langle j_S(x_2,t_2)_A\rangle &= \textstyle\int_{-\infty}^{t_2} \mathrm{d}t_1 \left\langle \frac{i}{\hbar}\left[\Delta H(t_1)_1, j_S(x_2,t_2)_A\right]\right\rangle_{eq} \\ &= \textstyle\int_{-\infty}^{t_2} \mathrm{d}t_1 \left\langle \frac{i}{\hbar}\left[\Delta H(t_1), j_S(x_2,t_2)_A\right]\right\rangle_{eq} \\ &\quad - \textstyle\frac{1}{i\omega}\Delta V_S(t_2)^B \left\langle \int_{x_1}^{\infty} \mathrm{d}x\ \frac{i}{\hbar}\left[\rho_S(x,t_2)_B, j_S(x_2,t_2)_A\right]\right\rangle_{eq}.\end{aligned} \tag{37}$$

If $x_2 < x_1$, then the equal-time commutator in the second term is zero, because the operators are separated in space. On the other hand, if $x_2 > x_1$, then

$$\begin{aligned}\left\langle \int_{x_1}^{\infty} \mathrm{d}x\ \frac{i}{\hbar}\left[\rho_S(x,t_2)_B, j_S(x_2,t_2)_A\right]\right\rangle_{eq} &= k\beta\left\langle \int_{x_1}^{\infty} \mathrm{d}x\ \frac{i}{\hbar}\left[T_t^t(x,t_2)_B, j_S(x_2,t_2)_A\right]\right\rangle_{eq} \\ &= k\beta\delta_{AB}\left\langle \partial_t j_S(x_2,t_2)_A\right\rangle_{eq}\end{aligned} \tag{38}$$

which vanishes, since nothing changes with time when the system is in equilibrium. Since the second term on the rhs of (37) always vanishes,

$$\begin{aligned}\Delta\langle j_S(x_2,t_2)_A\rangle = \sum_{B=1}^{N}\int_{-\infty}^{t_2} \mathrm{d}t_1 \\ \times \frac{1}{i\omega}\Delta V_S(t_1)^B\left\langle \frac{i}{\hbar}\left[j_S(x_1,t_1)_B, j_S(x_2,t_2)_A\right]\right\rangle_{eq}\end{aligned} \tag{39}$$

and the entropic admittance matrix is

$$\begin{aligned}Y_S(\omega)_{AB} = \lim_{x_1,x_2\to 0} \frac{1}{i\omega}\int_{-\infty}^{t_2} \mathrm{d}t_1 \\ \times\, \mathrm{e}^{-i\omega(t_1-t_2)}\left\langle \frac{i}{\hbar}\left[j_S(x_1,t_1)_B, j_S(x_2,t_2)_A\right]\right\rangle_{eq}.\end{aligned} \tag{40}$$

The change in the junction entropy is

$$\partial_t s(t) = \partial_t\langle q_S(t)\rangle = \sum_{A=1}^{N}\langle -j_S(0,t)_A\rangle \tag{41}$$

so

$$\Delta s(t) = \frac{1}{i\omega}\sum_{A=1}^{N}\sum_{B=1}^{N} Y_S(\omega)_{AB}\Delta V_S(t)^B. \tag{42}$$

In the static limit,

$$\Delta s = \sum_{B=1}^{N} C_{SB}\Delta V_S^B \tag{43}$$

where

$$C_{SB} = \lim_{\omega\to 0}\frac{1}{i\omega}\sum_{A=1}^{N} Y_S(\omega)_{AB}. \tag{44}$$

C_{SB} is the entropic "capacitance" of the junction (still abusing the electrical analogy). Again, raising the temperature of the junction by ΔT is equivalent to raising the entropic potential everywhere outside the junction by $\Delta V_S^B = \Delta T$, so

$$\frac{\mathrm{d}s}{\mathrm{d}T} = C_S = \sum_{B=1}^{N} C_{SB} = \lim_{\omega \to 0} \frac{1}{i\omega} \sum_{A,B=1}^{N} Y_S(\omega)_{AB} \tag{45}$$

where C_S is the total entropic "capacitance" of the junction.

3 The Continuity Equation at a Junction

The change in the junction entropy is given by the time evolution equation

$$\partial_t \langle q_S(t) \rangle = \langle \partial_t q_S(t) \rangle + \left\langle \frac{i}{\hbar} \left[\Delta H(t), q_S(t) \right] \right\rangle. \tag{46}$$

The equal-time commutator vanishes, because the entropic potential, $\Phi_S(x,t)^A = \Delta V_S(t)^A$, is imposed slightly outside the junction, in the region $x > x_1$. But the change in the junction entropy should properly include also the changes in entropy in the wires very near the junction:

$$\partial_t \left\langle \sum_{A=1}^{N} \int_0^{x_2} \mathrm{d}x \;\; \rho_S(x,t)_A \right\rangle. \tag{47}$$

This can be calculated using the continuity equation for entropy flow in wires derived in [7]:

$$\begin{aligned} \partial_t \langle \rho_S(x,t) \rangle + \partial_x \langle j_S(x,t) \rangle = -k\beta \partial_x \Phi_S(x,t) \langle j_S(x,t) \rangle \\ -k\beta \partial_x \left[\Phi_S(x,t) \langle j_S(x,t) \rangle \right]. \end{aligned} \tag{48}$$

The result is

$$\begin{aligned} \partial_t \left\langle q_S(t) + \textstyle\sum_{A=1}^{N} \int_0^{x_2} \mathrm{d}x \;\; \rho_S(x,t\;)_A \right\rangle + \left\langle \textstyle\sum_{A=1}^{N} j_S(x_2,t)_A \right\rangle = \\ -k\beta \Delta V_S(t)^A \left\langle j_S(x_1,t)_A + j_S(x_2,t)_A \right\rangle. \end{aligned} \tag{49}$$

Note that this is still a local calculation at the junction. The result depends only on the values of the entropic potential in the wires near the junction. Now take $x_1, x_2 \to 0$, including in the change of junction entropy the changes of entropy in the wires near the junction:

$$\partial_t \langle q_S(t) \rangle + \sum_{A=1}^{N} \langle j_S(0,t)_A \rangle = -2k\beta \sum_{A=1}^{N} \Delta V_S(t)^A \langle j_S(0,t)_A \rangle. \tag{50}$$

This is the continuity equation for entropy at the junction. It is an exact equation. There is no linear response approximation, no assumption that $\Delta V_S(t)^A$ is small, so it is better written

$$\partial_t \langle q_S(t) \rangle + \sum_{A=1}^{N} \langle j_S(0,t)_A \rangle = -2k\beta \sum_{A=1}^{N} V_S(t)^A \langle j_S(0,t)_A \rangle \tag{51}$$

or

$$\partial_t s(t) + \sum_{A=1}^{N} I_S(t)_A = -2k\beta \sum_{A=1}^{N} V_S(t)^A I_S(t)_A \tag{52}$$

where $I_S(t)_A = \langle j_S(0,t)_A \rangle$.

