



Cauchy Conformal Fields in Dimensions $d > 2$

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Abstract: Holomorphic fields play an important role in 2d conformal field theory. We generalize them to $d > 2$ by introducing the notion of Cauchy conformal fields, which satisfy a first order differential equation such that they are determined everywhere once we know their value on a codimension 1 surface. We classify all the unitary Cauchy fields. By analyzing the mode expansion on the unit sphere, we show that all unitary Cauchy fields are free in the sense that their correlation functions factorize on the 2-point function. We also discuss the possibility of non-unitary Cauchy fields and classify them in $d = 3$ and 4.

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1. Introduction

1.1. *Generalizing holomorphic fields to $d > 2$ —the Cauchy condition.* Holomorphic quantum fields play a major role in two dimensional conformal field theory. Their defining property is the Cauchy–Riemann equation

$$(\partial_x + i\partial_y)\phi(z) = 0. \quad (1.1)$$

Equation (1.1) defines a Cauchy problem. Namely, if we know the field ϕ on some contour, say for instance $x = \text{const}$, or the circle $|x + iy| = \text{const}$, then (1.1) uniquely determines ϕ everywhere. That ϕ depends holomorphically on the single complex variable $z = x + iy$ is equivalent to the Cauchy–Riemann equation (1.1). A 2d holomorphic field has a mode expansion

$$\phi_n = \int_C dz f_n(z)\phi(z), \quad \phi(z) = \sum_n f_n^*(z)\phi_n, \quad \int_C f_m(z)f_n^*(z) = \delta_{m,n}, \quad (1.2)$$

where the smearing functions $f_n(z)$ are a complete set of functions holomorphic in some neighborhood of the contour C (for example, a circle in the 2d space-time). The Cauchy property allows us to deform the contour without changing the result, within the region where $\phi(z)$ and $f_n(z)$ are non-singular. By appropriately deforming the contour, we

can thus obtain the commutation relations of the modes from just the singular terms in the operator products of the holomorphic fields. The representation theory of the algebra of the modes becomes a powerful tool for analyzing the quantum field theory. In a complementary vein, the global properties of the Cauchy–Riemann equation put stringent constraints on the correlation functions of holomorphic fields, making possible exact solutions.

Holomorphic quantum fields were first constructed and studied in string theory [1–4]. Early uses of contour deformation techniques can be found in [5, 6]. P. Goddard wrote “People gradually realized through this time (1971–1973) that one could more profitably use the analytic properties of the fields in this way than think in terms of distributions on the unit circle” (P. Goddard, private communication). Holomorphic fields and the contour deformation technique were later rediscovered in [7].

In this paper our goal is to generalize the notion of holomorphic conformal fields to higher dimensions, and to attempt a classification of such fields. Our hope—unrealized—was that there might be as rich a variety of such fields as in two dimensions. To mimic the two dimensional case, we search for fields ϕ that satisfy a first order differential equation that has the Cauchy property—the property of uniquely determining ϕ everywhere once ϕ is known on some codimension 1 surface S . This Cauchy property generalizes to $d > 2$, along with the techniques of mode expansion and contour deformation. The property of depending on a single complex variable on the other hand does not seem to generalize to conformal fields in $d > 2$.

Not every field satisfying a first order differential has the Cauchy property. For example, a conserved spin 1 current $j^\mu(x)$ satisfies the first order equation $\partial_\mu j^\mu = 0$, but knowing j^μ on the hyperplane $x^d = 0$ is not enough to determine j^μ on nearby hyperplanes. We are thus looking for fields that satisfy a special type of first order differential equation.

An example of a field that does have the Cauchy property is a self-dual two form $\phi^{\mu\nu}(x)$ in $d = 4$ dimensions. It satisfies the self-duality condition

$$\phi^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu}{}_{\mu'\nu'} \phi^{\mu'\nu'} \quad (1.3)$$

and the first order differential equation

$$\partial_\mu \phi^{\mu\nu} = 0. \quad (1.4)$$

Using the self-duality condition to eliminate ϕ^{ij} in favor of ϕ^{kd} , the first order equation becomes

$$\partial_d \phi^{id} = -\epsilon^{ij}{}_k \partial_j \phi^{kd}. \quad (1.5)$$

If we know ϕ on a surface $x^d = \text{const}$, then we can integrate this first order equation to get ϕ everywhere. Self-dual 2-forms in 4d are thus Cauchy fields. Our goal is to identify the other fields of this type in all dimensions $d > 2$.

1.2. Level 1 short representations. Differential equations for ϕ correspond to shortened representations of the conformal algebra $\mathfrak{so}(d, 2)$. In the radial quantization of a CFT, each conformal primary field $\phi(x)$ corresponds to an irreducible lowest weight representation of the conformal algebra $\mathfrak{so}(d, 2)$. The lowest weight Δ is the lowest eigenvalue of the dilation generator D . The lowest weight states $|\phi\rangle$ form an irreducible representation V of the euclidean rotation Lie algebra $\mathfrak{so}(d)$. The field $\phi(x)$ is a vector-valued field of

spin V and conformal weight (or scaling dimension) Δ . The conformal transformation properties of $\phi(x)$ are completely determined by the data (V, Δ) .

The field $\phi(x)$ satisfies a first order differential equation iff the states $|\phi\rangle$ are annihilated by some linear combination of the translation generators P_μ . The P_μ act as raising operators in the conformal algebra, adding 1 to the weight. If a linear combination of the P_μ annihilates $|\phi\rangle$, this means that there is a null subspace of states at level 1 in the representation. The representation is said to be a *level 1 short representation*. In $d = 2$ for instance, the Cauchy–Riemann differential equation (1.1) comes from the null state condition $\bar{L}_{-1}|\phi\rangle = 0$. For general d , classifying the primary fields that satisfy first order differential equations is equivalent to classifying the level 1 short representations of the conformal algebra.

The inner product matrix of the conformal representation at level 1 is easily computed in terms of the commutation relations of the conformal algebra. There is a null state on level 1 iff the determinant of the level 1 inner product matrix equals 0. This is a polynomial equation on the conformal weight Δ with coefficients that depend on the $\mathfrak{so}(d)$ representation V . This type of argument is of course very familiar from the derivation of unitarity bounds. A necessary condition for the conformal representation to be unitary is that the inner product matrix at level 1 be non negative. For a given $\mathfrak{so}(d)$ representation V , the inner product matrix is positive definite for large values of Δ . As Δ decreases, it eventually reaches a value where one or more null states appear. That is the unitarity bound on Δ . Below that value of Δ , the level 1 inner product matrix has at least one negative eigenvalue. Thus, if we insist on unitarity, the only level 1 short representations that can occur are at the level 1 unitarity bound. If we do not require unitarity, more possibilities open up, as there are in general several values of Δ where null states appear on level 1. The conserved spin 1 current discussed above does have null states at level 1, which leads to its conservation equation, but it does not have the Cauchy property. The central question of this work is thus: which level 1 null states give Cauchy differential equations?

1.3. Summary of results. Let us summarize our results. For unitary fields, we find a complete answer to this question.

In even dimensions $d = 2n$, a unitary Cauchy field is a conformal field whose spin is an $\mathfrak{so}(d)$ representation V_s with highest weight λ_s ,

$$V_s = V_{\lambda_s}, \quad \lambda_s = (|s|, \dots, |s|, s), \quad s \in \frac{1}{2}\mathbb{Z}, \quad (1.6)$$

and whose conformal weight is

$$\Delta_s = \begin{cases} n - 1 + |s| & s \neq 0 \\ 0 & s = 0. \end{cases} \quad (1.7)$$

The case $s = 0$ is the trivial case $\phi = 1$, the identity operator, which of course satisfies the first order equation $\partial_\mu \phi = 0$.

In odd dimensions $d = 2n + 1$, the unitary Cauchy fields are the primary fields (V_s, Δ_s)

$$V_s = V_{\lambda_s}, \quad \lambda_s = (s, \dots, s), \quad \text{with } s = 0 \text{ or } s = \frac{1}{2}, \quad (1.8)$$

$$\Delta_0 = 0, \quad \Delta_{1/2} = n - \frac{1}{2}. \quad (1.9)$$

The case $s = 0$ is again the identity operator.

We prove that these lists exhaust all unitary Cauchy fields in $d > 2$.

It was shown in [8] that the conformal primary fields which satisfy the massless Klein–Gordon equation

$$\partial^\mu \partial_\mu \phi(x) = 0 \tag{1.10}$$

consist exactly of the free massless scalar field and the fields (V_s, Δ_s) . For this reason the fields in representations V_s were called “free” in [8]. This was a misnomer, since the Klein–Gordon equation did not immediately imply that the correlation functions factorize as free-field correlation functions into products of 2-point functions according to the Wick contraction rule. For example, non-abelian current algebras in $d = 2$, and, more generally, W -algebras, satisfy the Klein–Gordon equation (1.10) but are certainly not free.¹

Here we prove that all unitary Cauchy fields in $d > 2$ are indeed free fields by using the Cauchy property to constrain the possible modes of the field. From the mode expansions, we find constraints on the singular part of the operator product expansion of the field with its adjoint field which imply, for $d > 2$, that the commutators of the modes are multiples of the identity operator, which establishes that indeed the unitary Cauchy fields all have free-field correlation functions. We find that only the massless spinor and the self-dual n -form field can have a local energy-momentum tensor, in accord with the Weinberg–Witten theorem [10].

It would have been very nice to find non-free Cauchy fields that could be used to construct non-trivial conformal field theories. In a sense however the negative result is not unexpected, as it is related to results on conserved higher spin currents in quantum field theory in $d > 2$. Note that short representations lead to conserved currents: suppose we have a field $\phi_a(x)$ in a representation V of $\mathfrak{so}(d)$, with a being an index for V , and suppose that it satisfies a first-order differential equation $A_b^{\mu a} \partial_\mu \phi_a(x) = 0$. Then $J_b^\mu = A_b^{\mu a} \phi_a(x)$ is a conserved current. J will be a higher spin conserved current for all but the smallest representations V . Note that this is somewhat different from the usual construction of higher spin currents as bilinears, since here J is linear in the underlying fields. Nonetheless, the Coleman–Mandula theorem [11] states that higher spin conserved currents in $d > 2$ dimensions must be free. The original theorem only applies to theories with a mass gap, so it is not directly applicable here. It is believed however that a similar theorem also holds for conformal field theories. There has been recent progress towards proving such a conformal Coleman–Mandula theorem, although it seems that nothing to date has been fully proven. It was argued in [12] for conserved higher spin currents in $d = 3$, in [13] for symmetric traceless fields in $d = 4$, and in [14] for 4-, 5- and 6-pt functions of the energy-momentum tensor in $d = 4$. For higher dimensions there has been work [15] on the construction of higher spin algebras, arguing that for $d = 3$ and $d > 5$ under certain assumptions free theories are the only such higher spin algebras. From this point of view, we are proving here a partial Coleman–Mandula theorem in arbitrary dimension.

In the final section, we show that the contour deformation technique can be used for any Cauchy conformal field, in any dimension d . We show that there always exist enough smearing functions to capture all the modes of the field on a surface of codimension 1.

Our results for non-unitary conformal Cauchy fields are incomplete. We classify all possible spins and scaling dimensions for $d = 3$ and 4. For $d > 4$, we find a restricted

¹ For scalar fields with $\partial^\mu \partial_\mu \phi(x) = 0$ in $d > 2$ factorization was argued in [9]. The argument assumes however that the conformal Coleman–Mandula theorem (see below) holds. We thank Sasha Zhiboedov for discussion of this and related matters.

list of possible spins and scaling dimensions, but we do not prove that all the possibilities on the list do in fact have the Cauchy property. We derive the mode expansions only for the non-unitary cases where the spin is one of the $\mathfrak{so}(d)$ representations V_s but the scaling dimension is not the unitary value. We do not show that the non-unitary theories are free. It is thus possible that there are interesting interacting non-unitary Cauchy fields in dimensions $d > 2$, whose properties could be investigated by a straightforward application of the methods used here.

2. The Conformal Algebra and Conformal Fields

2.1. The conformal algebra and its representations. We consider euclidean CFTs in the radial quantization. The conformal operator algebra is $\mathfrak{so}(d, 2)$. It has $(d+2)(d+1)/2$ generators: the generators P_μ of translations, the generators $L_{\mu\nu}$ of $\mathfrak{so}(d)$, the euclidean rotations, the generator D of dilations, and the generators K_μ of the special conformal transformations. The commutation relations are

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [K_\mu, K_\nu] &= 0, & [K_\mu, P_\nu] &= 2\delta_{\mu\nu}D - 2L_{\mu\nu} \\ [D, P_\mu] &= P_\mu, & [D, L_{\mu\nu}] &= 0, & [D, K_\mu] &= -K_\mu \\ [L_{\mu\nu}, P_\sigma] &= \delta_{\nu\sigma}P_\mu - \delta_{\mu\sigma}P_\nu, & [L_{\mu\nu}, K_\sigma] &= \delta_{\nu\sigma}K_\mu - \delta_{\mu\sigma}K_\nu \\ [L_{\mu\nu}, L_{\rho\sigma}] &= \delta_{\nu\rho}L_{\mu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} + \delta_{\mu\sigma}L_{\nu\rho} - \delta_{\mu\rho}L_{\nu\sigma} \end{aligned} \quad (2.1)$$

The adjointness relations are

$$P_\mu^\dagger = K_\mu, \quad D^\dagger = D, \quad L_{\mu\nu}^\dagger = -L_{\mu\nu}. \quad (2.2)$$

Our generators differ by a factor of i from the usual ones in the physics literature.

The conformal generators implement the conformal vector fields:

generator	vector field	
P_μ	∂_μ	
D	$x^\mu \partial_\mu$	
$L_{\mu\nu}$	$(\delta_\mu^\sigma x_\nu - \delta_\nu^\sigma x_\mu) \partial_\sigma$	(2.3)
K_μ	$(2x_\mu x^\sigma - x^2 \delta_\mu^\sigma) \partial_\sigma$	

The ground state is annihilated by all the conformal generators, so the correlation functions are invariant under the complex conformal algebra $\mathfrak{so}(d+2, \mathbb{C})$. The conformal symmetries of euclidean space form the real subalgebra $\mathfrak{so}(d+1, 1)$.

The space of states of the radial quantization is constructed from the correlation functions with respect to the reflection in the unit sphere, the inversion $R : x^\mu \mapsto x^{-2}x^\mu$. Writing $X[v]$ for the operator generator implementing the conformal vector field v , the adjoint is $X[v]^\dagger = -X[Rv]$, thus the adjointness relations of (2.2) above. Extending $X[v]$ to complex vector fields v , the vector fields satisfying $Rv = \bar{v}$ have skew-adjoint generators. The usual Minkowski space quantization is constructed with respect to the reflection in the hyperplane $x^d = 0$, $R_{Mink} : (\vec{x}, x^d) \mapsto (\vec{x}, -x^d)$. So, in the Minkowski space quantization, $P_d^\dagger = P_d$ and $P_i^\dagger = -P_i$ (recall that our generators differ from the usual ones by a factor of i). The two reflections, R and R_{Mink} are conjugate to each other in the euclidean conformal group, so the two Lie algebras of skew-adjoint generators are not the same, but they are isomorphic to each other, both isomorphic to $\mathfrak{so}(d, 2)$.