4 Junctions in Bulk-Critical Quantum Circuits

Assume now that all the wires are critical in the bulk:

$$\Theta(x,t)_A = -T_\mu^\mu(x,t)_A = 0. \tag{53}$$

Any departure from criticality is in the junction. The entropy current in each bulk-critical wire separates into chiral entropy currents (see Appendix 2):

$$j_S(x,t)_A = j_R(x,t)_A - j_L(x,t)_A \tag{54}$$

$$\rho_S(x,t)_A = v^{-1} j_R(x,t)_A + v^{-1} j_L(x,t)_A \tag{55}$$

where

$$j_R(x,t)_A = j_R(x - vt)_A \tag{56}$$

is the right-moving entropy current, and

$$j_L(x,t)_A = j_L(x + vt)_A \tag{57}$$

is the left-moving entropy current. The operators $j_R(x - vt)_A$ and $j_L(x + vt)_A$ are defined on the entire real line, because, although x cannot be negative, the time t can range from $-\infty$ to $+\infty$. The rate of change of the junction entropy is the net rate of inflow,

$$\partial_t q_S = \sum_{A=1}^{N} j_L(0,t)_A - \sum_{A=1}^{N} j_R(0,t)_A. \tag{58}$$

The equilibrium expectation values of commutators of the chiral currents vanish in regions of space-time where effects are forbidden by causality and chirality:

$$\left\langle \left[j_R(x_1,t_1)_B, j_R(x_2,t_2)_A \right] \right\rangle_{eq} = 0 \quad \text{if } A \neq B \text{ or } t_2 - t_1 \neq (x_2 - x_1)/v \tag{59}$$

$$\left\langle \left[j_L(x_1,t_1)_B, j_L(x_2,t_2)_A \right] \right\rangle_{eq} = 0 \quad \text{if } A \neq B \text{ or } t_2 - t_1 \neq (x_1 - x_2)/v \tag{60}$$

$$\left\langle \left[j_R(x_1,t_1)_B, j_L(x_2,t_2)_A \right] \right\rangle_{eq} = 0 \quad \text{if } t_1 - t_2 < (x_1 + x_2)/v \tag{61}$$

$$\left\langle \left[j_L(x_1,t_1)_B, j_R(x_2,t_2)_A \right] \right\rangle_{eq} = 0 \quad \text{if } t_2 - t_1 < (x_1 + x_2)/v. \tag{62}$$

The first two identities express the fact that any commutator of two currents of the same chirality is an equal-time commutator. The last two identities express the fact that the left-moving current at x_1 cannot affect the right-moving current at x_2 before a signal has time to travel from x_1 to the junction then back to x_2, and the fact that the right-moving current cannot affect the left-moving current at all. For the first identity, use time-translation invariance and chirality to write

$$\left\langle\left[j_R(x_1,t_1)_B,\, j_R(x_2,t_2)_A\right]\right\rangle_{eq} = \left\langle\left[j_R(x_1-vt_1+vt_2+vt,t_2)_B,\, j_R(x_2+vt,t_2)_A\right]\right\rangle_{eq} \tag{63}$$

for any t sufficiently large, such that $x_2 + vt > 0$ and $x_1 - vt_1 + vt_2 + vt > 0$. The commutator on the rhs is then an equal-time commutator, which vanishes if $A \neq B$ or if $x_1 - x_2 - vt_1 + vt_2 \neq 0$. The second identity, (60), is derived in the same way. For the third identity, (61), write

$$\left\langle\left[j_R(x_1,t_1)_B,\, j_L(x_2,t_2)_A\right]\right\rangle_{eq} = \left\langle\left[j_R(x_1-vt_1+vt_2-vt,t_2)_B,\, j_L(x_2+vt,t_2)_A\right]\right\rangle_{eq} \tag{64}$$

whenever $x_1 - vt_1 + vt_2 - vt > 0$ and $x_2 + vt > 0$, which is possible if and only if $t_1 - t_2 < (x_2 + x_1)/v$. The commutator on the rhs is then an equal-time commutator at nonzero spatial separation, so is zero. The fourth identity, (62), follows in the same way.

For $t_1 < t_2$, Eqs. (59), (60), and (61) imply that

$$\left\langle\left[j_S(x_1,t_1)_B,\, j_S(x_2,t_2)_A\right]\right\rangle_{eq} = \begin{cases} \left\langle\left[j_R(x_1,t_1)_B - j_L(x_1,t_1)_B,\, j_R(x_2,t_2)_A\right]\right\rangle_{eq} & \text{if } x_1 < x_2 \\ \left\langle\left[-j_L(x_1,t_1)_B,\, j_R(x_2,t_2)_A - j_L(x_2,t_2)_A\right]\right\rangle_{eq} & \text{if } x_1 > x_2. \end{cases} \tag{65}$$

Then

$$\frac{1}{i\omega}\int_{-\infty}^{t_2} \mathrm{d}t_1\; \mathrm{e}^{-i\omega(t_1-t_2)} \left\langle \frac{i}{\hbar}\left[j_S(x_1,t_1)_B,\, j_S(x_2,t_2)_A\right]\right\rangle_{eq} = \frac{1}{i\omega}\int_{-\infty}^{\infty} \mathrm{d}t_1\; \mathrm{e}^{-i\omega(t_1-t_2)} \begin{cases} \left\langle \frac{i}{\hbar}\left[j_R(x_1,t_1)_B - j_L(x_1,t_1)_B,\, j_R(x_2,t_2)_A\right]\right\rangle_{eq} & \text{if } x_1 < x_2 \\ \left\langle \frac{i}{\hbar}\left[-j_L(x_1,t_1)_B,\, j_R(x_2,t_2)_A - j_L(x_2,t_2)_A\right]\right\rangle_{eq} & \text{if } x_1 > x_2 \end{cases} \tag{66}$$

where the restriction to $t_1 \leq t_2$ in the time integrals on the rhs has been dropped because the integrands vanish when $t_1 > t_2$, by (59), (60), and (62).

A possible ambiguity in Eq. (40) for $Y_S(\omega)_{AB}$ is now apparent. Taking the limit $x_1, x_2 \to 0$ with $x_1 < x_2$ gives one expression for $Y_S(\omega)_{AB}$:

$$Y_S(\omega)^{12}_{AB} = \frac{1}{i\omega}\int_{-\infty}^{\infty} \mathrm{d}t_1\; \mathrm{e}^{-i\omega(t_1-t_2)} \times \left\langle \frac{i}{\hbar}\left[j_R(0,t_1)_B - j_L(0,t_1)_B,\, j_R(0,t_2)_A\right]\right\rangle_{eq}. \tag{67}$$

Taking the limit with $x_1 > x_2$ gives another expression:

$$Y_S(\omega)^{21}_{AB} = \frac{1}{i\omega}\int_{-\infty}^{\infty} \mathrm{d}t_1\; \mathrm{e}^{-i\omega(t_1-t_2)} \times \left\langle \frac{i}{\hbar}\left[-j_L(0,t_1)_B,\, j_R(0,t_2)_A - j_L(0,t_2)_A\right]\right\rangle_{eq}. \tag{68}$$

The difference involves only the commutators of currents of the same chirality. These commutators are completely determined by bulk conformal invariance [(see (171) and (172) of Appendix 2], with the result that

$$Y_S(\omega)^{21}_{AB} - Y_S(\omega)^{12}_{AB} = \delta_{AB} 2k\beta \langle j_S(0,t)_A \rangle, \tag{69}$$

so the ambiguity only exists when there are nonzero entropy currents flowing in equilibrium. As explained in [7] at the end of Sect. 1, persistent equilibrium currents are possible because of the isolation of the near-critical degrees of freedom from the environment. In the circuit laws for entropy, the choice between $Y_S(\omega)^{12}_{AB}$ and $Y_S(\omega)^{21}_{AB}$ is a matter of convention, as long as the convention is used consistently. For the local analysis of a junction connected to open wires, the appropriate choice depends on the situation. To describe the change in the entropy currents due to a change $\Delta T(t)$ in the junction temperature, the appropriate choice is $Y_S(\omega)^{12}_{AB}$:

$$\Delta I_S(t)_A = \sum_{B=1}^{N} Y_S(\omega)^{12}_{AB} \Delta T(t). \tag{70}$$

The entropic potential on the junction is changed by $-\Delta T(t)$, then the resulting currents are observed outside the junction, i.e., $x_1 < x_2$. On the other hand, to describe the change in the junction entropy due to changes, $\Delta T(t)^B = -\Delta V_S(t)^B$, in the temperatures of the wires, the appropriate choice is $Y_S(\omega)^{21}_{AB}$:

$$\Delta s(t) = \sum_{B=1}^{N} \left(\frac{1}{i\omega} \sum_{A=1}^{N} Y_S(\omega)^{21}_{AB} \right) \Delta V_S(t)^B. \tag{71}$$

The entropic potential is changed in the wires outside the junction, then the net flow of entropy into the junction in measured at the junction, i.e., $x_1 > x_2$.

For either choice of convention, $Y_S(\omega)_{AB}$ is manifestly a well-defined analytic function of the frequency, given by (67) or (68). The original Kubo formula, (40), has the possibility of a short-time divergence, because of the restriction to $t_1 \leq t_2$ in the integral over time. There is no such possibility in (67) or (68), since $Y_S(\omega)_{AB}$ is simply the Fourier transform in time of a commutator of local operators.

Define the Fourier modes of the chiral entropy currents by

$$j_R(x,t)_A = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \; \mathrm{e}^{-i\omega(t-x/v)} \; \tilde{j}_R(\omega)_A \tag{72}$$

$$j_L(x,t)_A = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \; \mathrm{e}^{-i\omega(t+x/v)} \; \tilde{j}_L(\omega)_A. \tag{73}$$

Equations (67) and (68) are equivalent to

$$2\pi\delta(\omega'+\omega)\,\hbar\omega Y_S(\omega)^{12}_{AB} = \Big\langle \big[\tilde{j}_R(\omega')_B - \tilde{j}_L(\omega')_B, \tilde{j}_R(\omega)_A \big] \Big\rangle_{eq} \tag{74}$$

$$2\pi\delta(\omega'+\omega)\,\hbar\omega Y_S(\omega)^{21}_{AB} = \Big\langle \big[-\tilde{j}_L(\omega')_B, \tilde{j}_R(\omega)_A - \tilde{j}_L(\omega)_A \big] \Big\rangle_{eq}. \tag{75}$$

These formulas are used below to calculate $Y_S(\omega)_{AB}$ for junctions in bulk-critical Ising circuits.

From now on, to simplify the discussion, assume that no equilibrium entropy currents are flowing through the junction. With this assumption,

$$Y_S(\omega)_{AB} = Y_S(\omega)^{12}_{AB} = Y_S(\omega)^{21}_{AB}. \tag{76}$$

When there are no equilibrium entropy currents, bulk conformal invariance implies (see (190) of Appendix 2):

$$\left\langle\left[\tilde{\jmath}_R(\omega')_A,\, \tilde{\jmath}_R(\omega)_B\right]\right\rangle_{eq} = -\delta_{AB}\; 2\pi\,\delta(\omega'+\omega)\frac{c}{12}2\pi k^2\omega\left[1+\left(\frac{\hbar\beta}{2\pi}\right)^2\omega^2\right] \quad (77)$$

$$\left\langle\left[\tilde{\jmath}_L(\omega')_A,\, \tilde{\jmath}_L(\omega)_B\right]\right\rangle_{eq} = -\delta_{AB}\; 2\pi\,\delta(\omega'+\omega)\frac{c}{12}2\pi k^2\omega\left[1+\left(\frac{\hbar\beta}{2\pi}\right)^2\omega^2\right] \quad (78)$$

This allows (66) to be re-written:

$$\begin{aligned}\frac{1}{i\omega}\int_{-\infty}^{t_2} \mathrm{d}t_1\; \mathrm{e}^{-i\omega(t_1-t_2)} \quad &\left\langle \frac{i}{\hbar}\left[j_S(x_1,t_1)_B,\, j_S(x_2,t_2)_A\right]\right\rangle_{eq} \\ &= \mathrm{e}^{i\omega(x_1+x_2)/v}\, Y_S(\omega)_{AB} - \left(\mathrm{e}^{i\omega|x_1-x_2|/v} - \mathrm{e}^{i\omega(x_1+x_2)/v}\right) \\ &\quad \times \delta_{AB}\,\frac{c}{12}\,\frac{2\pi k^2}{\hbar}\left[1+\left(\frac{\hbar\beta}{2\pi}\right)^2\omega^2\right] \end{aligned} \quad (79)$$

for any $x_1, x_2 \geq 0$. The lhs of this equation describes an experiment in which the system is perturbed at some arbitrary point, x_1, away from the junction, then the response is detected at a second arbitrary point, x_2, also away from the junction. Equation (79) says that the entropic admittance of the junction can be extracted from this measurement.

5 Properties of $Y_S(\omega)_{AB}$ in a Bulk-Critical Quantum Circuit

The entropic admittance of a junction in a bulk-critical circuit satisfies:

A. $Y_S(\omega)_{AB}$ is analytic in the upper half-plane.

B. $\overline{Y_S(\omega)}_{AB} = Y_S(-\bar{\omega})_{AB}$.

C. $Y_S(\omega) + Y_S(\omega)^\dagger \leq 0$ for all real ω.

D. $\sum_{A=1}^N Y_S(0)_{AB} = \sum_{B=1}^N Y_S(0)_{AB} = 0$.

E. The only singularities of $Y_S(\omega)_{AB}$ are poles lying below the real axis.

F. $Y_S(2\pi i/\hbar\beta)_{AB} = 0$.

All of these properties can be checked in the explicit example given by (142) below.

Property A expresses causality. The reality condition, property B, follows directly from (74) and (75) and the fact that the currents are self-adjoint. For the positivity condition, property C, write (74) and (75) in terms of the equilibrium two-point functions:

$$2\pi\,\delta(\omega'+\omega)\,\frac{\hbar\omega Y_S(\omega)^{12}_{AB}}{1-\mathrm{e}^{\beta\hbar\omega}} = \left\langle \tilde{\jmath}_R(\omega')_B\, \tilde{\jmath}_R(\omega)_A\right\rangle_{eq} - \left\langle \tilde{\jmath}_L(\omega')_B\, \tilde{\jmath}_R(\omega)_A\right\rangle_{eq} \quad (80)$$

$$2\pi\,\delta(\omega'+\omega)\,\frac{\hbar\omega Y_S(\omega)^{21}_{AB}}{1-\mathrm{e}^{\beta\hbar\omega}} = \left\langle \tilde{\jmath}_L(\omega')_B\, \tilde{\jmath}_L(\omega)_A\right\rangle_{eq} - \left\langle \tilde{\jmath}_L(\omega')_B\, \tilde{\jmath}_R(\omega)_A\right\rangle_{eq}. \quad (81)$$

Take the complex conjugate of (81), use the self-adjointness of the currents and the the reality condition, property B, and exchange A with B and ω with ω' to get

$$2\pi\,\delta(\omega'+\omega)\,\frac{\hbar\omega \overline{Y_S(\omega)^{21}_{BA}}}{1-\mathrm{e}^{\beta\hbar\omega}} = \left\langle \tilde{\jmath}_L(\omega')_B\, \tilde{\jmath}_L(\omega)_A\right\rangle_{eq} - \left\langle \tilde{\jmath}_R(\omega')_B\, \tilde{\jmath}_L(\omega)_A\right\rangle_{eq} \quad (82)$$

Add (80) and (82) and use the self-adjointness of the currents to get property C (still assuming no equilibrium entropy currents, so $Y_S(\omega)^{21}_{AB} = Y_S(\omega)^{12}_{AB} = Y_S(\omega)_{AB}$).