A conformal field theory has a complete set of scaling fields $\phi_i(x)$ with scaling dimensions Δ_i , satisfying

$$[P_\mu, \phi_i(x)] = \partial_\mu \phi_i(x), \quad [D, \phi_i(x)] = (x^\mu \partial_\mu + \Delta_i) \phi_i(x). \quad (2.4)$$

The radial quantization gives a space of states in one-to-one correspondence with the scaling fields. The operator-state correspondence maps the scaling field $\phi_i(x)$ to an eigenstate of the dilation generator D ,

$$\phi_i(x) \leftrightarrow |\phi_i\rangle = \phi_i(0)|0\rangle, \quad D|\phi_i\rangle = \Delta_i|\phi_i\rangle, \quad (2.5)$$

where $|0\rangle$ is the ground state, corresponding to the identity field 1. The ground state $|0\rangle$ is annihilated by all the conformal generators. Correlation functions are given by ground state expectation values of radially ordered products of fields.

The generators of the conformal algebra act as operators on the state space. The eigenvalue of D is the conformal weight. The generators P_μ raise the weight by 1, and the generators K_μ lower the weight by 1. The conformal lowest weight states are the states that are killed by the lowering operators K_μ ,

$$K_\mu|\phi\rangle = 0, \quad D|\phi\rangle = \Delta|\phi\rangle. \quad (2.6)$$

The full space of states is generated from the lowest weight states by the action of the raising operators P_μ . The $\mathfrak{so}(d)$ generators $L_{\mu\nu}$ commute with D , so they take lowest weight states to lowest weight states. The space of lowest weight states thus decomposes into a sum of irreducible representations of $\mathfrak{so}(d)$. We write $|\phi\rangle$ for the finite dimensional vector space of lowest weight states of weight Δ in an irreducible $\mathfrak{so}(d)$ representation V . The raising operators P_μ acting on $|\phi\rangle$ generate an irreducible lowest weight representation of the conformal algebra, with lowest weight Δ .

The conformal primary fields $\phi(x)$ are in one-to-one correspondence with the conformal lowest weight states $|\phi\rangle$ and are labeled by the same data (V, Δ) . The representation V is called the *spin* of the field $\phi(x)$. Writing $\phi_a(x)$ for the component fields of the representation V , the generators $L_{\mu\nu}$ act on the lowest weight states by matrices $M_{\mu\nu}$ on V ,

$$L_{\mu\nu}|\phi_b\rangle = M_{\mu\nu}^a_b|\phi_a\rangle. \quad (2.7)$$

The matrices $M_{\mu\nu}$ satisfy the same commutation relations as the $L_{\mu\nu}$,

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\nu\sigma} M_{\mu\rho} + \delta_{\mu\sigma} M_{\nu\rho} - \delta_{\mu\rho} M_{\nu\sigma}. \quad (2.8)$$

There is a unique hermitian inner product on the representation V such that

$$M_{\mu\nu}^\dagger = -M_{\mu\nu}. \quad (2.9)$$

The inner product on the entire conformal representation is the unique inner product determined by the adjointness relations (2.2).

The action of the conformal generators on the conformal fields can be derived from the operator state correspondence, the action of the conformal generators on the lowest weight states, and the translation covariance

$$\phi(x) = e^{x^\mu P_\mu} \phi(0) e^{-x^\mu P_\mu}, \quad \phi(x)|0\rangle = e^{x^\mu P_\mu} |\phi\rangle. \quad (2.10)$$

The results are

$$[P_\mu, \phi(x)] = \partial_\mu \phi(x), \quad (2.11)$$

$$[D, \phi(x)] = x^\mu \partial_\mu \phi(x) + \Delta \phi(x), \quad (2.12)$$

$$[L_{\mu\nu}, \phi_b(x)] = (x_\nu \partial_\mu - x_\mu \partial_\nu) \phi_b(x) + M_{\mu\nu}^a \phi_a(x), \quad (2.13)$$

$$[K_\mu, \phi_b(x)] = (2x_\mu x^\sigma - x^2 \delta_\mu^\sigma) \partial_\sigma \phi_b(x) + 2x^\sigma (\Delta \delta_{\mu\sigma} \delta_b^a - M_{\mu\sigma}^a) \phi_a(x). \quad (2.14)$$

2.2. Level 1 short representations. Let us now study the irreducible lowest weight representations of the conformal algebra in more detail. Let $\phi(x)$ be a conformal field of dimension Δ in the $\mathfrak{so}(d)$ representation V . As we argued above, the full representation is obtained by acting with raising operators P_μ . For a generic representation, we expect all those *descendent* states to be linearly independent. Such a representation is called *long*. We are interested in *short* or *degenerate* representations, where there are linear relations among some of the descendent states.

In mathematical language, the *Verma module* is the vector space spanned by all the formal products $P_{\mu_1} \cdots P_{\mu_N} |\phi_a\rangle$ symmetric in the indices μ_i . We have a linear map from the Verma module to the physical state space, taking the formal product to the physical product of operators acting on the lowest weight state. The kernel of this linear map is the space of null states—the linear relations in the Verma module.

The weight space $D = \Delta + N$ is called *level N*. Let us consider level 1, spanned by the states $P_\mu |\phi_a\rangle$. Suppose the conformal representation is degenerate on level 1. Then there are identities of the form

$$A_a^{\mu b} P_\mu |\phi_b\rangle = 0. \quad (2.15)$$

This is equivalent to

$$A_a^{\mu b} \partial_\mu \phi_b(0) |0\rangle = 0 \quad (2.16)$$

which, by a standard quantum field theory argument from translation invariance of correlation functions, is equivalent to

$$A_a^{\mu b} \partial_\mu \phi_b(x) = 0, \quad (2.17)$$

a first order differential equation with constant coefficients.

Conversely, if $\phi(x)$ satisfies a first order differential equation with constant coefficients, the differential equation can always be written as a set of equations in the form (2.17), which implies (2.16) and (2.15). So the conformal representation is degenerate on level 1.

In fact, we do not need the condition of constant coefficients. If $\phi(x)$ is a conformal field, then a first order differential equation with non-constant coefficients is equivalent to a first order equation with constant coefficients. Suppose ϕ satisfies

$$A_a^{\mu b}(x) \partial_\mu \phi_b(x) = 0. \quad (2.18)$$

For each x , conjugate with translation operators to get

$$A_a^{\mu b}(x) \partial_\mu \phi_b(0) |0\rangle = 0. \quad (2.19)$$

This is equivalent to performing the radial quantization with x as origin. Equation (2.19) asserts that a certain subspace $\mathcal{N}(x)$ of level 1 states is null. Let \mathcal{N} be the span of all the $\mathcal{N}(x)$ for all x ,

$$\mathcal{N} = \bigoplus_x \mathcal{N}(x) \subset \mathbb{C} \otimes V. \quad (2.20)$$

All the constraints on ϕ from the original differential equation (2.18) are expressed in the fact that \mathcal{N} is a null subspace. The fact that \mathcal{N} is a null subspace is expressed as well by the differential equation with constant coefficients

$$P_{\mathcal{N}}^\mu \partial_\mu \phi(x) = 0, \quad (2.21)$$

where $P_{\mathcal{N}}$ is the projection on the subspace \mathcal{N} . So the original first order differential equation on $\phi(x)$ with non-constant coefficients is equivalent to a differential equation with constant coefficients.

Therefore we can say that the conformal field $\phi(x)$ satisfies a first order differential equation iff the conformal representation is degenerate on level 1.

To see when level 1 is degenerate, we make the standard calculation of the matrix of inner products of the level 1 states,

$$\langle \phi_{b'} | P_{v'}^\dagger P_v | \phi_b \rangle = \langle \phi_{b'} | K_{v'} P_v | \phi_b \rangle = \langle \phi_{b'} | [K_{v'}, P_v] | \phi_b \rangle \quad (2.22)$$

$$= \langle \phi_{b'} | (2\delta_{v'v} D - 2L_{v'v}) | \phi_b \rangle \quad (2.23)$$

$$= \langle \phi_{b'} | (2\delta_{v'v} \Delta \delta_b^a - 2M_{v'v}^a) | \phi_a \rangle \quad (2.24)$$

The formal states $P_\mu | \phi_a \rangle$ of the Verma module form the $\mathfrak{so}(d)$ representation $\mathbb{C}^d \otimes V$, where \mathbb{C}^d is the fundamental representation. Define the self-adjoint matrix \hat{M} on $\mathbb{C}^d \otimes V$,

$$\hat{M}_{vb}^{\mu a} = M_{vb}^{\mu a}. \quad (2.25)$$

Then the inner product on level 1 states given by Eq. (2.24) is the hermitian quadratic form on $\mathbb{C}^d \otimes V$ corresponding to the self-adjoint matrix $2(\Delta - \hat{M})$. Therefore the representation is degenerate on level 1 iff Δ is an eigenvalue of \hat{M} [16]. The null vectors are then the eigenspace $\hat{M} = \Delta$.

A primary field $\phi(x)$ thus satisfies a first order differential equation iff Δ is an eigenvalue of the self-adjoint matrix \hat{M} acting on $\mathbb{C}^d \otimes V$.

The conformal representation is unitary on level 1 iff $\Delta - \hat{M} \geq 0$ which is to say that Δ cannot be smaller than the largest eigenvalue of \hat{M} . The largest eigenvalue of \hat{M} is the level 1 *unitarity bound*, the lower bound on Δ imposed by unitarity.

For example, consider a scalar field—a field in the trivial representation of $\mathfrak{so}(d)$. The trivial representation has $\hat{M} = 0$, so the level 1 unitarity bound is at $\Delta = 0$. At the unitarity bound, at $\Delta = 0$, the level 1 matrix of inner products is identically zero, so all the states on level 1 are null, so the field satisfies the first order differential equation $\partial_\mu \phi(x) = 0$, which implies that $\phi = 1$, the identity field.

For scalar fields, there is an additional unitarity condition at level 2,

$$\Delta(\Delta - (d-2)/2) \geq 0. \quad (2.26)$$

This of course allows the $\Delta = 0$ identity field—the ground state—but it forces any non-trivial scalar field to have dimension at least that of the free scalar field, $(d-2)/2$.

The complete set of unitarity conditions, taking account of all levels of the conformal representation, are known. For scalar fields, the level 1 and level 2 conditions are necessary and sufficient for unitarity. For fields in a non-trivial representation V , the level 1 unitarity condition is necessary and sufficient. This was established first for $d = 3$ [17] and $d = 4$ [18] and, finally, for any dimension [19].

2.3. The matrix \hat{M} in terms of Casimir invariants. Since finding the eigenvalues of the matrix \hat{M} is the central issue, let us rewrite it in terms of basic Lie algebra theoretic objects. Normalize the quadratic Casimir invariant of a representation V of $\mathfrak{so}(d)$ as

$$C_2^d(V) = -\frac{1}{4}M_{\mu\nu}M^{\mu\nu}. \quad (2.27)$$

The matrix \hat{M} on $\mathbb{C}^d \otimes V$ is then simply

$$\hat{M} = \mathbf{1} \otimes C_2^d(V) + C_2^d(\mathbb{C}^d) \otimes \mathbf{1} - C_2^d(\mathbb{C}^d \otimes V). \quad (2.28)$$

To see this, note that the fundamental representation, which is given in Eq. (2.1), is generated by the matrices

$$M_{\mu\nu\sigma}^{F\rho} = \delta_{\mu}^{\rho}\delta_{\nu\sigma} - \delta_{\nu}^{\rho}\delta_{\mu\sigma} \quad (2.29)$$

so

$$M_{\mu\nu\sigma}^{F\rho} M^{\mu\nu a}_b = 2M^{\rho}_\sigma{}^a_b = 2\hat{M}_{\sigma b}^{\rho a} \quad (2.30)$$

so

$$\hat{M} = \frac{1}{2}M_{\mu\nu}^F \otimes M^{\mu\nu} = \mathbf{1} \otimes C_2^d(V) + C_2^d(\mathbb{C}^d) \otimes \mathbf{1} - C_2^d(\mathbb{C}^d \otimes V). \quad (2.31)$$

So the eigenvalues of \hat{M} are gotten by decomposing $\mathbb{C}^d \otimes V$ into irreducible components and finding their quadratic Casimir invariants. The classical representation theory needed for this is collected in the appendices. We postpone calculating until we have found how to tell which $\mathfrak{so}(d)$ representations V give degenerate conformal representations that give first order differential equations with the Cauchy property.

3. First Order Differential Equations and the Cauchy Property

3.1. The first order differential equation. A conformal field $\phi(x)$ of spin V satisfies a first order differential equation iff Δ is an eigenvalue of \hat{M} . We want to know for what $\mathfrak{so}(d)$ representations V and scaling dimensions Δ does this first order differential equation have the Cauchy property.

The central object for that analysis is \hat{P}_{Δ} , the projection matrix acting on $\mathbb{C}^d \otimes V$ that projects on the eigenspace $(\mathbb{C}^d \otimes V)_{\Delta}$ where $\hat{M} = \Delta$,

$$\hat{P}_{\Delta} : \mathbb{C}^d \otimes V \rightarrow (\mathbb{C}^d \otimes V)_{\Delta}. \quad (3.1)$$

To simplify notations, we will often just write \hat{P} for \hat{P}_{Δ} . The matrix elements of \hat{P} are $\hat{P}_{vs}^{\mu r}$. The eigenspace $(\mathbb{C}^d \otimes V)_{\Delta}$ is the space of null states on level 1, so

$$\hat{P}_{vb}^{\mu a} P_{\mu}|\phi_a\rangle = 0. \quad (3.2)$$

Suppressing the indices for V , write \hat{P}_v^μ for the matrix on V with matrix elements $\hat{P}_{vb}^{\mu a}$. Then the null state conditions are written

$$\hat{P}_v^\mu P_\mu |\phi\rangle = 0, \quad (3.3)$$

equivalent to the differential equation

$$\hat{P}_v^\mu \partial_\mu \phi(x) = 0. \quad (3.4)$$

3.2. The Cauchy property as an algebraic condition. The Cauchy property is the condition on the differential equation that the values of $\phi(x)$ on a codimension 1 submanifold completely determine $\phi(x)$ everywhere. For a rotationally invariant first order equation with constant coefficients, this is simply the condition that $\partial_d \phi(x)$ is completely determined by the $\partial_i \phi(x)$, which is an algebraic condition.

Suppose $\phi(x)$ and $\tilde{\phi}(x)$ have the same spatial derivatives at x , $\partial_i \phi(x) = \partial_i \tilde{\phi}(x)$. We need the differential equation to imply that $\partial_d \phi(x) = \partial_d \tilde{\phi}(x)$. Writing $\delta\phi = \tilde{\phi} - \phi$, we need

$$\partial_i \delta\phi(x) = 0 \implies \partial_d \delta\phi(x) = 0. \quad (3.5)$$

Both $\phi(x)$ and $\tilde{\phi}(x)$ satisfy the differential equation (3.4) and the equation is linear, so $\delta\phi(x)$ also satisfies it,

$$0 = \hat{P}_v^\mu \partial_\mu \delta\phi(x). \quad (3.6)$$

Writing the indices for V explicitly, this is

$$\hat{P}_{vb}^{da} \partial_d \delta\phi_a(x) + \hat{P}_{vb}^{ia} \partial_i \delta\phi_a(x) = 0. \quad (3.7)$$

But the spatial derivatives are zero, so the differential equation gives

$$\hat{P}_{vb}^{da} \partial_d \delta\phi_a(x) = 0. \quad (3.8)$$

We need this to imply that $\partial_d \delta\phi_r(x) = 0$.