For property D, note that conservation at the junction, (58), implies

$$0 = \sum_{A=1}^{N} \left[\tilde{\jmath}_L(0)_A - \tilde{\jmath}_R(0)_A \right]. \tag{83}$$

In (80), set $\omega' = 0$ to get

$$2\pi\delta(\omega) \left(\frac{-1}{\beta}\right) \sum_{B=1}^{N} Y_S(0)^{12}_{AB} = \left\langle \sum_{B=1}^{N} \left[\tilde{\jmath}_R(0)_B - \tilde{\jmath}_L(0)_B \right] \tilde{\jmath}_R(\omega)_A \right\rangle_{eq} = 0 \tag{84}$$

so $\sum_{B=1}^{N} Y_S(0)^{12}_{AB} = 0$. In (81), set $\omega = 0$ to get

$$2\pi\delta(\omega') \left(\frac{-1}{\beta}\right) \sum_{A=1}^{N} Y_S(0)^{21}_{AB} = \left\langle \tilde{\jmath}_L(\omega')_B \sum_{A=1}^{N} \left[\tilde{\jmath}_L(0)_A - \tilde{\jmath}_R(0)_A \right] \right\rangle_{eq} = 0 \tag{85}$$

so $\sum_{A=1}^{N} Y_S(0)^{21}_{AB} = 0$. These results are exactly what is required in order that Eqs. (70) and 71 be physically sensible at $\omega = 0$.

Property E follows from (67) or (68), each of which displays $Y_S(\omega)_{AB}$ as the Fourier transform in time of the equilibrium expectation value of a commutator of local operators. The junction is a bounded system at nonzero temperature, so such an expectation value is a meromorphic function of ω. Causality implies that all its singularities lie below the real axis.

Property F, $Y_S(2\pi i/\hbar\beta)_{AB} = 0$, is derived from bulk conformal invariance, by Wick rotating to euclidean time $\tau = it$. The euclidean space-time of each bulk wire is a semi-infinite cylinder: $x > 0$, $\tau \sim \tau + \hbar\beta$. This euclidean space-time can be reinterpreted with x/v as the euclidean time coordinate and and $v\tau$ as the spatial coordinate. Space is then a circle of circumference $\hbar v\beta$. In this quantization, correlation functions on the semi-infinite cylinder are expectation values between the conformally invariant ground state at late time, at large x/v, and a boundary state at time $x/v = 0$.

In this quantization,

$$T_R(x + iv\tau)_A = \frac{-\hbar v^2}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi n(x+iv\tau)/\hbar v\beta} L_n \tag{86}$$

where the L_n are the Virsaro operators of the bulk conformal field theory in wire A (the constant of proportionality is found in Appendix 1). The L_n for $n \leq 1$, annihilate the conformally invariant ground state when they act to the left, at large time x/v. In a correlation function containing $T_R(x + iv\tau)_A$ for sufficiently large x, the contributions of the operators L_n, $n \leq 1$, vanish, because they are acting on the conformally invariant ground state. The two-point function given by the Fourier transform of (80) is:

$$(k\beta)^2 \left\langle [T_R(0)_B - T_L(0)_B]\; T_R(x - vt)_A \right\rangle_{eq} = \frac{1}{2\pi} \int \mathrm{d}\omega\; \mathrm{e}^{-i\omega(t-x/v)} \frac{\hbar\omega Y_S(\omega)_{AB}}{1 - \mathrm{e}^{\beta\hbar\omega}}. \tag{87}$$

The product of operators on the lhs is time-ordered for $t < 0$ (in the original, physical quantization). The Wick rotation of this equation is sensible in the region $-\hbar\beta < \tau < 0$,

$$(k\beta)^2\Big\langle [T_R(0)_B - T_L(0)_B]\ T_R(x + iv\tau)_A\Big\rangle_{eq} = \frac{1}{2\pi}\int \mathrm{d}\omega\ \mathrm{e}^{-\omega(\tau+ix/v)}\frac{\hbar\omega Y_S(\omega)_{AB}}{1-\mathrm{e}^{\beta\hbar\omega}}, \tag{88}$$

because the integrand decays exponentially when $\omega \to \pm\infty$, as long as $-\hbar\beta < \tau < 0$. Deforming the integration contour into the upper half-plane yields a sum over the thermal poles:

$$(k\beta)^2\Big\langle [T_R(0)_B - T_L(0)_B]\ T_R(x + iv\tau)_A\Big\rangle_{eq} = \sum_{n=1}^{\infty}\frac{2\pi n}{\beta} \times Y_S(2\pi i n/\hbar\beta)_{AB}\ \mathrm{e}^{-2\pi n(x+iv\tau)/\hbar v\beta} \tag{89}$$

The $n = 1$ term must vanish because it is due to the conformal generator L_1 acting on the conformally invariant ground state. So bulk conformal invariance implies $Y_S(2\pi i/\hbar\beta)_{AB} = 0$.

6 "Mass Gaps" for Junctions

The real-time response functions

$$\frac{1}{2\pi}\int \mathrm{d}\omega\ \mathrm{e}^{-i\omega(t_2-t_1)}Y_S(\omega)_{AB} \tag{90}$$

describe the entropy flowing out of the junction at time t_2 in wire A, in response to a perturbation in wire B at time $t_1 < t_2$. The integral can be evaluated by deforming the contour of integration into the lower half-plane, since $t_2 - t_1 > 0$. The response decays as e^{-t/t_r}, where $1/t_r$ is the gap between the real ω-axis and the nearest singularity in the lower half-plane. The time t_r is the the characteristic response time of the junction in the channel AB. At nonzero temperature, there will always be a gap. In the Ising example, (142) below, the characteristic response time of a 1-wire junction is $t_r = \lambda^2 + \pi/\hbar\beta$, where λ is the boundary coupling constant (the boundary magnetic field).

The interesting question is, what happens to these gaps in the limit $T \to 0$? The question is better posed in terms of the energy response functions

$$\Big\langle \frac{i}{\hbar}\big[T_t^x(0,t_1)_B, T_t^x(0,t_2)_A\big]\Big\rangle \tag{91}$$

because of the extra factor $1/T$ in the entropy current. When the junction is non-critical, then some of the $T = 0$ response functions will decay exponentially in time. If the junction is critical, then the $T = 0$ response functions will decay as powers of the time, or else vanish identically. The gaps, $1/t_r$, in each channel might be interpreted as the "mass gaps" of the junction, analogous to the mass gap in a bulk system.

7 Example: Junctions in Bulk-Critical Ising Circuits

As an example, consider the general junction in a bulk-critical Ising circuit. The wires are in the universality class of the 1+1 dimensional Ising model at its critical point. The circuit is described by the quantum field theory of a free Majorana fermion, massless in the bulk. There are two real, chiral, free fermion fields in each wire:

$$\psi_R(x,t)^A = \psi_R(x-vt)^A \tag{92}$$
$$\psi_L(x,t)^A = \psi_L(x+vt)^A. \tag{93}$$

The operators $\psi_R(x-vt)^A$ and $\psi_L(x+vt)^A$ are defined on the entire real line, because, although x cannot be negative, the time t can range from $-\infty$ to $+\infty$. The fermion fields satisfy canonical equal-time anti-commutation relations in the bulk. For $x, x' > 0$:

$$\left[\psi_R(x',t)^A, \psi_R(x,t)^B\right]_+ = \delta^{AB}\, \hbar v\, \delta(x'-x) \tag{94}$$
$$\left[\psi_L(x',t)^A, \psi_L(x,t)^B\right]_+ = \delta^{AB}\, \hbar v\, \delta(x'-x) \tag{95}$$
$$\left[\psi_R(x',t)^A, \psi_L(x,t)^B\right]_+ = 0. \tag{96}$$

Their Fourier transforms are

$$\psi_R(x-vt)^A = \frac{1}{2\pi}\int d\omega\; e^{-i\omega(t-x/v)}\tilde{\psi}_R(\omega)^A \tag{97}$$
$$\psi_L(x+vt)^A = \frac{1}{2\pi}\int d\omega\; e^{-i\omega(t+x/v)}\tilde{\psi}_L(\omega)^A \tag{98}$$