Define \hat{P}^d to be the matrix with matrix elements \hat{P}_{vb}^{da} . It is a linear map from V to the $\hat{M} = \Delta$ eigenspace,

$$\hat{P}^d : V \rightarrow (\mathbb{C}^d \otimes V)_\Delta, \quad \hat{P}^d v = \hat{P}(\hat{e}_d \otimes v), \quad (3.9)$$

where \hat{e}_d is the unit vector in the d -direction in \mathbb{C}^d .

We need

$$\hat{P}^d \partial_d \delta\phi(x) = 0 \implies \partial_d \delta\phi(x) = 0. \quad (3.10)$$

But $\partial_d \delta\phi(x)$ can be any vector in V . So this is simply the condition

$$\forall v \in V, \hat{P}^d v = 0 \implies v = 0, \quad (3.11)$$

which is the condition that \hat{P}^d is injective. So we have shown that the Cauchy property is equivalent to the condition that \hat{P}^d is injective,

A1 The first order differential equation satisfied by $\phi(x)$ has the Cauchy property iff the matrix $\hat{P}^d : V \rightarrow (\mathbb{C}^d \otimes V)_\Delta$ is injective.

Now define \hat{P}_d^d to be the matrix with matrix elements \hat{P}_{db}^{da} . It is a linear map from V to V ,

$$\hat{P}_d^d : V \rightarrow V, \quad \hat{P}_d^d v = \text{Proj}_{\mathbb{C}\hat{e}_d \otimes V}(\hat{P}^d v). \quad (3.12)$$

We will argue that **A1** is equivalent to

A2 The first order differential equation satisfied by $\phi(x)$ has the Cauchy property iff the matrix $\hat{P}_d^d : V \rightarrow V$ is invertible.

To see the equivalence, first note that if \hat{P}^d is not injective, then there is a non-zero vector v with $\hat{P}^d v = 0$, which implies $\hat{P}_d^d v = 0$, so \hat{P}_d^d cannot be invertible. Now suppose \hat{P}_d^d is not invertible. We will use the invariant inner-products on V and on the tensor product space $\mathbb{C}^d \otimes V$. If \hat{P}_d^d is not invertible, there must be a non-zero $v \in V$ such that $\hat{P}_d^d v = 0$, so

$$(v, \hat{P}_d^d v) = 0. \quad (3.13)$$

But $(v, \hat{P}_d^d v) = (\hat{e}_d \otimes v, \hat{P} \hat{e}_d \otimes v)$, so

$$(\hat{e}_d \otimes v, \hat{P} \hat{e}_d \otimes v) = 0. \quad (3.14)$$

\hat{P} is a self-adjoint projection, so

$$(\hat{P} \hat{e}_d \otimes v, \hat{P} \hat{e}_d \otimes v) = (\hat{e}_d \otimes v, \hat{P}^2 \hat{e}_d \otimes v) = (\hat{e}_d \otimes v, \hat{P} \hat{e}_d \otimes v) = 0. \quad (3.15)$$

The invariant inner-product is positive definite, so we have

$$\hat{P}^d v = \hat{P} \hat{e}_d \otimes v = 0,$$

so we have shown that \hat{P}_d^d not invertible implies \hat{P}^d not injective. So the algebraic Cauchy conditions are equivalent.

The *symbol* of the first order differential operator $\hat{P}_v^\mu \partial_\mu$ in the differential equation (3.4) is the map from unit co-vectors to matrices,

$$\hat{n}_\mu \mapsto \hat{n}_\mu \hat{P}_{vb}^{\mu a}. \quad (3.16)$$

By rotational invariance, we might as well choose \hat{n} in the d -direction, in which case the symbol is the matrix \hat{P}^d . So the symbol for any value of \hat{n}_μ is conjugate to \hat{P}^d by some rotation. So our Cauchy condition is exactly the condition that the symbol of the first order differential operator is injective.

A differential operator is said to be *elliptic* when its symbol is invertible. A differential operator is said to be *overdetermined elliptic* when its symbol is injective but not invertible. (Sometimes the label *elliptic* is used for both.)

It seems to us that the natural generalization of $\hat{\partial}$ to conformal fields in $d > 2$ is the Cauchy property as we have defined it, the property of having an injective symbol, which is the property that allows integrating the first order differential equation from data on a codimension 1 submanifold.

The label *overdetermined* refers to the fact that not all initial data on the codimension 1 submanifold is possible. The differential equation imposes linear constraints on the value of $\phi(x)$ on the submanifold. We discuss those constraints in Sect. 8.3 below.

As an illustration, let us explain in more detail how the first order equation satisfied by a conserved current, $\partial_\mu j^\mu(x) = 0$, fails to satisfy the conditions **A1** and **A2**. The first order differential equation (3.4) is

$$\hat{P}_{v\beta}^{\mu\alpha} \partial_\mu j_\alpha(x) = 0 \quad (3.17)$$

with the projection \hat{P} being

$$\hat{P}_{v\beta}^{\mu\alpha} = \delta^{\mu\alpha} \delta_{v\beta}. \quad (3.18)$$

Then

$$\hat{P}_{v\beta}^{d\alpha} = \delta^{d\alpha} \delta_{v\beta}, \quad (\hat{P}v)_{v\beta} = \hat{P}_{v\beta}^{d\alpha} v_\alpha = v^d \delta_{v\beta}. \quad (3.19)$$

If $v^d = 0$ then $\hat{P}^d v = 0$. So \hat{P}^d is not injective. The matrix \hat{P}_d^d is

$$P_{d\beta}^{d\alpha} = \delta^{d\alpha} \delta_{d\beta} \quad (3.20)$$

which is the projection matrix on the d -direction, which is obviously not invertible.

4. Cauchy Representations

From our discussion in the previous section, we define a field ϕ to be a Cauchy field if it satisfies a first order equation with injective symbol. This property then determines ϕ everywhere uniquely once we know its value on some codimension 1 submanifold.

We argued in Sect. 2.2 that short representations lead to differential equations. However, as we illustrated with the conserved current, not every short representation leads to a differential equation which satisfies the Cauchy condition. Cauchy representations are thus a proper subset of short representations, and we want to classify them.

In this section, we present all the unitary Cauchy representations. We show here that all of them have the Cauchy property. The proof that there are no other unitary Cauchy representations is given in Appendix B.1.

4.1. The identity field. The most obvious example is the trivial representation. If V is the trivial $\mathfrak{so}(d)$ representation, $M_{\mu\nu} = 0$, then $\Delta = 0$ is the level 1 degeneracy condition. If $\Delta = 0$, then all of level 1 is null. The first order differential equation is $\partial_\mu \phi(x) = 0$, whose solution is the identity field $\phi(x) = 1$. In the following we will thus only consider non-scalar (non-trivial) $\mathfrak{so}(d)$ representations V .

4.2. V_s for $d = 2n$. We saw at the beginning that the (anti-)self-dual n -forms Λ_\pm^n are Cauchy fields in even dimensions $d = 2n$. Let us now describe a larger set of examples in $d = 2n$, namely the representations (1.6),

$$V_s = V_{\lambda_s}, \quad \lambda_s = (|s|, \dots, |s|, s), \quad s \neq 0. \quad (4.1)$$

In particular, $V_{\pm\frac{1}{2}}$ are the chiral spinors, $V_{\pm 1}$ are the (anti-)self-dual n -forms.

For these examples we can make the rather abstract discussion in (3.2) much more concrete by giving very explicit expressions for the projectors $(\hat{P}_\Delta)_d^d$. First note from (A.12) that the tensor product $\mathbb{C}^d \otimes V_s$ decomposes into exactly two components,

$$\mathbb{C}^d \otimes V_s = V_{\lambda_s + \epsilon_1} \oplus V_{\lambda_s \mp \epsilon_n}, \quad \text{for } s = \pm |s|, \quad (4.2)$$

so that \hat{M} has two eigenspaces. Its eigenvalues can be calculated using its expression in terms of quadratic Casimirs (2.28) and formula (A.11) for the quadratic Casimirs,

$$\hat{M} = \begin{cases} -|s| & \text{on } V_{\lambda_s + \epsilon_1} \\ n - 1 + |s| & \text{on } V_{\lambda_s \mp \epsilon_n} \end{cases}. \quad (4.3)$$

Since \hat{M} has two distinct eigenvalues, and since the sum of the eigenvalues is $n - 1$, the projection on the subspace $\hat{M} = \Delta$ is simply

$$\hat{P}_\Delta = \frac{\hat{M} - (n - 1 - \Delta)}{\Delta - (n - 1 - \Delta)}. \quad (4.4)$$

Since $\hat{M}_d^d = 0$ we have

$$(\hat{P}_\Delta)_d^d = \frac{-(n - 1 - \Delta)}{\Delta - (n - 1 - \Delta)} \quad (4.5)$$

which is a non-zero multiple of the identity matrix. So \hat{P}_Δ^d is invertible. Therefore (V_s, Δ) is a Cauchy representation for Δ either of the two eigenvalues of \hat{M} .

The largest eigenvalue of \hat{M} is $n - 1 + |s|$, which is therefore the level 1 unitarity bound. In fact, since we are considering $s \neq 0$ only, it is the exact unitarity bound. So $V = V_s, \Delta = n - 1 + |s|$ gives a unitary Cauchy field, while $V = V_s, \Delta = -|s|$ gives a non-unitary Cauchy field. The Cauchy differential equation (3.4) is

$$\left[\hat{M}_v^\mu - (m - \Delta) \delta_v^\mu \right] \partial_\mu \phi(x) = 0, \quad m = \frac{1}{2}(d - 2). \quad (4.6)$$

For the unitary field, $\Delta = |s| + m$, the differential equation is

$$\left(\hat{M}_v^\mu + |s| \delta_v^\mu \right) \partial_\mu \phi(x) = 0. \quad (4.7)$$

4.3. V_s for $d = 2n + 1$. The Cauchy fields in odd dimensions are even more restricted. From (A.12) we see that for $s > 1/2$ we have

$$\mathbb{C}^d \otimes V_s = \begin{cases} V_{\lambda_s} \oplus V_{\lambda_s + \epsilon_1} \oplus V_{\lambda_s - \epsilon_n}, & \text{for } s > 1/2, \\ V_{\lambda_s} \oplus V_{\lambda_s + \epsilon_1}, & \text{for } s = 1/2. \end{cases} \quad (4.8)$$

Only for $s = 1/2$ are there again only two representations in the tensor product, so only for $s = 1/2$ can we repeat the above argument for the Cauchy property. Indeed, in Appendix B.1 we show that $s > 1/2$ gives non-Cauchy fields.

For $s = 1/2$, the eigenvalues of \hat{M} are $\frac{1}{2}(d - 1)$ and $-s$. The unitarity bound is $\Delta = \frac{1}{2}(d - 1)$. This is the massless spinor field. The Cauchy differential equation for either the unitary or non-unitary Cauchy field is

$$\left[\hat{M}_v^\mu - (m - \Delta) \delta_v^\mu \right] \partial_\mu \phi(x) = 0, \quad m = \frac{1}{2}(d - 2). \quad (4.9)$$

For the unitary field, $\Delta = m + |s|$, the differential equation is

$$\left(\hat{M}_v^\mu + |s| \delta_v^\mu \right) \partial_\mu \phi(x) = 0. \quad (4.10)$$

It can be shown that this equation is equivalent to the massless Dirac equation.

4.4. Summary of the Cauchy fields. We now have the classification of the non-trivial Cauchy conformal fields in unitary conformal field theories. Write $m = \frac{1}{2}(d - 2)$.

In even dimensions d , the non-trivial Cauchy conformal fields are the conformal primary fields of spin V and conformal weight Δ in the set

$$\left\{ (V, \Delta) = (V_s, |s| + m) : s \in \frac{1}{2}\mathbb{Z}, s \neq 0 \right\} \quad (4.11)$$

In odd dimensions $d = 2n$, the only non-trivial Cauchy conformal field is the massless spinor field, $V = V_s, s = \frac{1}{2}, \Delta = |s| + m$.

For both d even and for d odd, the Cauchy differential equation is

$$\left[\hat{M}_\nu^\mu - (m - \Delta)\delta_\nu^\mu \right] \partial_\mu \phi(x) = 0, \quad m = \frac{1}{2}(d - 2). \quad (4.12)$$

For the unitary cases,

$$\Delta = |s| + m, \quad \left(\hat{M}_\nu^\mu + |s|\delta_\nu^\mu \right) \partial_\mu \phi(x) = 0. \quad (4.13)$$

4.5. The spin V_s Cauchy fields as “free fields”. Conformal fields in the representations V_s were previously studied in somewhat different context [8], as a subset of the so-called “free field” conformal representations—the conformal fields $\phi(x)$ satisfying the massless Klein–Gordon equation

$$\partial^\mu \partial_\mu \phi(x) = 0. \quad (4.14)$$

Indeed, applying ∂^ν to Eq. (4.12) gives $\partial^\mu \partial_\mu \phi(x) = 0$. Conversely, if $P^\nu P_\nu |\phi\rangle = 0$, then the identity $[K_\mu, P^\nu P_\nu]|\phi\rangle = 0$ gives exactly the non-trivial first order differential equation (4.6), except for the case of a scalar field with the scaling dimension of the free massless scalar field. A conformal field with non-zero spin satisfying Eq. (4.6) necessarily belongs to an $\mathfrak{so}(d)$ representation V such that \hat{M} has at most two eigenvalues. So the “free field” representations are the representations V_s plus the representation of the free massless scalar field.

The label “free field” for conformal fields satisfying $\partial^\mu \partial_\mu \phi(x) = 0$ was a misnomer. Additional work is needed to show that $\phi(x)$ is actually a free field, *i.e.* a field whose correlation functions are gaussian, given in terms of its two-point function by Wick contractions. For $d = 2$, this is obviously not the case, as shown by the many non-free examples. In the following two sections we will develop the tools to show that for $d > 2$, all unitary Cauchy fields are indeed free.

5. Mode Expansions

In this and the following section we prove two negative results about the Cauchy conformal fields in unitary theories in dimensions $d > 2$: First, they are free fields—their correlation functions are the products of the 2-point function following the free-field Wick contraction rules. Second, they have a local stress-energy tensor only for $|s| \leq 1$.

To establish this result, we go through the following steps:

1. We construct a mode expansion for the field $\phi(x)$, expanding in spherical harmonics on the $(d - 1)$ -sphere.

2. We use the mode expansion and the branching rules for tensor products of $\mathfrak{so}(d)$ representations to show that the singular part of the operator product expansion contains only the identity field.
3. We then point out that the commutator depends only on the singular part of the operator product expansion. It follows that the commutator is a multiple of the identity, so that the correlation functions are indeed given by free-field Wick contractions.
4. We finally point out that, for $|s| > 1$, there is no field in the operator product expansion with the spin and scaling dimension of the stress-energy tensor.