The hamiltonian, H_0, acts by

$$\left[H_0, \tilde{\psi}_R(\omega)^A\right] = -\hbar\omega\tilde{\psi}_R(\omega)^A \tag{99}$$
$$\left[H_0, \tilde{\psi}_L(\omega)^A\right] = -\hbar\omega\tilde{\psi}_L(\omega)^A. \tag{100}$$

The junction dynamics is encoded in the reflection matrix, $R(\omega)^A_B$:

$$\tilde{\psi}_R(\omega)^A = \sum_{B=1}^{N} R(\omega)^A_B \tilde{\psi}_L(\omega)^B \tag{101}$$

or, suppressing indices,

$$\tilde{\psi}_R(\omega) = R(\omega)\tilde{\psi}_L(\omega). \tag{102}$$

The Fourier modes satisfy the anti-commutation relations

$$\left[\tilde{\psi}_R(\omega')^A, \tilde{\psi}_R(\omega)^B\right]_+ = \delta^{AB}\, 2\pi\hbar\, \delta(\omega'+\omega) \tag{103}$$
$$\left[\tilde{\psi}_L(\omega')^A, \tilde{\psi}_L(\omega)^B\right]_+ = \delta^{AB}\, 2\pi\hbar\, \delta(\omega'+\omega) \tag{104}$$
$$\left[\tilde{\psi}_L(\omega')^A, \tilde{\psi}_R(\omega)^B\right]_+ = \sum_{B'=1}^{N} \delta^{AB'} R(\omega)^B_{B'}\, 2\pi\hbar\, \delta(\omega'+\omega). \tag{105}$$

The matrix $R(\omega)$ is analytic in the upper half-plane, by (96), and satisfies the reality and unitarity conditions

$$R(-\omega) = \overline{R(\omega)} \tag{106}$$

$$R(-\omega)^T R(\omega) = 1. \tag{107}$$

The anti-commutation relations and the action of the hamiltonian determine the equilibrium two-point functions:

$$\left\langle \tilde{\psi}_R(\omega')^A \tilde{\psi}_R(\omega)^B \right\rangle_{eq} = (1 + \mathrm{e}^{\beta\hbar\omega})^{-1} \delta^{AB}\, 2\pi\hbar\delta(\omega' + \omega) \tag{108}$$

$$\left\langle \tilde{\psi}_L(\omega')^A \tilde{\psi}_L(\omega)^B \right\rangle_{eq} = (1 + \mathrm{e}^{\beta\hbar\omega})^{-1} \delta^{AB}\, 2\pi\hbar\delta(\omega' + \omega) \tag{109}$$

$$\left\langle \tilde{\psi}_L(\omega')^A \tilde{\psi}_R(\omega)^B \right\rangle_{eq} = (1 + \mathrm{e}^{\beta\hbar\omega})^{-1} \sum_{B'=1}^{N} \delta^{AB'} R(\omega)^B_{B'}\, 2\pi\hbar\delta(\omega' + \omega). \tag{110}$$

The chiral energy–momentum currents are

$$T_R(x - vt)_A = \frac{1}{2} v T_t^t(x,t)_A + \frac{1}{2} T_t^x(x,t)_A = \frac{i}{2} {:}\psi_R(x,t)^A \partial_t \psi_R(x,t)^A{:} \tag{111}$$

$$T_L(x + vt)_A = \frac{1}{2} v T_t^t(x,t)_A - \frac{1}{2} T_t^x(x,t)_A = \frac{i}{2} {:}\psi_L(x,t)^A \partial_t \psi_L(x,t)^A{:} \tag{112}$$

and their Fourier transforms are

$$T_R(x - vt)_A = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\; \mathrm{e}^{-i\omega(t - x/v)}\, \tilde{T}_R(\omega)_A \tag{113}$$

$$T_L(x + vt)_A = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\; \mathrm{e}^{-i\omega(t + x/v)}\, \tilde{T}_L(\omega)_A \tag{114}$$

$$\tilde{T}_R(\omega)_A = \frac{1}{2\pi} \int\!\!\int \mathrm{d}\omega_1 \mathrm{d}\omega_2\, \delta(\omega_1 + \omega_2 - \omega)\, \frac{\omega_2}{2} {:}\tilde{\psi}_R(\omega_1)^A \tilde{\psi}_R(\omega_2)^A{:} \tag{115}$$

$$\tilde{T}_L(\omega)_A = \frac{1}{2\pi} \int\!\!\int \mathrm{d}\omega_1 \mathrm{d}\omega_2\, \delta(\omega_1 + \omega_2 - \omega)\, \frac{\omega_2}{2} {:}\tilde{\psi}_L(\omega_1)^A \tilde{\psi}_L(\omega_2)^A{:}\,. \tag{116}$$

Their commutation relations with the fermion fields are

$$\left[\tilde{T}_R(\omega')_A, \tilde{\psi}_R(\omega)^B\right] = -\delta_A^B \hbar (\frac{1}{2}\omega' + \omega) \tilde{\psi}_R(\omega' + \omega)^B \tag{117}$$

$$\left[\tilde{T}_L(\omega')_A, \tilde{\psi}_L(\omega)^B\right] = -\delta_A^B \hbar (\frac{1}{2}\omega' + \omega) \tilde{\psi}_L(\omega' + \omega)^B\,. \tag{118}$$

The hamiltonian is

$$H_0 = \sum_{A=1}^{N} \tilde{T}_R(0)_A = \sum_{A=1}^{N} \tilde{T}_L(0)_A\,. \tag{119}$$

Equation (75) for $Y_S(\omega)_{AB}$ can now be evaluated, using the entropy currents

$$\tilde{j}_R(\omega)_A = k\beta \tilde{T}_R(\omega)_A \tag{120}$$

$$\tilde{j}_L(\omega)_A = k\beta \tilde{T}_L(\omega)_A \tag{121}$$

and Eqs. (115–116), (103–105), and (108–110). The result is

$$Y_S(\omega)_{AB} = \frac{k^2\hbar\beta^2}{\omega} \frac{1}{2\pi} \int\!\!\int \mathrm{d}\omega_1 \mathrm{d}\omega_2\, \delta(\omega_1 + \omega_2 - \omega) \left[R(\omega_1)^A_B R(\omega_2)^A_B - \delta^A_B \right] \frac{1}{8}(\omega_1 - \omega_2)^2 \left(\frac{1}{1 + \mathrm{e}^{-\beta\hbar\omega_2}} - \frac{1}{\mathrm{e}^{\beta\hbar\omega_1} + 1} \right). \tag{122}$$

The entropic "capacitance" is

$$\begin{aligned} C_{SB} &= \lim_{\omega\to 0} \tfrac{1}{i\omega} \textstyle\sum_{A=1}^{N} Y_S(\omega)_{AB} \\ &= \tfrac{k}{T}\tfrac{1}{2\pi i} \int \mathrm{d}\eta \left[R^{-1}R'(\eta)\right]_B^B \ \tfrac{(\beta\hbar\eta)^2 \mathrm{e}^{\beta\hbar\eta}}{2(\mathrm{e}^{\beta\hbar\eta}+1)^2}\ . \end{aligned} \tag{123}$$

The total entropic "capacitance" is

$$C_S = \frac{\mathrm{d}s}{\mathrm{d}T} = \sum_{B=1}^{N} C_{SB} = \frac{k}{T}\frac{1}{2\pi i}\int \mathrm{d}\eta \ \mathrm{Tr}\left[R^{-1}R'(\eta)\right] \ \frac{(\beta\hbar\eta)^2\mathrm{e}^{\beta\hbar\eta}}{2\left(\mathrm{e}^{\beta\hbar\eta}+1\right)^2}\ . \tag{124}$$

Integrating with respect to T gives the junction entropy, up to a constant. When $R(\omega)$ is independent of temperature, the integral can be done explicitly, giving

$$s(T) - s(0) = \frac{k}{2\pi i}\int \mathrm{d}\eta \ \mathrm{Tr}\left[R^{-1}R'(\eta)\right] \ \frac{1}{2}\left[\frac{\beta\hbar\eta}{\mathrm{e}^{\beta\hbar\eta}+1} + \ln\left(1+\mathrm{e}^{-\beta\hbar\eta}\right)\right]. \tag{125}$$

The information capacity of the junction is then

$$s(\infty) - s(0) = k\ln 2 \ \frac{1}{2\pi i}\int \mathrm{d}\eta \ \frac{1}{2}\mathrm{Tr}\left[R^{-1}R'(\eta)\right]. \tag{126}$$

The information capacity of the junction thus consists of a half-bit for each pole of $R(\omega)_B^A$ in the lower half-plane (or for each zero in the upper half-plane).