This is perhaps not the most elegant proof, but it has the virtue of efficiency. We deviate from the spirit of $d = 2$ holomorphic fields, since we do not calculate the commutators by contour deformation. In fact we have not yet established that contour deformation is valid. We return to the question of deformability in Sect. 8. It would be more elegant to use those techniques: construct the field modes by smearing $\phi(x)$ over a codimension 1 surface against smearing functions that satisfy the dual Cauchy equation, then calculate the commutator of the field modes by contour deformation. This is a priori a more powerful technique, because the commutator so derived would depend only on the part of the operator product expansion with singularity at least $O(r^{1-d})$, not on the entire singular part.

As it happens, our cruder approach is powerful enough to establish the two results. We can easily enforce a subset of the constraints implied by the Cauchy differential equation to restrict the modes enough to control the singular part of the operator product expansion. We do not need the precise list of field modes—some of the modes on our list might be zero. We will return to this question in Sect. 7.1 to show that they are all in fact non-vanishing. For the moment however our list of the possible modes is small enough to give our two negative results on the unitary Cauchy fields.

5.1. Expand in spherical harmonics. We describe the operator representation of a Cauchy field in the radial quantization by expanding the field in spherical harmonics. Suppose $\phi(x) = \phi(r\hat{x})$ is a Cauchy conformal field with spin V_s and conformal weight Δ .

The rotation Lie algebra $\mathfrak{so}(d)$ acts on the field by the generators

$$M_{\mu\nu}^{tot} \phi(x) = (M_{\mu\nu}^{orb} + M_{\mu\nu}^{spin}) \phi(x) \quad (5.1)$$

where, as in (2.13), the $M_{\mu\nu}^{spin}$ generate the spin representation V_s and the orbital generators

$$M_{\mu\nu}^{orb} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad (5.2)$$

generate the rotations acting on the unit sphere, on functions of \hat{x} . The functions on the unit sphere decompose into irreducible representations of the $M_{\mu\nu}^{orb}$, the spherical harmonics,

$$L_2(S^{d-1}) = \bigoplus_{l=0}^{\infty} S_l, \quad (5.3)$$

where

$$S_l = V_{(l,0,\dots,0)} \quad (5.4)$$

is the irreducible representation of $\mathfrak{so}(d)$ on traceless symmetric l -tensors.

Expand the field ϕ in spherical harmonics

$$\phi(r\hat{x}) = \sum_{l=0}^{\infty} Y_l(\hat{x})\phi_l(r), \quad Y_l \in S_l, \quad (5.5)$$

$$Y_l(\hat{x})\phi_l(r) = \frac{1}{l!} \hat{x}^{\mu_1} \dots \hat{x}^{\mu_k} \phi_l(r)_{\mu_1 \dots \mu_k}. \quad (5.6)$$

The mode $\phi_l(r)$ is a traceless symmetric l -tensor with values in V_s . Note that we are suppressing the V_s index in our labeling of the field $\phi(x)$ and its modes $\phi_l(x)$.

The field mode $\phi_l(r)$ lies in a sub-representation of $S_l \otimes V_s$. The next steps are to use the Cauchy equation to put constraints on this sub-representation.

5.2. *Constrain the modes using $\partial^\mu \partial_\mu \phi = 0$.* Again, applying ∂^ν to Eq. (4.12) gives $\partial^\mu \partial_\mu \phi = 0$. So $\phi(x)$ is a harmonic function, with the usual expansion in spherical harmonics.

To be explicit, the laplacian in polar coordinates is

$$\partial_\mu \partial^\mu = r^{-2} \left[(r \partial_r)^2 + (d-2)r \partial_r - 2C_2^{d,orb} \right] \quad (5.7)$$

where

$$C_2^{d,orb} = -\frac{1}{4} M_{\mu\nu}^{orb} M_{orb}^{\mu\nu} \quad (5.8)$$

is the quadratic Casimir operator of the orbital representation. Applying the laplacian to the mode expansion (5.5), using the quadratic Casimir invariants

$$C_2^d(S_l) = \frac{1}{2} l(l+d-2) \quad (5.9)$$

gives

$$(r \partial_r - l)(r \partial_r + d - 2 + l)\phi_l(r) = 0. \quad (5.10)$$

The general solution is

$$\phi_l(r) = r^l \phi_l^+ + r^{-l-2m} \phi_l^-. \quad (5.11)$$

Recall that we defined $m = \frac{d-2}{2}$.

We can read off the scaling dimensions directly, or use (2.4),

$$\dim(\phi_l^+) = \Delta + l, \quad \dim(\phi_l^-) = \Delta - l - 2m. \quad (5.12)$$

Now we follow the usual convention of labeling the field modes by minus the scaling dimension,

$$\phi_k = \begin{cases} \phi_{-k-\Delta}^+ & = \phi_{|k+\Delta-m|-m}^+ & : & k + \Delta \leq 0, \\ 0 & & : & 0 < k + \Delta < 2m, \\ \phi_{k-2m+\Delta}^- & = \phi_{|k+\Delta-m|-m}^- & : & 2m \leq k + \Delta, \end{cases} \quad (5.13)$$

$$k + \Delta \in \mathbb{Z}, \quad \phi_k \in S_{|k+\Delta-m|-m} \otimes V_s, \quad \dim(\phi_k) = -k, \quad [D, \phi_k] = -k\phi_k. \quad (5.14)$$

The mode expansion is now

$$\phi(x) = \sum_{\substack{k+\Delta \in \mathbb{Z} \\ |k+\Delta-m| \geq m}} r^{-k-\Delta} Y_{|k+\Delta-m|-m}(\hat{x}) \phi_k. \quad (5.15)$$

Next note that the correlation functions must be regular as $r \rightarrow 0$, so $\phi(x)|0\rangle$ must be regular, where $|0\rangle$ is the ground state of the radial quantization. Therefore,

$$\phi_k|0\rangle = 0, \quad k > -\Delta \quad (5.16)$$

and the operator state correspondence is

$$|\phi\rangle = \phi(0)|0\rangle = \phi_{-\Delta}|0\rangle. \quad (5.17)$$

We will thus identify the ϕ_k with $k \geq 2m - \Delta$ with annihilation modes, and the ϕ_k with $k \leq -\Delta$ with creation modes.

5.3. Constrain the modes using the radial Cauchy equation. So far, we have only used the fact that any field satisfying $\partial^\mu \partial_\mu \phi = 0$ has a discrete expansion in harmonic functions to get the mode expansion (5.15). The mode ϕ_k forms a subrepresentation of $S_{|k+\Delta-m|-m} \otimes V_s$. The next step is to show that the radial component of the first order differential equation (4.12),

$$x^\nu \left[\hat{M}_\nu^\mu - (m - \Delta) \delta_\nu^\mu \right] \partial_\mu \phi(x) = 0, \quad (5.18)$$

fixes the quadratic Casimir invariant of ϕ_k . Then we use the known decomposition of the tensor product $S_{|k+\Delta-m|-m} \otimes V_s$ into irreducible representations to find that the Casimir invariant of ϕ_k either does not occur in the decomposition, or identifies a unique irreducible. Thus certain of the modes ϕ_k must vanish identically, and each of the rest of the ϕ_k are in specific irreducible $\mathfrak{so}(d)$ representations.

The representation of $\mathfrak{so}(d)$ on $\phi(x)$ is the tensor product of the spin and orbital representations, as given in Eq. (5.1). Using Eq. (5.2) for the orbital generators,

$$\frac{1}{2} M_{\mu\nu}^{orb} M_{spin}^{\mu\nu} \phi_s(x) = x^\nu \partial^\mu M_{\mu\nu}^r \phi_r(x) = x^\nu \hat{M}_\nu^{\mu r} \partial_\mu \phi_r(x). \quad (5.19)$$

The $\mathfrak{so}(d)$ quadratic Casimir operators are

$$C_2^{d,orb} = -\frac{1}{4} M_{\mu\nu}^{orb} M_{orb}^{\mu\nu}, \quad C_2^{d,spin} = -\frac{1}{4} M_{\mu\nu}^{spin} M_{spin}^{\mu\nu}, \quad C_2^{d,tot} = -\frac{1}{4} M_{\mu\nu}^{tot} M_{tot}^{\mu\nu}, \quad (5.20)$$

so

$$\frac{1}{2} M_{\mu\nu}^{orb} M_{spin}^{\mu\nu} = C_2^{d,orb} + C_2^{d,spin} - C_2^{d,tot}, \quad (5.21)$$

so Eq. (5.18), the radial component of the Cauchy differential equation, can be written

$$\left[C_2^{d,orb} + C_2^{d,spin} - C_2^{d,tot} - (m - \Delta) r \partial_r \right] \phi(x) = 0 \quad (5.22)$$

Substituting the mode expansion (5.15) and taking account of the independence of the terms in the expansion, we get

$$\left[C_2^{d,orb} + C_2^{d,spin} - C_2^{d,tot} + (m - \Delta)(k + \Delta) \right] \phi_k = 0, \quad (5.23)$$

which, since ϕ_k lies in $S_{|k+\Delta-m|-m} \otimes V_s$, fixes the quadratic Casimir invariant of ϕ_k to be

$$C_2^{d,tot} \phi_k = \left[C_2^d(S_{|k+\Delta-m|-m}) + C_2^d(V_s) + (m - \Delta)(k + \Delta) \right] \phi_k. \quad (5.24)$$

Using

$$C_2^d(S_l) \phi_k = \frac{1}{2} l(l + d - 2) \phi_k, \quad (5.25)$$

with

$$l = |k + \Delta - m| - m, \quad (5.26)$$

we get a simple formula for the quadratic Casimir of ϕ_k ,

$$C_2^{d,tot} \phi_k = \left[C_2^d(V_s) + \frac{1}{2}(-\Delta^2 + k^2) \right] \phi_k. \quad (5.27)$$

The second step is to compare these Casimir values with the Casimirs of the irreducible representations that occur in the decomposition of the tensor product $S_{|k+\Delta-m|-m} \otimes V_s$. That decomposition is given in [20, Props 9.4 and 9.5]. For d even,

$$S_l \otimes V_{(|s|, \dots, |s|, \pm|s|)} = \bigoplus_{s'=\max(-|s|, -l+|s|)}^{|s|} V_{(l+s', |s|, \dots, |s|, \pm s')}. \quad (5.28)$$

For d odd, the decomposition of the tensor product is, for $s = 0$ and $s = 1/2$,

$$S_l \otimes V_{(|s|, \dots, |s|, |s|)} = \bigoplus_{s'=\max(-|s|, -l+|s|)}^{|s|} V_{(l+s', |s|, \dots, |s|, |s'|)}. \quad (5.29)$$

For d even, the Casimir invariants of the individual components are

$$\begin{aligned} & C_2^d(V_{(l+s', |s|, \dots, |s|, \pm s')}) \\ &= C_2^d(V_s) + \frac{1}{2} \left[(l + s')(l + s' + d - 2) + s'^2 - |s|(|s| + d - 2) - s^2 \right]. \end{aligned} \quad (5.30)$$

The Casimir of the component increases monotonically in s' , given the inequalities on s' in the decomposition (5.28). Therefore the Casimir of ϕ_k can match the Casimir of at most one component, so ϕ_k will lie in an irreducible representation, or will vanish.

The Casimir of the component labelled by s' agrees with the Casimir of ϕ_k given in Eq. (5.27) iff s' satisfies the quadratic equation

$$(l + s')(l + s' + d - 2) + s'^2 - |s|(|s| + d - 2) - s^2 = -\Delta^2 + k^2, \quad (5.31)$$

which can be re-arranged as

$$\left[s' + \frac{1}{2}(l + m) \right]^2 - \frac{1}{4}(k - \Delta + m)^2 = (|s| + m - \Delta)(\Delta + |s|). \quad (5.32)$$

The rhs vanishes, because Δ has one of the two values: $|s| + m$ in the unitary case, $-|s|$ in the non-unitary case. Since $l = |k + \Delta - m| - m$, the two roots are

$$s' = -\frac{1}{2} |k + \Delta - m| \pm \frac{1}{2} (k - \Delta + m). \quad (5.33)$$

We also need s' to satisfy the inequalities dictated by the decomposition (5.28),

$$-|s| \leq s', \quad -l + |s| \leq s', \quad s' \leq |s|. \quad (5.34)$$

For both the annihilation modes with $k + \Delta - m \geq m$ and the creation modes with $k + \Delta - m \leq -m$, we calculate the two roots s' , then check for which of the two values of Δ and for which values of k each root satisfies all the inequalities (assuming $d > 2$ so that $m > 0$):

	Root	Solutions
$k + \Delta - m \geq m$	$s' = -k$	none
	$s' = m - \Delta$	$\Delta = m + s , \quad k \geq \Delta$
$k + \Delta - m \leq -m$	$s' = -m + \Delta$	$\Delta = m + s , \quad k \leq -\Delta$
	$s' = k$	$\Delta = - s , \quad \Delta \leq k \leq -\Delta$

In the unitary case $\Delta = m + |s|$, we see that the modes ϕ_k with $m - |s| \leq k < \Delta$ have been eliminated, because their Casimirs do not occur in the decomposition of the tensor product. The mode expansion in the unitary case is

$$\phi(x) = \sum_{k+\Delta \in \mathbb{Z}, |k| \geq \Delta} r^{-k-\Delta} Y_l(\hat{x}) \phi_k. \quad (5.35)$$

where

$$\Delta = m + |s|, \quad l = |k + \Delta - m| - m = \begin{cases} k - \Delta + 2|s|, & k \geq \Delta \\ -k - \Delta, & k \leq -\Delta. \end{cases} \quad (5.36)$$

and the mode ϕ_k is in the irreducible representation

$$\phi_k \in V_{(|k|-\Delta+|s|, |s|, \dots, |s|, -\epsilon s)} \quad \epsilon = \text{sgn}(k). \quad (5.37)$$

At this point, it remains possible that some of these modes are identically zero, since we have only enforced the radial component of the first order differential equation. When we calculate the singular operator product expansion, we will find that all these modes are non-zero.

For the non-unitary case, $\Delta = -|s|$, only a finite number of modes can be non-zero,

$$\phi(x) = \sum_{k-|s| \in \mathbb{Z}, |k| \leq |s|} r^{-k+|s|} Y_{|s|-k}(\hat{x}) \phi_k. \quad (5.38)$$

$$\phi_k \in V_{(|s|, |s|, \dots, |s|, \epsilon' k)}, \quad \epsilon' = \text{sgn}(s). \quad (5.39)$$

Note that in the non-unitary case, the expansion in r is a polynomial. We will return to the consequences of this observation in the next section.

Finally, let us discuss the modes in odd dimension d . In this case, there is only one non-trivial Cauchy representation, $V_{(s, \dots, s)}$ with $s = 1/2$. The decomposition (5.29) is

$$S_l \otimes V_{(s, \dots, s)} = \bigoplus_{s' = \pm 1/2} V_{(l+s', s, \dots, s)}, \quad s = \frac{1}{2}, \quad l \geq 1. \quad (5.40)$$

The quadratic Casimir invariants of the components are

$$C_2^d(V_{(l+s',s,\dots,s)}) = C_2^d(V_{(s,\dots,s)}) + \frac{1}{2} [(l+s')(l+s'+d-2) - s(s+d-2)]. \quad (5.41)$$

Matching to the Casimirs of the ϕ_k , Eq. (5.27), gives the condition

$$(l+s')(l+s'+d-2) - s(s+d-2) = -\Delta^2 + k^2, \quad (5.42)$$

which can be re-written, since $s' = \pm \frac{1}{2}$,

$$s'|k + \Delta - m| = (m - \Delta)(k + \Delta - m) + m \left(m + \frac{1}{2} - \Delta \right). \quad (5.43)$$

For the unitary case, $\Delta = \frac{1}{2}(d-1) = m + \frac{1}{2}$, this is

$$s' \left| k + \frac{1}{2} \right| = -\frac{1}{2} \left(k + \frac{1}{2} \right), \quad (5.44)$$

The conformal dimension $\Delta = m + \frac{1}{2}$ is an integer, so the weights k of the modes are integers, so the solutions are : $s' = \frac{1}{2}$ for $k \leq -1$ and $s' = -\frac{1}{2}$ for $k \geq 0$, $l > 0$.

The mode expansion for d odd, in the unitary case, is obtained by combining with the results of the harmonic expansion, Eq. (5.15),

$$\phi(x) = \sum_{|k| \geq \Delta} r^{-k-\Delta} Y_l(\hat{x}) \phi_k, \quad (5.45)$$

$$\Delta = m + \frac{1}{2}, \quad l = |k + \Delta - m| - m = \begin{cases} k - \Delta + 1, & k \geq \Delta \\ -k - \Delta, & k \leq -\Delta, \end{cases} \quad (5.46)$$

$$\phi_k \in V_{(l-\epsilon s, s, \dots, s)} = V_{(|k|-\Delta+s, s, \dots, s)}, \quad \epsilon = \text{sgn}(k). \quad (5.47)$$

For the non-unitary case, $\Delta = -\frac{1}{2}$, the matching equation (5.43) is

$$s' \left| k - \frac{1}{2} - m \right| = \left(m + \frac{1}{2} \right) k - \frac{1}{4}. \quad (5.48)$$

The only solutions are $k = \frac{1}{2}$, $s' = \frac{1}{2}$ with $l = 0$ and $k = -\frac{1}{2}$, $s' = -\frac{1}{2}$ with $l = 1$. The mode expansion is thus

$$\phi(x) = Y_0(\hat{x}) \phi_{1/2} + r^1 Y_1(\hat{x}) \phi_{-1/2} \quad (5.49)$$

$$\phi_{\pm 1/2} \in V_{(s, s, \dots, s)}. \quad (5.50)$$

5.4. *Summary of the mode expansions.* We have established that the non-trivial unitary Cauchy fields are the fields with spin V_s ,

$$\text{for } d \text{ odd: } s = \frac{1}{2}, \quad (5.51)$$

$$\text{for } d = 2n \text{ even: } s \in \left\{ \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \dots \right\}, \quad (5.52)$$

and scaling dimension

$$\Delta = |s| + m, \quad m = \frac{1}{2}(d - 2), \quad (5.53)$$

and with mode expansion

$$\phi(x) = \sum_{|k| \geq \Delta} r^{-k-\Delta} Y_{|k+\Delta-m|-m}(\hat{x}) \phi_k. \quad (5.54)$$

$$\text{for } d \text{ odd: } \phi_k \in V_{(|k+s|-m-\epsilon, s, \dots, s)} \quad \epsilon = \text{sgn}(k), \quad (5.55)$$

$$\text{for } d = 2n \text{ even: } \phi_k \in V_{(|k|-\Delta+|s|, |s|, \dots, |s|, -\epsilon s)} \quad \epsilon = \text{sgn}(k). \quad (5.56)$$

6. Operator Products, Commutators, and Stress-Energy Tensor

Using the mode expansions, we show now that the commutators of the modes of unitary Cauchy fields are multiples of the identity operator. Therefore the correlation functions can be calculated by Wick contractions, so the unitary Cauchy fields are free fields.

In a first step we will show that the commutator of two Cauchy fields is completely determined by the singular part of the operator product expansion of the two fields. Then we show that the singular part of the operator product expansion of two Cauchy fields contains only the identity operator.

We also show that the operator product of two Cauchy fields contains a field with the spin and scaling dimension of the stress-energy tensor iff d is even and $s = \pm 1/2$ or $s = \pm 1$ or d is odd and $s = 1/2$. So the only Cauchy fields with local stress-energy tensors are the massless spinor fields in any dimension and the free (anti-)self-dual n -form fields in even dimensions $d = 2n$.

6.1. *Commutators and the singular part of the OPE.* The first step is to show that the commutator of a mode ϕ_k of a Cauchy field $\phi(x)$ with any field $\psi(x)$ is completely determined by the singular part of the operator product expansion of the two fields.

Let us extract the mode ϕ_k of the Cauchy field $\phi(x)$ by smearing over the sphere of radius r with the appropriate vector-valued spherical harmonic,

$$\int_{S_r^{d-1}} d\Omega F_k(\hat{x}) \phi(x) = r^{-k-\Delta} \phi_k. \quad (6.1)$$

Suppose $\psi(y)$ is any local field. Let $R(\phi(x)\psi(y))$ be the radially ordered operator product. The commutator of ϕ_k with $\psi(y)$ can be calculated by evaluating (6.1) at radius $r = |y| + \epsilon$ and at radius $r = |y| - \epsilon$,

$$\begin{aligned} \int_{S_{|y|+\epsilon}^{d-1}} - \int_{S_{|y|-\epsilon}^{d-1}} d\Omega F_k(\hat{x}) R(\phi(x)\psi(y)) &= (|y| + \epsilon)^{-k-\Delta} \phi_k \psi(y) - (|y| - \epsilon)^{-k-\Delta} \psi(y) \phi_k \\ &= |y|^{-k-\Delta} [\phi_k, \psi(y)] + O(\epsilon). \end{aligned} \quad (6.2)$$

If the OPE of $\phi(x)\psi(y)$ is non-singular, then $R(\phi(x)\psi(y))$ is bounded in the integrals on the left-hand side, so that the left-hand side is $O(\epsilon)$. Sending $\epsilon \rightarrow 0$, we see that the commutator $[\phi_k, \psi(y)]$ vanishes. Therefore the commutator only depends on the singular part of the OPE.

In fact, the deformability property of Cauchy fields as described in Sect. 8 will allow us to deform the integration domain on the lhs of (6.2) to a small sphere centered at y . We then see that the commutator depends only on the singular part of the OPE of $\phi(x)\psi(y)$ that is at least as singular as $|x - y|^{-(d-1)}$. For the present purpose we do not need this stronger result.

6.2. Commutators of Cauchy fields. The analysis in Sect. 5 showed that there are no modes ϕ_k in the range $-\Delta < k < \Delta$. This has drastic consequences for the operator product expansions of Cauchy fields.

Suppose $\phi_1(x)$ and $\phi_2(x)$ are Cauchy fields. We can suppose, without loss of generality, that $\Delta_1 \geq \Delta_2$ (exchanging $\phi_1 \leftrightarrow \phi_2$ if necessary).

The conformal highest weight state corresponding to $\phi_2(x)$ is

$$\phi_2(0)|0\rangle = \phi_{2,-\Delta_2}|0\rangle \quad (6.3)$$

The operator product $\phi_1(x)\phi_2(0)$ is given by

$$\phi_1(x) \phi_2(0)|0\rangle = \sum_{|k| \geq \Delta_1} r^{-k-\Delta_1} Y_{|k+\Delta_1-m|-m}(\hat{x}) \phi_{1,k} \phi_{2,-\Delta_2}|0\rangle. \quad (6.4)$$

The singular part is

$$(\phi_1(x) \phi_2(0))_{sing} |0\rangle = \sum_{k \geq \Delta_1} r^{-k-\Delta_1} Y_{k+\Delta_1-2m}(\hat{x}) \phi_{1,k} \phi_{2,-\Delta_2}|0\rangle. \quad (6.5)$$

The state

$$|\phi'_{k-\Delta_2}\rangle = \phi_{1,k} \phi_{2,-\Delta_2}|0\rangle \quad (6.6)$$

has conformal weight $-k + \Delta_2$. But $k \geq \Delta_1 \geq \Delta_2$. By unitarity, there are no states with conformal weight < 0 . Therefore $|\phi'_{k-\Delta_2}\rangle = 0$ unless $k = \Delta_1 = \Delta_2$, in which case $|\phi'_{k-\Delta_2}\rangle$ has weight 0, so must be proportional to the ground state,

$$\phi_{1,k} \phi_{2,-\Delta_2}|0\rangle = \delta_{k,\Delta_1} C_{12}|0\rangle, \quad \Delta_1 = \Delta_2. \quad (6.7)$$

Therefore, using $\Delta_1 = |s_1| + m$,

$$(\phi_1(x) \phi_2(0))_{sing} = \begin{cases} 0, & \Delta_1 \neq \Delta_2, \\ r^{-2\Delta_1} Y_{2|s_1|}(\hat{x}) C_{12} \mathbf{1}, & \Delta_1 = \Delta_2. \end{cases} \quad (6.8)$$

From the argument in Sect. 6.1 it thus follows that all commutators between Cauchy fields of different scaling dimensions must vanish, and all commutators of Cauchy fields of the same scaling dimension must be proportional to the identity operator.

We have a decomposition of any Cauchy field $\phi(x)$ into creation and annihilation operators, so we can evaluate any correlation function by applying Wick's theorem, commuting destruction operators to the right and creation operators to the left. It thus follows that all correlation functions of unitary Cauchy fields factorize into two point functions. All unitary Cauchy fields are thus indeed free fields.

6.3. Existence of stress-energy tensor. Suppose $\phi(x)$ is a Cauchy field of dimension Δ . If there is a local energy momentum tensor $T_{\mu\nu}(x)$, it must have dimension d . The operator product $T_{\mu\nu}(x) \phi(0)$ must contain the field $\phi(0)$ with a canonical non-zero coefficient. Therefore

$$\langle \phi^\dagger(\infty) T_{\mu\nu}(x) \phi(0) \rangle \neq 0. \quad (6.9)$$

Therefore, by a global conformal transformation

$$\langle T_{\mu\nu}(\infty) \phi^\dagger(x) \phi(0) \rangle \neq 0. \quad (6.10)$$

Therefore the operator product of $\phi^\dagger(x) \phi(0)$ must contain a dimension d field. But the lowest dimension field occurring in the operator product $\phi^\dagger(x) \phi(0)$, besides the identity, has dimension 2Δ . So there cannot be a local energy momentum tensor $T_{\mu\nu}(x)$ unless $2\Delta \leq d$. Since $\Delta = |s| + (d-2)/2$, this is $|s| \leq 1$.

So a Cauchy field with $|s| > 1$ cannot occur in a conformal field theory with a local energy-momentum tensor. This is a manifestation of the Weinberg–Witten theorem [10] which says that massless fields with spin $j > 1$ cannot couple to a local stress-energy tensor.

7. More on Modes

Now that we know that every unitary Cauchy conformal field is free, we can finish the analysis of the mode expansion.

7.1. The modes ϕ_k , $|k| \geq \Delta$ are all non-zero. Suppose $\phi(x)$ is a unitary Cauchy field of spin V_s and scaling dimension $\Delta = |s| + n - 1$. We show that all the modes ϕ_k , $|k| \geq \Delta$ are non-zero by calculating the commutators of the modes recursively using the raising operators P_μ , and seeing that the commutators are non-zero.

Let $\phi^\dagger(x)$ be the adjoint field, so $\langle \phi^\dagger(x) \phi(0) \rangle \neq 0$. The scaling dimension of $\phi^\dagger(x)$ is also Δ . The spin of $\phi^\dagger(x)$ must be one of $V_{\pm s}$. Which one will be determined later.

Write the mode expansions (5.35) in the form

$$\phi(x) = \sum_{k+\Delta \in \mathbb{Z}, |k| \geq \Delta} r^{-k-\Delta} \phi_k(\hat{x}), \quad \phi^\dagger(x) = \sum_{k+\Delta \in \mathbb{Z}, |k| \geq \Delta} r^{-k-\Delta} \phi_k^\dagger(\hat{x}). \quad (7.1)$$

The (anti-)commutators of the modes are multiples of the identity, so

$$[\phi_k^\dagger(\hat{x}), \phi_k(\hat{x})]_{\mp} = C_k(\hat{x}) \mathbf{1}. \quad (7.2)$$

The commutator is used for $s \in \mathbb{Z}$, the anti-commutator for $s \in \frac{1}{2} + \mathbb{Z}$.

The commutators of P_μ with the modes are obtained by calculating

$$[x^\mu P_\mu, \phi(x)] = x^\mu \partial_\mu \phi(x) \quad (7.3)$$

$$\sum_{k+\Delta \in \mathbb{Z}, |k| \geq \Delta} r^{-k-\Delta} x^\mu [P_\mu, \phi_k(\hat{x})] = \sum_{k+\Delta \in \mathbb{Z}, |k| \geq \Delta} r^{-k-\Delta} (-k - \Delta) \phi_k(\hat{x}) \quad (7.4)$$

which gives

$$[\hat{x}^\mu P_\mu, \phi_{k+1}(\hat{x})] = (-k - \Delta) \phi_k(\hat{x}). \quad (7.5)$$

Then we derive the recursion relation:

$$0 = \left[\hat{x}^\mu P_\mu, [\phi_{k+1}^\dagger(\hat{x}), \phi_{-k}(\hat{x})]_{\mp} \right] \quad (7.6)$$

$$= \left[\hat{x}^\mu P_\mu, \phi_{k+1}^\dagger(\hat{x}), \phi_{-k}(\hat{x}) \right]_{\mp} + \left[\phi_{k+1}^\dagger(\hat{x}), [\hat{x}^\mu P_\mu, \phi_{-k}(\hat{x})] \right]_{\mp} \quad (7.7)$$

$$= \left[(-k - \Delta) \phi_k^\dagger(\hat{x}), \phi_{-k}(\hat{x}) \right]_{\mp} + \left[\phi_{k+1}^\dagger(\hat{x}), (k + 1 - \Delta) \phi_{-k-1}(\hat{x}) \right]_{\mp} \quad (7.8)$$

which is

$$(k + 1 - \Delta) C_{k+1}(\hat{x}) = (k + \Delta) C_k(\hat{x}) \quad (7.9)$$

which we solve to get

$$C_k(\hat{x}) = \binom{k + \Delta - 1}{2\Delta - 1} C_\Delta(\hat{x}) \quad k \geq \Delta \quad (7.10)$$

$$C_k(\hat{x}) = \binom{-k + \Delta - 1}{2\Delta - 1} C_{-\Delta}(\hat{x}) \quad k \leq -\Delta. \quad (7.11)$$

The second equation is equivalent to the first under $\phi \leftrightarrow \phi^\dagger$, so we only need to consider $k \geq \Delta$.