An elementary Ising junction is a junction without substructure. It has a fermionic degree of freedom, $\chi(t)^A$, at the end of each wire, satisfying

$$\partial_t \chi(t) = -\lambda(\psi_R + \psi_L)(0,t) \tag{127}$$

$$\lambda^T \chi(t) = (\psi_R - \psi_L)(0,t) \tag{128}$$

where λ is a real $N \times N$ matrix. The matrix elements λ_B^A are the junction coupling constants. Eliminating $\chi(t)$ gives boundary conditions on the fermion fields:

$$(\partial_t + \lambda^T\lambda)\psi_R(0,t) = (\partial_t - \lambda^T\lambda)\psi_L(0,t) \tag{129}$$

or, equivalently,

$$R(\omega) = \frac{\omega - i\lambda^T\lambda}{\omega + i\lambda^T\lambda}. \tag{130}$$

The information capacity of the elementary Ising junction is given immediately by (126):

$$s(\infty) - s(0) = \frac{N}{2}k\ln 2 \tag{131}$$

Equation (123) for the entropic "capacitance" is evaluated by analytic continuation into the upper half-plane, picking up the residues at the thermal poles. The result is

$$C_{SB} = \frac{\mathrm{d}s}{\mathrm{d}T} = k^2\beta F_1(\sigma)_B^B \tag{132}$$

where

$$\sigma = \frac{\beta\hbar\lambda^T\lambda}{2\pi} \tag{133}$$

$$F_1(\sigma) = \sigma - \sigma^2\psi'(\frac{1}{2}+\sigma) \tag{134}$$

$$\psi'(z) = (\ln\Gamma)''(z) = \sum_{n=0}^{\infty}\frac{1}{(n+z)^2}. \tag{135}$$

The total entropic "capacitance" is

$$C_S = \frac{\mathrm{d}s}{\mathrm{d}T} = \sum_{B=1}^{N} C_{SB} = k^2\beta \ \mathrm{Tr}\ F_1(\sigma). \tag{136}$$

Integrating with respect to T gives the junction entropy, up to a constant:

$$s(T) - s(0) = k\mathrm{Tr}\left[F_2(\sigma) - \sigma F_2'(\sigma)\right] = k\left(1 - \beta\frac{\partial}{\partial\beta}\right)\mathrm{Tr}F_2(\sigma) \tag{137}$$

where

$$F_2(\sigma) = \sigma \ln\sigma - \sigma - \ln\Gamma\left(\frac{1}{2} + \sigma\right) + \ln\sqrt{2\pi}. \tag{138}$$

Therefore the boundary free energy, up to a constant, is

$$\ln z(T) - \ln z(0) = \mathrm{Tr}F_2(\sigma). \tag{139}$$

This agrees, in the case $N = 1$, with the direct calculation of the boundary partition function and the boundary entropy of a bulk-critical Ising wire in a boundary magnetic field [4,11]. The junction coupling constant, λ, is the boundary magnetic field.

To have an explicit example of the entropic admittance, for illustration, calculate $Y_S(\omega)_{AB}$ for the case $N = 1$. The wire labels, $A, B = 1$, are suppressed. Equation (130) for the reflection matrix is

$$R(\omega) = \frac{\omega - i\lambda^2}{\omega + i\lambda^2} \tag{140}$$

In (122), use the identity

$$R(\omega_1)R(\omega_2) - 1 = \left(\frac{\omega_1 + \omega_2}{\omega_2 - \omega_1}\right)[R(\omega_1) - R(\omega_2)] \tag{141}$$

then evaluate the integral by deforming the contour, to obtain the entropic admittance of the boundary of a bulk-critical quantum Ising wire:

$$\begin{aligned} Y_S(\omega) &= k^2\frac{\hbar}{2\pi}\beta^2\lambda^2 i\omega - \frac{1}{2}k^2\left(\frac{\hbar}{2\pi}\right)^2\beta^3\lambda^2\omega(\omega + 2i\lambda^2) \\ &\quad\times \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1}{2} + \frac{\beta\hbar\lambda^2}{2\pi}\right)\left(n + \frac{1}{2} + \frac{\beta\hbar\lambda^2}{2\pi} - \frac{i\beta\hbar\omega}{2\pi}\right)} \\ &= k^2\frac{\hbar}{2\pi}\beta^2\lambda^2 i\omega + k^2\frac{\hbar}{2\pi}\beta^2\lambda^2\left(\lambda^2 - \frac{i\omega}{2}\right) \\ &\quad\times\left[\psi\left(\frac{1}{2} + \frac{\beta\hbar}{2\pi}(\lambda^2 - i\omega)\right) - \psi\left(\frac{1}{2} + \frac{\beta\hbar}{2\pi}\lambda^2\right)\right]. \end{aligned} \tag{142}$$

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Appendix 1: The Energy–Momentum Tensor at a Junction

Consider N wires connected at a junction. Label the wires $A = 1, \ldots, N$. Each wire is parametrized by a spatial coordinate, $x \geq 0$. All the end-points, $x = 0$, are identified to a single point, which is the junction. The time coordinate is t; the euclidean time is $\tau = it$. The euclidean space-time of each wire is parametrized by the complex coordinate $z = x + iv\tau = x - vt$, $\bar{z} = x - iv\tau = x + vt$. The space-time metric on wire A, in the bulk, is

$$g_{\mu\nu}(x)_A \mathrm{d}x^\mu \mathrm{d}x^\nu = -v^2(\mathrm{d}t)^2 + (\mathrm{d}x)^2 = |\mathrm{d}z|^2 . \tag{143}$$

At the junction, there is only the metric on time, $g_{tt} = -v^2$. Varying the space-time metric gives the correlation functions of the energy–momentum tensor:

$$\delta\langle\cdots\rangle = -\frac{1}{2}\left\langle \frac{i}{\hbar}\int \mathrm{d}t \left[\delta g_{tt}(t) T^{tt}(t)_{junct} + \sum_{A=1}^{N}\int_0^\infty \mathrm{d}x\ \delta g_{\mu\nu}(x,t)_A\, T^{\mu\nu}(x,t)_A\right]\cdots\right\rangle . \tag{144}$$

An infinitesimal localized reparametrization of space-time is a collection of vector fields, $v^\nu(x,t)_A$, one on each wire, all supported within a bounded region of space-time. They must agree at the junction,

$$v^\nu(0,t)_A = v^\nu(t)_{junct} , \tag{145}$$

and must leave the junction in place,

$$v^x(0,t)_A = v^x(t)_{junct} = 0. \tag{146}$$

Such a reparametrization of space-time is equivalent to changing the space-time metric by

$$\delta g_{\mu\nu}(x,t)_A = \partial_\mu v_\nu(x,t)_A + \partial_\nu v_\mu(x,t)_A \tag{147}$$

$$\delta g_{tt}(t)_{junct} = 2\partial_t v_t(t)_{junct}. \tag{148}$$

The physics does not depend on the parametrization of space-time, so

$$0 = \int \mathrm{d}t \left[\partial_t v^t(t)_{junct} T_t^t(t)_{junct} + \sum_{A=1}^{N}\int_0^\infty \mathrm{d}x\ \partial_\mu v^\nu(x,t)_A T_\nu^\mu(x,t)_A\right] \tag{149}$$

for all localized reparametrizations of space-time. Integrating by parts gives the local conservation of energy and momentum:

$$0 = \partial_\mu T_\nu^\mu(x,t)_A \tag{150}$$

$$0 = \partial_t T_t^t(t)_{junct} + \sum_{A=1}^{N} T_t^x(0,t)_A. \tag{151}$$

Taking $v^\mu(x,t)$ to be infinitesimal time translation identifies the hamiltonian:

$$H_0 = T_t^t(t)_{junct} + \sum_{A=1}^{N}\int_0^\infty \mathrm{d}x\ T_t^t(x,t)_A. \tag{152}$$

$T_t^t(x,t)_A$ is the bulk energy density in wire A. $T_t^t(t)_{junct}$ is the energy of the junction.

The departure from scale invariance is the trace of the energy–momentum tensor, $\Theta(x,t) = -T^{\mu}_{\mu}(x,t)$. The junction contribution is conventionally written $\theta(t)$. That is, the junction energy is written

$$T^t_t(t)_{junct} = -\theta(t) \tag{153}$$

and conservation of energy at the junction is written

$$0 = -\partial_t\theta(t) + \sum_{A=1}^{N} T^x_t(0,t)_A. \tag{154}$$

Appendix 2: The Chiral Energy and Entropy Currents

When the quantum wires are bulk-critical, the bulk energy density and current are linear combinations of chiral energy currents:

$$T_R(x,t)_A = T_R(x-vt)_A = \frac{1}{2}vT^t_t(x,t)_A + \frac{1}{2}T^x_t(x,t)_A \tag{155}$$

$$T_L(x,t)_A = T_L(x+vt)_A = \frac{1}{2}vT^t_t(x,t)_A - \frac{1}{2}T^x_t(x,t)_A. \tag{156}$$

The bulk entropy current and density operators are linear combinations of chiral entropy currents:

$$j_S(x,t)_A = j_R(x,t)_A - j_L(x,t)_A \tag{157}$$

$$\rho_S(x,t)_A = \frac{1}{v}j_R(x,t)_A + \frac{1}{v}j_L(x,t)_A \tag{158}$$

where

$$j_R(x,t)_A = j_R(x-vt)_A = k\beta T_R(x,t)_A - \frac{1}{2}k\beta\big\langle T_R(x,t)_A + T_L(x,t)_A\big\rangle_{eq} \tag{159}$$

$$j_L(x,t)_A = j_L(x+vt)_A = k\beta T_L(x,t)_A - \frac{1}{2}k\beta\big\langle T_R(x,t)_A + T_L(x,t)_A\big\rangle_{eq}. \tag{160}$$

$j_R(x,t)_A$ flows to the right $j_L(x,t)_A$ flows to the left, both at the speed of "light", v: $j_R(x,t+\delta t)_A = j_R(x-v\delta t,t)_A$, $j_L(x,t+\delta t)_A = j_L(x+v\delta t,t)_A$. The equilibrium expectation values in (159) and (160) are subtracted so that $\big\langle\rho_S(x,t)_A\big\rangle_{eq} = 0$. Then

$$\big\langle j_R(x,t)_A\big\rangle_{eq} = -\big\langle j_L(x,t)_A\big\rangle_{eq} = \frac{1}{2}\big\langle j_S(x,t)_A\big\rangle_{eq}. \tag{161}$$

The goal is to calculate the commutators $\big[j_R(x',t')_A, j_R(x,t)_B\big]$ and $\big[j_L(x',t')_A, j_L(x,t)_B\big]$. Because of the chirality of the currents, these commutators of two currents of the same chirality are completely determined by the equal-time commutators. The equal-time commutators vanish if $A \neq B$, so the commutators need only be calculated for $A = B$. The labels A, B are dropped during the calculation, to simplify the notation, then restored at the end.

In 1+1 dimensional conformal field theory, the chiral components of the energy–momentum tensor are usually written $T(z)$, $\bar{T}(\bar{z})$, where

$$T(z) = -\frac{2\pi}{\hbar v^2}T_R(x-vt) \tag{162}$$

$$\bar{T}(\bar{z}) = -\frac{2\pi}{\hbar v^2}T_L(x+vt). \tag{163}$$

The constant of proportionality, $-2\pi/\hbar v^2$, will be confirmed shortly. In euclidean space-time, $T(z)$ and $\bar{T}(\bar{z})$ satisfy operator product expansions

$$T(z')\,T(z) \sim (z'-z)^{-4}\frac{c}{2} + (z'-z)^{-2}2T(z) + (z'-z)^{-1}\partial T(z) \tag{164}$$

$$\bar{T}(\bar{z}')\,\bar{T}(\bar{z}) \sim (\bar{z}'-\bar{z})^{-4}\frac{c}{2} + (\bar{z}'-\bar{z})^{-2}2\bar{T}(\bar{z}) + (\bar{z}'-\bar{z})^{-1}\bar{\partial}\bar{T}(\bar{z}). \tag{165}$$

Write the equal-time commutators as τ-ordered operator products,

$$\left[T(x'), T(x)\right] = \tau\{T(x'+i\epsilon)\,T(x)\} - \tau\{T(x'-i\epsilon)\,T(x)\} \tag{166}$$

$$\left[\bar{T}(x'), \bar{T}(x)\right] = \tau\{\bar{T}(x'-i\epsilon)\,\bar{T}(x)\} - \tau\{\bar{T}(x'+i\epsilon)\,\bar{T}(x)\}\,, \tag{167}$$

and evaluate them using the operator product expansions, getting

$$\frac{1}{2\pi i}\left[T(x'), T(x)\right] = \frac{c}{12}\delta^{(3)}(x'-x) + (\partial_{x'}-\partial_x)\,\left[\delta(x'-x)\,T(x)\right] \tag{168}$$

$$\frac{-1}{2\pi i}\left[\bar{T}(x'), \bar{T}(x)\right] = \frac{c}{12}\delta^{(3)}(x'-x) + (\partial_{x'}-\partial_x)\,\left[\delta(x'-x)\,\bar{T}(x)\right]. \tag{169}$$

Integrate over x' to identify the energy density as

$$T_t^t(x,t) = -\frac{\hbar v}{2\pi}T(x-vt) - \frac{\hbar v}{2\pi}\bar{T}(x+vt)\,, \tag{170}$$

confirming the constant of proportionality in (162–163), between $T(z)$ and $T_R(x-vt)$ and between $\bar{T}(\bar{z})$ and $T_L(x+vt)$. These equal-time commutation relations could have been derived equally well by combining conformal invariance with the general equal-time commutation relations derived in Appendix 1 of [7]. The equal-time commutation relations, (168) and (169), can be written

$$\frac{i}{\hbar}\left[T_R(0,t'), T_R(0,t)\right] = -\frac{\hbar}{2\pi}\frac{c}{12}\partial_t^3\delta(t-t') + (\partial_t-\partial_{t'})\left[\delta(t-t')T_R(0,t)\right] \tag{171}$$

$$\frac{i}{\hbar}\left[T_L(0,t'), T_L(0,t)\right] = -\frac{\hbar}{2\pi}\frac{c}{12}\partial_t^3\delta(t-t') + (\partial_t-\partial_{t'})\left[\delta(t-t')T_L(0,t)\right]. \tag{172}$$