The mode $\phi_{-\Delta}$ has $l = 0$, so $\phi_{-\Delta}(\hat{x})$ is independent of \hat{x} , so the two-point function is

$$\langle \phi^\dagger(x) \phi(0) \rangle = \langle 0 | |x|^{-2\Delta} \phi_\Delta^\dagger(\hat{x}) \phi_{-\Delta}(\hat{x}) | 0 \rangle = |x|^{-2\Delta} C_\Delta(\hat{x}) \quad (7.12)$$

therefore $C_\Delta(\hat{x})$ is not identically zero, therefore, by the recursion relation (7.10), none of the $C_k(\hat{x})$, $k \geq \Delta$ are identically zero, therefore none of the modes ϕ_k , ϕ_{-k}^\dagger , $k \geq \Delta$, are identically zero. The same is true exchanging $\phi \leftrightarrow \phi^\dagger$. So all the modes ϕ_k , ϕ_k^\dagger , $|k| \geq \Delta$, are non-zero.

7.2. *The spin of $\phi^\dagger(x)$.* The two point function is

$$\langle \phi^\dagger(x) \phi(0) \rangle = \langle 0 | \phi_\Delta^\dagger \phi_{-\Delta} | 0 \rangle \quad (7.13)$$

with

$$\phi_{-\Delta} \in V_s, \quad (7.14)$$

so ϕ_Δ^\dagger must be in V_s^* , the dual space to V_s . The representation V_s is unitary, $V_s = V_s^\dagger = \bar{V}_s^*$, so the dual space is the same as the complex conjugate, $V_s^* = \bar{V}_s$. For $d = 2n$ even,

all of the V_s are generated by tensor products of $V_{\pm\frac{1}{2}}$ with itself, and $V_{\pm\frac{1}{2}}$ is the chiral spinor representation. For d odd, $V_{\frac{1}{2}}$ is the spinor representation. So, from the reality properties of spinors,

$$\phi_{\Delta}^{\dagger} \in V_s^* = \bar{V}_s = \begin{cases} V_s & d = 2n, n \text{ even} \\ V_{-s} & d = 2n, n \text{ odd} \\ V_s & d \text{ odd.} \end{cases} \quad (7.15)$$

Comparing to the representation of ϕ_{Δ}^{\dagger} as given by (5.55) and (5.56), we find that the spin of the adjoint field $\phi^{\dagger}(x)$ must be

$$V_{\bar{s}} = \begin{cases} V_{-s} & d = 2n, n \text{ even} \\ V_s & d = 2n, n \text{ odd} \\ V_s & d \text{ odd.} \end{cases} \quad (7.16)$$

7.3. Invariant (anti-)commutators of the modes. The mode ϕ_{-k} lies in a representation given by (5.55) or (5.56), and the mode ϕ_k^{\dagger} lies in the dual representation. Using upper indices α, β for the representation of ϕ_{-k} , and lower indices α, β for the dual representation, the (anti-)commutator of the modes takes the form

$$[\phi_{k,\beta}^{\dagger}, \phi_{-k}^{\alpha}]_{\mp} = c_k \delta_{\beta}^{\alpha} \quad (7.17)$$

where c_k is a number.

The terms in the mode expansions (7.1) take the form

$$\phi_{-k}(\hat{x}) = Y_l(s, -k; \hat{x})_{\alpha} \phi_{-k}^{\alpha}, \quad \phi_k^{\dagger}(\hat{x}) = Y_l(\bar{s}, k; \hat{x})^{\beta} \phi_{k,\beta}^{\dagger}, \quad (7.18)$$

where $Y_l(s, -k; \hat{x})_{\alpha}$ is the vector spherical harmonic expressing the Clebsch–Gordan coefficients between S_l, V_s , and the representation of ϕ_{-k} , only the last of which is labeled by an explicit index, α , and similarly for $Y_l(\bar{s}, k; \hat{x})^{\beta}$.

Equation (7.2) becomes

$$C_k(\hat{x}) = c_k Y_l(\bar{s}, k; \hat{x})^{\alpha} Y_l(s, -k; \hat{x})_{\alpha}. \quad (7.19)$$

In particular, the starting point of the recursion formula is

$$C_{\Delta}(\hat{x}) = c_{\Delta} Y_{2|s|}(\bar{s}, \Delta; \hat{x})^{\alpha} Y_0(s, -\Delta; \hat{x})_{\alpha} = c_{\Delta} Y_{2|s|}(\bar{s}, \Delta; \hat{x}), \quad (7.20)$$

where $Y_{2|s|}(\bar{s}, \Delta; \hat{x})$ is the Clebsch–Gordan for $S_{2|s|}, V_{\bar{s}}$, and V_s . The 2-point function is

$$\begin{aligned} \langle \phi^{\dagger}(x) \phi(0) \rangle &= \langle 0 || x |^{-2\Delta} \phi_{\Delta}^{\dagger}(\hat{x}) \phi_{-\Delta}(\hat{x}) | 0 \rangle \\ &= |x|^{-2\Delta} C_{\Delta}(\hat{x}) = |x|^{-2\Delta} c_{\Delta} Y_{2|s|}(\bar{s}, \Delta; \hat{x}). \end{aligned} \quad (7.21)$$

The result (7.10) of the recursion gives us

$$c_k Y_l(\bar{s}, k; \hat{x})^{\alpha} Y_l(s, -k; \hat{x})_{\alpha} = \binom{k + \Delta - 1}{2\Delta - 1} c_{\Delta} Y_{2|s|}(\bar{s}, \Delta; \hat{x}), \quad k \geq \Delta, \quad (7.22)$$

which determines the numbers c_k , after some group-theoretic work which we refrain from doing.

8. Deformability

8.1. Smearing currents. Suppose $\phi(x)$ is a Cauchy field in the Cauchy conformal representation (V, Δ) , not necessarily unitary. We want to be able to use the same deformation of contour arguments for $\phi(x)$ as for a holomorphic field in 2 dimensions. In particular, we would like to calculate a commutator $[\phi_k, \psi(y)]$ by deforming the integrals for the mode ϕ_k to an integral over a small sphere centered at y , so that the commutator can be extracted from the operator product expansion of $\phi(x) \psi(y)$.

A deformable integral over codimension 1 surfaces in space-time is given by a conserved current. So we want to represent each mode of $\phi(x)$ by smearing over a codimension 1 surface S by a vector-valued current $f^\mu(x)$, by

$$\phi[f, S] = \int_S d^{d-1}x \hat{n}_\mu \langle f^\mu(x), \phi(x) \rangle, \quad f^\mu(x) \in V \quad (8.1)$$

where the inner-product on the rhs is the invariant hermitian inner-product on V . We can deform S to any cobounding S' by Stokes' theorem, if

$$\partial_\mu \langle f^\mu(x), \phi(x) \rangle = 0. \quad (8.2)$$

Since $\phi(x)$ satisfies a linear first order differential equation that contains only a linear combination of derivatives of ϕ , $f^\mu(x)$ must separately satisfy

$$\partial_\mu f^\mu(x) = 0 \quad (8.3)$$

and

$$\langle f^\mu(x), \partial_\mu \phi(x) \rangle = 0. \quad (8.4)$$

The Cauchy differential equation (3.4) can be written

$$\partial_\nu \phi = (1 - \hat{P})_\nu^\mu \partial_\mu \phi \quad (8.5)$$

where \hat{P} is the self-adjoint projection on the null subspace in level 1 of the Verma module. So Eq. (8.4) is equivalent to

$$(1 - \hat{P})_\nu^\mu f^\nu(x) = 0. \quad (8.6)$$

The smearing currents $f^\mu(x)$ satisfying (8.3) and (8.6) are exactly those defining a mode $\phi[f, S]$ that can be evaluated on any deformation S' of S .

8.2. Enough smearing currents. We want to show now that there are enough smearing currents satisfying (8.3) and (8.6) to capture all the modes of $\phi(x)$. This means that the contour deformation technique can be used to calculate with the modes.

For simplicity, we take the codimension 1 surface S to be \mathbb{R}^{d-1} . The Cauchy differential equation is first order, so it will certainly be possible to covariantize. The complexified tangent space of space-time is \mathbb{C}^d . Decompose it into the tangential and normal subspaces,

$$\mathbb{C}^d = \mathbb{C}^{d-1} \oplus \mathbb{C}, \quad \partial_\mu = (\vec{\partial}, \partial_d). \quad (8.7)$$

The projection matrix

$$\hat{P} : \mathbb{C}^d \otimes V \rightarrow \mathbb{C}^d \otimes V \quad (8.8)$$

that enters in the Cauchy first order differential equation (3.4) and in Eq. (8.6) for the smearing current decomposes into a block matrix

$$\hat{P} = \left(\begin{array}{c|c} \overleftarrow{P} & \overrightarrow{P} \\ \hline \overrightarrow{P}^\dagger & \hat{P}_d^d \end{array} \right). \quad (8.9)$$

\hat{P} is a self-adjoint projector, so the operators \hat{P}_d^d and \overleftarrow{P} are self-adjoint, but by themselves they are not projectors. The projector condition $\hat{P}^2 = \hat{P}$ is

$$\overleftarrow{P}^2 + \overrightarrow{P} \overrightarrow{P}^\dagger = \overleftarrow{P} \quad \text{or} \quad \overrightarrow{P} \overrightarrow{P}^\dagger = \overleftarrow{P} (1 - \overleftarrow{P}), \quad (8.10)$$

$$\overleftarrow{P} \overrightarrow{P} + \overrightarrow{P} \hat{P}_d^d = \overrightarrow{P} \quad \text{or} \quad \overrightarrow{P} \hat{P}_d^d = (1 - \overleftarrow{P}) \overrightarrow{P}, \quad (8.11)$$

$$\overrightarrow{P}^\dagger \overleftarrow{P} + \hat{P}_d^d \overrightarrow{P}^\dagger = \overrightarrow{P}^\dagger \quad \text{or} \quad \overrightarrow{P}^\dagger \overleftarrow{P} = (1 - \hat{P}_d^d) \overrightarrow{P}^\dagger, \quad (8.12)$$

$$\overrightarrow{P}^\dagger \overrightarrow{P} + (\hat{P}_d^d)^2 = \hat{P}_d^d \quad \text{or} \quad \overrightarrow{P}^\dagger \overrightarrow{P} = \hat{P}_d^d (1 - \hat{P}_d^d). \quad (8.13)$$

Decomposing f into components (\vec{f}, f_d) and using the block matrix decomposition (8.9) of \hat{P} , Eq. (8.6) becomes

$$\overleftarrow{P} \vec{f}(x) + \overrightarrow{P} f_d(x) = \vec{f}(x) \quad (8.14)$$

$$\overrightarrow{P}^\dagger \vec{f}(x) + \hat{P}_d^d f_d(x) = f_d(x). \quad (8.15)$$

Our approach is to pick $f_d(x)$, and then use the first equation (8.14) to determine \vec{f} , solving

$$(1 - \overleftarrow{P}) \vec{f}(x) = \overrightarrow{P} f_d(x), \quad (8.16)$$

to express \vec{f} in terms of f_d . This has a solution because the projector condition (8.10) implies that the image of \overrightarrow{P} is orthogonal to the eigenspace $\overleftarrow{P} = 1$. We can thus define the inverse of $1 - \overleftarrow{P}$ to be zero on the eigenspace $\overleftarrow{P} = 1$ to obtain

$$\vec{f}(x) = (1 - \overleftarrow{P})^{-1} \overrightarrow{P} f_d(x) + \vec{f}(x)_1, \quad \overleftarrow{P} \vec{f}(x)_1 = \vec{f}(x)_1, \quad (8.17)$$

for some arbitrary $\vec{f}(x)_1$ in the eigenspace $\overleftarrow{P} = 1$. The spatial piece \vec{f} of the current is thus determined by f_d up to the component of eigenvalue 1 under \overleftarrow{P} . Using the projector conditions (8.10–8.13), it is straightforward to check that (8.15) is then automatically satisfied. This shows that (8.6) imposes no further constraints on our choice of f_d . The only remaining condition on $f^\mu(x)$ is (8.3),

$$\partial_d f_d(x) = -\vec{\partial} \cdot \vec{f}(x). \quad (8.18)$$

It follows that $\partial_d f_d$ as given by (8.18) and $\partial_d \vec{f}$ as given by (8.17) can be integrated for arbitrary initial data $f_d(x)$ on S . The corresponding mode is

$$\phi[f, S] = \int_S d^{d-1} \vec{x} \langle f_d(x), \phi(x) \rangle. \quad (8.19)$$

Since $f_d(x)$ is arbitrary on S , there are enough deformable smearing currents to capture every mode of $\phi(x)$. In principle, for $V = V_S$, we could now construct explicit smearing functions $f_k^\mu(x)$ such that

$$\phi[f_k, S^{d-1}] = \phi_k \quad (8.20)$$

for the modes ϕ_k defined in Sect. 5, but we will refrain from doing so here.

8.3. Duality. Equation (8.1) suggests a duality between the Cauchy fields $\phi(x)$ and the smearing currents $f^\mu(x)$. But not every initial data $\phi(x)$ on S can be integrated to a solution of the Cauchy equation. We will see below that the Cauchy equation is over-determined. So there must be an equivalence relation—a gauge symmetry—on the smearing currents.

In fact, Eqs. (8.3) and (8.6) on $f^\mu(x)$ have the gauge symmetries

$$f^\mu(x) \rightarrow f^\mu(x) + \delta f^\mu(x) \quad (8.21)$$

$$\delta f^\mu(x) = \partial_\sigma g^{\mu\sigma}(x), \quad g^{\mu\sigma}(x) = -g^{\sigma\mu}(x), \quad (1 - \hat{P})^\mu_\nu g^{\nu\sigma}(x) = 0. \quad (8.22)$$

The mode $\phi[f]$ is gauge-invariant because (8.22) implies

$$\phi[f + \delta f, S] - \phi[f, S] = \int_S d^{d-1}x \hat{n}_\mu \langle \partial_\sigma g^{\mu\sigma}(x), \phi(x) \rangle \quad (8.23)$$

$$= \int_S d^{d-1}x \hat{n}_\mu \langle g^{\sigma\mu}(x), \partial_\sigma \phi(x) \rangle \quad (8.24)$$

$$= 0. \quad (8.25)$$

We would want to show that the solutions of Eqs. (8.3) and (8.6) on $f^\mu(x)$ modulo the gauge transformations (8.22) are exactly dual to solutions of the Cauchy differential equation (3.4), with the pairing given by (8.1). We do not know how to do that. It requires describing the space of smearing functions $f_d(x)$ modulo gauge transformations, and the space of initial data $\phi(x)$ satisfying the overdetermination conditions. We suspect that [21] is relevant.

Here, we will only take a first look at the overdetermination conditions. Again taking the codimension 1 surface S to be \mathbb{R}^{d-1} , we ask for the conditions that $\phi(x)$ must satisfy on S in order to extend to a solution of the Cauchy equation (3.4) off of S . We derive only the first such over-determination condition, which is a first order differential equation on S . Using the block decomposition (8.9) of \hat{P} , the Cauchy differential equation (3.4) becomes

$$\overleftarrow{P} \vec{\partial} \phi + \vec{P} \partial_d \phi = 0 \quad (8.26)$$

$$\vec{P}^\dagger \vec{\partial} \phi + \hat{P}_d^d \partial_d \phi = 0. \quad (8.27)$$

The Cauchy condition is equivalent to the invertibility of \hat{P}_d^d , so (8.27) can be written

$$\partial_d \phi = -(\hat{P}_d^d)^{-1} \vec{P}^\dagger \vec{\partial} \phi, \quad (8.28)$$

which gives the normal derivative in terms of the data on S . This is the Cauchy property.