The equilibrium expectation values of the commutators are

$$\left\langle\frac{i}{\hbar}\left[T_R(0,t'), T_R(0,t)\right]\right\rangle_{eq} = -\frac{\hbar}{2\pi}\frac{c}{12}\partial_t^3\delta(t-t') + 2\partial_t\delta(t-t')\left\langle T_R(0,t)\right\rangle_{eq} \tag{173}$$

$$\left\langle\frac{i}{\hbar}\left[T_L(0,t'), T_L(0,t)\right]\right\rangle_{eq} = -\frac{\hbar}{2\pi}\frac{c}{12}\partial_t^3\delta(t-t') + 2\partial_t\delta(t-t')\left\langle T_L(0,t)\right\rangle_{eq}. \tag{174}$$

Now restore the wire labels, A, B. Define the Fourier modes of the chiral energy–momentum currents by

$$T_R(x,t)_A = T_R(x-vt)_A = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \mathrm{e}^{-i\omega(t-x/v)}\,\tilde{T}_R(\omega)_A \tag{175}$$

$$T_L(x,t)_A = T_L(x+vt)_A = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \mathrm{e}^{-i\omega(t+x/v)}\,\tilde{T}_L(\omega)_A \tag{176}$$

The commutation relations, (171) and (172), are equivalent to

$$\left[\tilde{T}_R(\omega')_B, \tilde{T}_R(\omega)_A\right] = \delta_{AB}\left[-\frac{c}{12}\hbar^2\omega^3\delta(\omega'+\omega) + \hbar(\omega'-\omega)\tilde{T}_R(\omega'+\omega)_A\right] \tag{177}$$

$$\left[\tilde{T}_L(\omega')_B, \tilde{T}_L(\omega)_A\right] = \delta_{AB}\left[-\frac{c}{12}\hbar^2\omega^3\delta(\omega'+\omega) + \hbar(\omega'-\omega)\tilde{T}_L(\omega'+\omega)_A\right]. \tag{178}$$

Conservation of energy at the junction, (13), implies that

$$\sum_A \tilde{T}_R(0)_A - \sum_A \tilde{T}_L(0)_A = 0. \tag{179}$$

The hamiltonian acts by

$$\left[H_0, \tilde{T}_R(\omega)_A\right] = -\hbar\omega \tilde{T}_R(\omega)_A \tag{180}$$
$$\left[H_0, \tilde{T}_L(\omega)_A\right] = -\hbar\omega \tilde{T}_L(\omega)_A \tag{181}$$

so the hamiltonian is

$$H_0 = \sum_A \tilde{T}_R(0)_A = \sum_A \tilde{T}_L(0)_A. \tag{182}$$

A straightforward calculation shows these expressions for the hamiltonian to be equal to the spatial integral of the energy density.

The equilibrium expectation values of the commutators are

$$\left\langle\left[\tilde{T}_R(\omega')_B, \tilde{T}_R(\omega)_A\right]\right\rangle_{eq} = \delta_{AB}\left[-\frac{c}{12}\frac{\hbar}{2\pi}\omega^3 - 2\omega\left\langle T_R(0,t)_A\right\rangle_{eq}\right]2\pi\hbar\delta(\omega'+\omega) \tag{183}$$
$$\left\langle\left[\tilde{T}_L(\omega')_B, \tilde{T}_L(\omega)_A\right]\right\rangle_{eq} = \delta_{AB}\left[-\frac{c}{12}\frac{\hbar}{2\pi}\omega^3 - 2\omega\left\langle T_L(0,t)_A\right\rangle_{eq}\right]2\pi\hbar\delta(\omega'+\omega). \tag{184}$$

The equilibrium one-point functions,

$$\left\langle T_R(x,t)_A\right\rangle_{eq} = \left\langle T_R(x-vt)_A\right\rangle_{eq} \tag{185}$$
$$\left\langle T_L(x,t)_A\right\rangle_{eq} = \left\langle T_L(x+vt)_A\right\rangle_{eq}, \tag{186}$$

are independent of t, so are constant in x. They can be evaluated far from the junction, where the system is conformally invariant. When no equilibrium currents are flowing, when $\left\langle T_t^x(0,t)_A\right\rangle_{eq} = 0$, the equilibrium bulk energy density is [1,3]

$$\left\langle T_t^t(x,t)_A\right\rangle_{eq} = \frac{c}{12}\frac{2\pi}{\hbar v\beta^2}. \tag{187}$$

Equivalently,

$$\left\langle T_R(x,t)_A\right\rangle_{eq} = \left\langle T_L(x,t)_A\right\rangle_{eq} = \frac{c}{24}\frac{2\pi}{\hbar\beta^2}. \tag{188}$$

The equilibrium expectation values of the commutators are then

$$\left\langle\left[\tilde{T}_R(\omega')_B, \tilde{T}_R(\omega)_A\right]\right\rangle_{eq} = -\delta_{AB}\delta(\omega'+\omega)\frac{c}{12}\hbar^2\left[\omega^3 + \left(\frac{2\pi}{\hbar\beta}\right)^2\omega\right] \tag{189}$$
$$\left\langle\left[\tilde{T}_L(\omega')_B, \tilde{T}_L(\omega)_A\right]\right\rangle_{eq} = -\delta_{AB}\delta(\omega'+\omega)\frac{c}{12}\hbar^2\left[\omega^3 + \left(\frac{2\pi}{\hbar\beta}\right)^2\omega\right]. \tag{190}$$

Bulk conformal invariance requires that this commutator vanish at $\omega = 2\pi i/\hbar\beta$. This condition gives another way to derive the equilibrium energy density.

References

1. Affleck, I.: Universal term in the free energy at a critical point and the conformal anomaly. Phys. Rev. Lett. **56**, 746–748 (1986)
2. Affleck, I., Ludwig, A.W.: Universal noninteger "ground state degeneracy" in critical quantum systems. Phys. Rev. Lett. **67**, 161 (1991)
3. Blöte, H., Cardy, J., Nightingale, M.: Conformal invariance, the central charge, and universal finite-size amplitudes at criticality. Phys. Rev. Lett. **56**, 742–745 (1986)
4. Chatterjee, R.: Exact partition function and boundary state of critical ising model with boundary magnetic field. Mod. Phys. Lett. A **10**, 973–984 (1995)
5. Friedan, D.: Entropy flow in near-critical quantum circuits (2005). arXiv:cond-mat/0505084
6. Friedan, D.: Entropy flow through near-critical circuit junctions (2005). arXiv:cond-mat/0505085
7. Friedan, D.: Entropy flow in near-critical quantum circuits. J. Stat. Phys. (2017). doi:10.1007/s10955-017-1751-9
8. Friedan, D., Konechny, A.: Boundary entropy of one-dimensional quantum systems at low temperature. Phys. Rev. Lett. **93**, 030,402 (2004)
9. Friedan, D., Konechny, A.: Infrared properties of boundaries in 1-d quantum systems. J. Stat. Phys. **0603**, P014 (2006)
10. Ghoshal, S., Zamolodchikov, A.: Boundary s-matrix and boundary state in two-dimensional integrable quantum field theory. Int. J. Mod. Phys. **A9**, 3841–3886 (1994). Erratum *ibid.***A9**, 4353 (1994)
11. Konechny, A.: Ising model with a boundary magnetic field: An example of a boundary flow. JHEP **12**, 058 (2004)
12. Tsvelick, A., Wiegmann, P.: Exact results in the theory of magnetic alloys. Adv. Phys. **32**(4), 453–713 (1983)