Now we substitute for $\partial_d \phi$ in Eq. (8.26), getting

$$\overleftrightarrow{P}_1 \overleftrightarrow{\partial} \phi = 0, \quad (8.29)$$

where

$$\overleftrightarrow{P}_1 = \overleftrightarrow{P} - \overleftrightarrow{P} (\hat{P}_d^d)^{-1} \overleftrightarrow{P}^\dagger. \quad (8.30)$$

From the identities (8.10)–(8.13) it follows that

$$(\overleftrightarrow{P}_1)^2 = \overleftrightarrow{P}_1, \quad \overleftrightarrow{P}_1 \overleftrightarrow{P} = \overleftrightarrow{P} \overleftrightarrow{P}_1 = \overleftrightarrow{P} \quad (8.31)$$

so \overleftrightarrow{P}_1 is the projection on the eigenspace $\overleftrightarrow{P} = 1$.

The first-order differential equation (8.29) is the first over-determination condition. It contains no derivatives in the normal direction, so it is a differential equation on S . The Cauchy differential equation (3.4) is equivalent to the combination of (8.28) and (8.29). In Appendix C we work out the first-order over-determination condition explicitly for the unitary Cauchy representations $V = V_s$.

The covariant form of the first-order over-determination condition is the differential equation on S ,

$$\hat{P}_1(x)^\mu \partial_\mu \phi(x) = 0, \quad (8.32)$$

where $\hat{P}_1(x)$ is the projection on the eigenspace with eigenvalue 1 of the operator

$$\overleftrightarrow{P}(x) = [P(T_x S) \otimes 1] \hat{P} [P(T_x S) \otimes 1] \quad (8.33)$$

on $\mathbb{C}^d \otimes V$, where $P(T_x S)$ is the projection on the complexified tangent space to S at x , which is a subspace of \mathbb{C}^d .

The first order over-determination condition (8.29) is a necessary condition on the initial data $\phi(x)$ on S , but it is not necessarily a sufficient condition for integrating the Cauchy equation off S . Equation (8.28) can be integrated to determine $\phi(x)$ uniquely on any nearby surface S' . But further integration requires $\phi(x)$ on S' to continue to satisfy (8.29). We need the integrability condition

$$\partial_d \overleftrightarrow{P}_1 \overleftrightarrow{\partial} \phi = 0. \quad (8.34)$$

Let us write (8.28) as

$$(\partial_d + A_d)\phi = 0, \quad A_d = A_d^j \partial_j = -(\hat{P}_d^d)^{-1} (\overleftrightarrow{P}^\dagger)^j \partial_i \quad (8.35)$$

and (8.29) as

$$A_i \phi = 0, \quad A_i = \overleftrightarrow{P}_1^k \partial_k \phi. \quad (8.36)$$

The integrability condition (8.34) is

$$0 = \partial_d A_i \phi = A_i \partial_d \phi = A_i (-A_d)\phi = [A_d, A_i]\phi - A_d A_i \phi = [A_d, A_i]\phi, \quad (8.37)$$

since we already have the first-order condition $A_i \phi = 0$. So we need the second-order over-determination condition

$$[A_d, A_i]\phi = [A_d^j, A_i^k] \partial_j \partial_k \phi = 0. \quad (8.38)$$

If the second-order condition over-determination condition follows from the first-order condition, then we are done. If not, then we have to impose the second-order condition in addition to the first-order condition, and then check the integrability of the second-order condition, and so on.

In the unitary case, whatever the complete set of over-determination conditions, we know that, when S is the unit sphere, all the solutions are the $\phi_k(\hat{x})$, because these are the only possible modes and all are non-zero.

The structure of the over-determination conditions is determined by the pattern of highest weight vectors in the Verma module. The null space $(P_d + A_d^j P_j)\phi$ generates a submodule of the Verma module—all null states, all perpendicular to the entire Verma module. Complementary to this null submodule is the submodule with basis

$$\{P_{i_1} \cdots P_{i_N} |\phi\rangle\}. \quad (8.39)$$

This is the $\mathfrak{so}(d-1, 2)$ Verma module generated by the $\mathfrak{so}(d)$ representation V considered as a representation of $\mathfrak{so}(d-1)$. The first-order over-determination condition corresponds to a null space $A_i^k P_k |\phi\rangle$ on level 1. The second-order condition corresponds to a null space on level 2. The second-order condition is independent if the corresponding level 2 null subspace is not contained in the sub-module generated by the level 1 null space. So specifying the over-determination conditions is equivalent to finding the minimal set of generators for the full null sub-module of the $\mathfrak{so}(d-1, 2)$ Verma module.

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Appendices

A. The irreducible representations of $\mathfrak{so}(d)$.

A.1. The highest weights representations. In this Appendix we collect some basic results in the representation theory of $\mathfrak{so}(d)$, taken from *e.g.* [22]. The irreducible representations of $\mathfrak{so}(d)$ are written V_λ , where λ is the highest weight. For $d = 2n$, the highest weights of the irreducible representations are

$$\lambda = (\lambda_1, \dots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|. \quad (A.1)$$

For $d = 2n + 1$, the highest weights of the irreducible representations

$$\lambda = (\lambda_1, \dots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0. \quad (A.2)$$

In both cases, the λ_i are all integers (for the vector representations) or all half-integers (for the spinor representations).

Some notable representations are given by:

for all d : trivial representation $V_0 = V_{(0,\dots,0)}$ (A.3)

 fundamental representation $\mathbb{C}^d = V_{(1,0,\dots,0)}$ (A.4)

 symmetric traceless l -tensors $S_l = V_{(l,0,\dots,0)}$ (A.5)

for $d = 2n$: p -forms, $0 \leq p \leq n - 1$ $\Lambda^p = V_{(\underbrace{1,\dots,1}_p, \underbrace{0,\dots,0}_{n-p})}$ (A.6)

 (anti-)self-dual n -forms $\Lambda_{\pm}^n = V_{(1,1,\dots,1,\pm 1)}$ (A.7)

 chiral spinors $S_{\pm} = V_{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})}$ (A.8)

for $d = 2n + 1$: p -forms, $0 \leq p \leq n$ $\Lambda^p = V_{(\underbrace{1,\dots,1}_p, \underbrace{0,\dots,0}_{n-p})}$ (A.9)

 spinors $S = V_{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})}$ (A.10)

The quadratic Casimir invariant—with the normalization given in Eq. (2.27)—is, for all d ,

$$C_2^d(V_{\lambda}) = \frac{1}{2} \sum_{i=1}^n \lambda_i (\lambda_i + d - 2i) \quad (\text{A.11})$$

Decomposing the tensor product $\mathbb{C}^d \otimes V_{\lambda}$ into irreducibles gives

$$\mathbb{C}^d \otimes V_{\lambda} = \begin{cases} \bigoplus_{\lambda'=\lambda \pm \epsilon_k} V_{\lambda'} & d = 2n \\ \bigoplus_{\lambda'=\lambda, \lambda \pm \epsilon_k} V_{\lambda'} & d = 2n + 1 \end{cases} \quad (\text{A.12})$$

where $\epsilon_k = (0, \dots, 0, 1, 0, \dots)$ is the weight that has 1 in the k -th position and 0 elsewhere, and the sums include all the $\lambda' = \lambda \pm \epsilon_k$ that are highest weights for $\mathfrak{so}(d)$.

Finally we will need the branching rules of an $\mathfrak{so}(d)$ irreducible V_{λ} decomposing into $\mathfrak{so}(d-1)$ irreducibles V_{μ}^{d-1} ,

$$V_{\lambda} = \bigoplus_{\mu} V_{\mu}^{d-1} \quad (\text{A.13})$$

where the sum ranges over all μ that are highest weights for $\mathfrak{so}(d-1)$ satisfying $\lambda_1 - \mu_1 \in \mathbb{Z}$ and

$$\begin{aligned} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq |\lambda_n|, & \quad d = 2n, \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \geq |\mu_n|, & \quad d = 2n + 1. \end{aligned} \quad (\text{A.14})$$

A.2. The eigenvalues of \hat{M} . In what follows we need to know the eigenvalues $\hat{M}_{\lambda',\lambda}$ of \hat{M} acting on the irreducible components $V_{\lambda'} \subset \mathbb{C}^d \otimes V_{\lambda}$,

$$\hat{M}_{\lambda',\lambda} = C_2^d(V_{\lambda}) + C_2^d(\mathbb{C}^d) - C_2^d(V_{\lambda'}). \quad (\text{A.15})$$

For the λ' of interest, the ones occurring in (A.12), this evaluates to

$$\hat{M}_{\lambda+\epsilon_k,\lambda} = k - 1 - \lambda_k \quad (\text{A.16})$$

$$\hat{M}_{\lambda-\epsilon_k,\lambda} = \lambda_k + d - 1 - k \quad (\text{A.17})$$

$$\hat{M}_{\lambda,\lambda} = \frac{1}{2}(d - 1) \quad (\text{A.18})$$

Recall that the number $\hat{M}_{\lambda',\lambda}$ occurs as an eigenvalue only if λ' is actually a highest weight.

To check unitarity, it is important to identify the largest eigenvalue of \hat{M} . For $d = 2n$, the numbers $\hat{M}_{\lambda',\lambda}$ satisfy the inequalities

for $\lambda_n > 0$:

$$\hat{M}_{\lambda+\epsilon_1,\lambda} < \cdots < \hat{M}_{\lambda+\epsilon_{n-1},\lambda} < \hat{M}_{\lambda+\epsilon_n,\lambda} < \hat{M}_{\lambda-\epsilon_n,\lambda} < \hat{M}_{\lambda-\epsilon_{n-1},\lambda} < \cdots < \hat{M}_{\lambda-\epsilon_1,\lambda}, \quad (\text{A.19})$$

for $\lambda_n = 0$:

$$\hat{M}_{\lambda+\epsilon_1,\lambda} < \cdots < \hat{M}_{\lambda+\epsilon_{n-1},\lambda} < \hat{M}_{\lambda+\epsilon_n,\lambda} = \hat{M}_{\lambda-\epsilon_n,\lambda} < \hat{M}_{\lambda-\epsilon_{n-1},\lambda} < \cdots < \hat{M}_{\lambda-\epsilon_1,\lambda}, \quad (\text{A.20})$$

for $\lambda_n < 0$:

$$\hat{M}_{\lambda+\epsilon_1,\lambda} < \cdots < \hat{M}_{\lambda+\epsilon_{n-1},\lambda} < \hat{M}_{\lambda-\epsilon_n,\lambda} < \hat{M}_{\lambda+\epsilon_n,\lambda} < \hat{M}_{\lambda-\epsilon_{n-1},\lambda} < \cdots < \hat{M}_{\lambda-\epsilon_1,\lambda}. \quad (\text{A.21})$$

For $d = 2n + 1$, the $\hat{M}_{\lambda',\lambda}$ satisfy

for $\lambda_n > 0$:

$$\hat{M}_{\lambda+\epsilon_1,\lambda} < \cdots < \hat{M}_{\lambda+\epsilon_n,\lambda} < \hat{M}_{\lambda,\lambda} < \hat{M}_{\lambda-\epsilon_n,\lambda} < \cdots < \hat{M}_{\lambda-\epsilon_1,\lambda}, \quad (\text{A.22})$$

for $\lambda_n = 0$:

$$\hat{M}_{\lambda+\epsilon_1,\lambda} < \cdots < \hat{M}_{\lambda+\epsilon_n,\lambda} < \hat{M}_{\lambda,\lambda} = \hat{M}_{\lambda-\epsilon_n,\lambda} < \cdots < \hat{M}_{\lambda-\epsilon_1,\lambda}. \quad (\text{A.23})$$

A.3. Example: $d = 4$. Let us give the more familiar expressions for $d = 4$ here. The general formula becomes

$$C_2^d(V_\lambda) = \frac{1}{2}\lambda_1(\lambda_1 + 2) + \frac{1}{2}\lambda_2^2. \quad (\text{A.24})$$

We usually write $\mathfrak{so}(4) = \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$. The irreducible representations of $\mathfrak{so}(4)$ are the tensor products of $\mathfrak{su}(2)_{L,R}$ irreducibles j_L, j_R . The corresponding $\mathfrak{so}(4)$ highest weight is

$$\lambda_1 = j_L + j_R, \quad \lambda_2 = j_L - j_R, \quad (\text{A.25})$$

giving

$$C_2^d(V_\lambda) = j_L(j_L + 1) + j_R(j_R + 1). \quad (\text{A.26})$$

B. Classification of Cauchy fields. Next, we want to classify the Cauchy conformal fields. That is, we want to determine which representations V_λ and which scaling dimensions Δ satisfy the algebraic condition **A1**, which is equivalent to the Cauchy property. Condition **A1** is the condition that the matrix $\hat{P}^d : V_\lambda \rightarrow (\mathbb{C}^d \otimes V_\lambda)_\Delta$ be injective. Since we picked out the direction d , \hat{P}^d is not fully $\mathfrak{so}(d)$ -invariant, but it is $\mathfrak{so}(d-1)$ -invariant. The idea is thus to decompose all $\mathfrak{so}(d)$ representations into $\mathfrak{so}(d-1)$ representations.

We can obtain necessary conditions for \hat{P}^d to be injective by using Schur's Lemma. Any non-trivial irreducible $\mathfrak{so}(d-1)$ representation V_μ that occurs in V_λ must also occur in $(\mathbb{C}^d \otimes V_\lambda)_\Delta$. Otherwise the $\mathfrak{so}(d-1)$ invariant map \hat{P}^d must map it to the trivial representation, which means that \hat{P}^d cannot be injective. Therefore, a necessary condition for \hat{P}^d to be injective is

C1 Every inequivalent irreducible $\mathfrak{so}(d-1)$ -representation V_μ that occurs in V_λ must also occur in $(\mathbb{C}^d \otimes V_\lambda)_\Delta$.

If we demand that the representation V_λ lead to a unitary representation of the full conformal group, it is necessary that Δ is the largest eigenvalue of \hat{M} .

C2 For (V_λ, Δ) to be a unitary conformal representation, Δ must be the largest eigenvalue of \hat{M} .

Let us first discuss condition **C1**. For λ non-trivial, the branching rules given in (A.14) imply that the only λ' which satisfy this necessary condition are

$$\begin{aligned}
 d = 2n : & \\
 & \mathbf{A}_{2n} : \lambda' = \lambda + \epsilon_1 \\
 & \mathbf{B}_{2n}^+ : \lambda' = \lambda - \epsilon_n, \quad \lambda_n > 0, \\
 & \mathbf{B}_{2n}^- : \lambda' = \lambda + \epsilon_n, \quad \lambda_n < 0, \\
 d = 2n + 1 : & \\
 & \mathbf{A}_{2n+1} : \lambda' = \lambda + \epsilon_1 \\
 & \mathbf{C}_{2n+1} : \lambda' = \lambda,
 \end{aligned} \tag{B.1}$$

To see this, note that for instance if $\lambda' = \lambda - \epsilon_j$, $j < n$, then $\mu = \lambda$ is not in λ' ; for $\lambda' = \lambda + \epsilon_j$, $\mu = (\lambda_1, \dots, \lambda_j, \lambda_j, \dots)$ is in λ but not in λ' . Similar arguments eliminate the other cases. For $\lambda = 0$, the only λ' is \mathbb{C}^d , which satisfies the necessary condition.

B.1. Assuming unitarity. Let us now show that the unitary Cauchy fields with spin V_s listed in Sect. 4.4 are the only unitary Cauchy fields that satisfy condition **C2**. This completes the classification of unitary Cauchy fields.

First note from (A.19)–(A.23) that unless $\lambda = 0$, $\lambda' = \lambda + \epsilon_1$ never leads to the largest eigenvalue. In the following always assume that $\lambda \neq 0$.

Next, for $d = 2n$ consider the case $\lambda_n > 0$, $\lambda' = \lambda - \epsilon_n$. From (A.19) we see that λ' can only be the largest eigenvalue if none of the $\lambda - \epsilon_j$ are representations of $\mathfrak{so}(d)$. This is only the case if $\lambda = (|s|, \dots, |s|, |s|)$. Similarly, for $\lambda_n < 0$, $\lambda' = \lambda + \epsilon_n$, we see from (A.21) that $\lambda' = \lambda + \epsilon_n$ is only the largest eigenvalue if $\lambda = (|s|, \dots, |s|, -|s|)$. This establishes the claim for even d .

For odd d , $\lambda' = \lambda$ has to lead to the largest eigenvalue. From (A.22) no representations with $\lambda - \epsilon_j$ can appear, so that $\lambda = (|s|, \dots, |s|, |s|)$. Moreover $\lambda - \epsilon_n$ must not appear either, which is only the case if $s = 1/2$. This establishes the claim for odd d .

B.2. The non-unitary cases: $d = 3$ and $d = 4$. Let us now investigate the injectivity condition **C1** if we do not require unitarity. We did not develop the general theory, but the special cases $d = 3$ and $d = 4$ are relatively straightforward to work out.

For $d = 3$, the problem reduces to decomposing $\mathfrak{so}(3)$ into $\mathfrak{so}(2)$ representations, which is decomposing $\mathfrak{su}(2)$ into $\mathfrak{u}(1)$ representations. Since the latter are 1 dimensional it is enough to check that $P_{\lambda, \lambda'}^d$ is non-vanishing on each $\mathfrak{u}(1)$ representation μ that occurs in λ , which is equivalent to the non-vanishing of the relevant Clebsch–Gordan coefficient between the fundamental \mathbb{C}^3 of $\mathfrak{so}(3)$, λ , and λ' . In terms of Wigner 3- j symbols the condition is thus

$$\begin{pmatrix} 1 & \lambda & \lambda' \\ 0 & \mu & -\mu \end{pmatrix} \neq 0, \quad |\mu| \leq \lambda, \quad \lambda - \mu \in \mathbb{Z}, \quad (\text{B.2})$$

where by assumption λ' occurs in the fusion of λ with \mathbb{C}^3 . There is then just one additional selection rule on the 3- j symbols, namely

$$\begin{pmatrix} 1 & \lambda & \lambda' \\ 0 & \mu & -\mu \end{pmatrix} = 0 \quad \text{iff} \quad 1 + \lambda + \lambda' \in 2\mathbb{Z} + 1 \quad \text{and} \quad \mu = 0. \quad (\text{B.3})$$

This never affects the case **A3**, but if $\lambda = \lambda'$ is integer, then $P_{\lambda, \lambda'}^d$ vanishes on $\mu = 0$. Condition **C1** is thus satisfied for

$$\begin{aligned} d = 3 \\ \mathbf{A}_3 : \lambda' = \lambda + \epsilon_1 & \quad \text{injective} \\ \mathbf{C}_3 : \lambda' = \lambda & \quad \text{injective iff } \lambda \in \frac{1}{2} + \mathbb{Z}. \end{aligned} \quad (\text{B.4})$$

For $d = 4$, we show that the map $P_{\lambda, \lambda'}^d$ is injective in all three of the non-unitary cases **A4**, **B4**⁺, **B4**⁻.

Use $\mathfrak{so}(4) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The representation (λ_1, λ_2) of $\mathfrak{so}(4)$ is the representation $(j) \otimes (k)$ of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, with

$$\lambda_1 = j + k, \quad \lambda_2 = j - k. \quad (\text{B.5})$$

The fundamental representation $(\frac{1}{2})$ of $\mathfrak{sl}(2)$ is \mathbb{C}^2 with invariant anti-symmetric tensor ϵ^{ab} , $a, b = 1, 2$. Take as basis for the $\mathfrak{sl}(2)$ representation (j) the symmetric tensors of rank $2j$ on \mathbb{C}^2 . For the second $\mathfrak{sl}(2)$, take $(\mathbb{C}^2)^*$ as the fundamental representation. This gives a basis for λ as tensors on \mathbb{C}^2 ,

$$t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} \quad (\text{B.6})$$

symmetric separately in the a_i and in the b_i . The representation \mathbb{C}^d has basis t_b^a . The $\mathfrak{so}(3)$ invariant is δ_b^a .

The case **A4** is $j' = j + \frac{1}{2}$, $k' = k + 1/2$. The projection \hat{P} is thus given by the map

$$\hat{P} : \mathbb{C}^d \otimes V_\lambda \rightarrow (\mathbb{C}^d \otimes V_\lambda)_{\lambda'}, \quad (t_b^a, t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}}) \mapsto \mathbf{Sym}_a \mathbf{Sym}_b \left(t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} t_b^a \right) \quad (\text{B.7})$$

The map $P_{\lambda, \lambda'}^d$ is

$$t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} \mapsto \mathbf{Sym}_a \mathbf{Sym}_b \left(t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} \delta_{b_{2k+1}}^{a_{2j+1}} \right) \quad (\text{B.8})$$

which is pretty clearly injective. To see this, note that the irreducible components under $\mathfrak{so}(d-1)$ have basis

$$t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} = \mathbf{Sym}_a \mathbf{Sym}_b \left(w_{b_1 \dots b_{2k-r}}^{a_1 \dots a_{2j-r}} \delta_{b_{2k-r+1}}^{a_{2j-r+1}} \dots \delta_{b_{2k}}^{a_{2j}} \right), \quad w_{b_1 \dots b_{2k-r-1}c}^{a_1 \dots a_{2j-r-1}c} = 0. \quad (\text{B.9})$$

In this basis, $P_{\lambda, \lambda'}^d$ is just the identity on each $\mathfrak{so}(d-1)$ component of λ .

The case \mathbf{B}_{2n}^+ is $j > k > 0$, $j' = j - \frac{1}{2}$, $k' = k + 1/2$. The map $P_{\lambda, \lambda'}^d$ is

$$t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} \mapsto \mathbf{Sym}_b \left(t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} \epsilon_{a_{2j} b_{2k+1}} \right). \quad (\text{B.10})$$

Again look at the action of $P_{\lambda, \lambda'}^d$ on the $\mathfrak{so}(d-1)$ component of λ with basis elements given in (B.9),

$$t_{b_1 \dots b_{2k}}^{a_1 \dots a_{2j}} \mapsto \mathbf{Sym}_a \mathbf{Sym}_b \left(w_{b_1 \dots b_{2k-r}}^{a_1 \dots a_{2j-r-1} a_{2j}} \epsilon_{a_{2j} b_{2k-r+1}} \delta_{b_{2k-r+2}}^{a_{2j-r}} \dots \delta_{b_{2k+1}}^{a_{2j-1}} \right). \quad (\text{B.11})$$

We only have to show that this is non-zero. Take the tensor w to have only one non-zero component, $w_{22 \dots}^{11 \dots} = 1$. Then $w_{22 \dots}^{11 \dots} \epsilon_{12} \neq 0$. So $P_{\lambda, \lambda'}^d$ is injective.

The case \mathbf{B}_4^- is the same as \mathbf{B}_4^+ with $j \leftrightarrow k$.

So, for $d = 4$, in all three of the non-unitary cases \mathbf{A}_4 , \mathbf{B}_4^+ , \mathbf{B}_4^- , the map $P_{\lambda, \lambda'}^d$ is injective.

B.3. The non-unitary case: $d > 4$. The question is, for V_λ an irreducible representation of $\mathfrak{so}(d)$, and $V_{\lambda'}$ one of the irreducible components of $\mathbb{C}^d \otimes V_\lambda$ listed in (B.1), is the map $P_{\lambda, \lambda'}^d : V_\lambda \rightarrow V_{\lambda'}$ injective, where

$$P_{\lambda, \lambda'}^d(v) = P_{\lambda, \lambda'}(\hat{e}_d \otimes v) \quad (\text{B.12})$$

where

$$P_{\lambda, \lambda'} : \mathbb{C}^d \otimes V_\lambda \rightarrow V_{\lambda'} \quad (\text{B.13})$$

is the projection on the component $V_{\lambda'}$. We cannot answer the question, but we convey a suggestion from N. Wallach:

In the case of $\lambda + \epsilon_1$ mapping into $\lambda \otimes \epsilon_1$, $P_{\lambda, \lambda'}^d$ is injective since it is just Cartan multiplication (which is multiplication in an integral domain). In the even dimensional case \mathbf{B}_{2n}^\pm , $P_{\lambda, \lambda'}^d$ is adjoint to Cartan multiplication (which is multiplication in an integral domain). That is we look at λ mapping into $(\lambda + \epsilon_1) \otimes \epsilon_1$, by realizing $\lambda + \epsilon_1$ in $\lambda \otimes \epsilon_1$ and contracting (N. Wallach, private communication).

C. First-order over-determination conditions for $V = V_s$. As an illustration, let us now work out the first-order over-determination conditions explicitly for the Cauchy representations V_s . Take $d = 2n$ and $V = V_s$, $s \neq 0$.

Let us first work out the eigenvalues of P_d^d and \overleftrightarrow{P} . Since they are self-adjoint, we can decompose $\mathbb{C} \otimes V$ into eigenspaces of \hat{P}_d^d and decompose $\mathbb{C}^{d-1} \otimes V$ into eigenspaces of \overleftrightarrow{P}

$$\mathbb{C} \otimes V = \bigoplus_{\lambda} V_{\lambda}, \quad \hat{P}_d^d \Big|_{V_{\lambda}} = \lambda \quad (\text{C.1})$$

$$\mathbb{C}^{d-1} \otimes V = \bigoplus_{\lambda} W_{\lambda}, \quad \overleftrightarrow{P} \Big|_{W_{\lambda}} = \lambda, \quad (\text{C.2})$$

but since they are not projectors, they can have eigenvalues different from $\lambda = 0, 1$. Equations (8.10)–(8.13) become, for $v_{\lambda} \in V_{\lambda}$ and $w_{\lambda} \in W_{\lambda}$,

$$\vec{P}^{\dagger} \vec{P} v_{\lambda} = \lambda(1 - \lambda)v_{\lambda} \quad (\text{C.3})$$

$$(\hat{P}_d^d + \lambda - 1)\vec{P}^{\dagger} w_{\lambda} = 0 \quad (\text{C.4})$$

$$(\overleftrightarrow{P} + \lambda - 1)\vec{P} v_{\lambda} = 0 \quad (\text{C.5})$$

$$\vec{P} \vec{P}^{\dagger} w_{\lambda} = \lambda(1 - \lambda)w_{\lambda}. \quad (\text{C.6})$$

The Cauchy condition $\text{Ker } \hat{P}_d^d = 0$ means that $V_0 = 0$, so that (C.3) and (C.6) respectively imply that

$$\text{Ker } \vec{P} = V_1, \quad \text{Im } \vec{P}^{\dagger} = \sum_{\lambda \neq 0,1} V_{\lambda} \quad (\text{C.7})$$

$$\text{Ker } \vec{P}^{\dagger} = W_0 \oplus W_1, \quad \text{Im } \vec{P} = \sum_{\lambda \neq 0,1} W_{\lambda} \quad (\text{C.8})$$

whereas the second and third equation are equivalent to

$$\vec{P} V_{\lambda} = W_{1-\lambda}, \quad \vec{P}^{\dagger} W_{\lambda} = V_{1-\lambda}, \quad \lambda \neq 0, 1. \quad (\text{C.9})$$

Let us now evaluate these expression for Cauchy representations V_s . The projection \hat{P} is

$$\hat{P} = \frac{\hat{M} - (n - 1 - \Delta)}{\Delta - (n - 1 - \Delta)}, \quad (\text{C.10})$$

with $\Delta = n - 1 + |s|$ in the unitary case and $\Delta = -|s|$ in the non-unitary case. Since $\hat{M}_d^d = 0$, \hat{P}_d^d has only one eigenvalue,

$$\hat{P}_d^d = \frac{-(n - 1 - \Delta)}{\Delta - (n - 1 - \Delta)}. \quad (\text{C.11})$$

From (C.9) it follows that \overleftrightarrow{P} has at most eigenvalues

$$\lambda = 1 - \frac{-(n - 1 - \Delta)}{\Delta - (n - 1 - \Delta)} = \frac{\Delta}{2\Delta - (n - 1)} \quad (\text{C.12})$$

and $\lambda = 0$ and 1. Having established that, let us work out the eigenspaces of \overleftrightarrow{P} explicitly. Using

$$\overleftrightarrow{P} = \frac{\hat{M}^{d-1} - (n - 1 - \Delta)}{\Delta - (n - 1 - \Delta)}, \quad (\text{C.13})$$

the strategy is of course to use the $\mathfrak{so}(d-1)$ invariance of \hat{M}^{d-1} to express it in terms of $\mathfrak{so}(d-1)$ Casimirs. Decomposing V_s into representations of $\mathfrak{so}(d-1)$,

$$V_s = V_{\mu_{|s|}}, \quad \mu_{|s|} = (|s|, \dots, |s|) \quad (\text{C.14})$$

$$\mathbb{C}^{d-1} \otimes V_s = V_{\mu_{|s|}} \oplus V_{\mu_{|s|}+\epsilon_1} \oplus V_{\mu_{|s|}-\epsilon_{n-1}} \quad (\text{C.15})$$

where the right-most summand does not occur for $|s| = 1/2$. The quadratic Casimirs are

$$C_2^{d-1}(V_\mu) = \frac{1}{2} \sum_{i=1}^{n-1} \mu_i (\mu_i + d - 1 - 2i) \quad (\text{C.16})$$

$$C_2^{d-1}(\mathbb{C}^{d-1}) = \frac{1}{2}(d-2) = n-1 \quad (\text{C.17})$$

$$C_2^{d-1}(V_{\mu_{|s|}}) = \frac{1}{2}(n-1)|s|(|s|+n-1) \quad (\text{C.18})$$

$$C_2^{d-1}(V_{\mu_{|s|}+\epsilon_1}) = C_2^{d-1}(V_{\mu_{|s|}}) + |s| + n - 1 \quad (\text{C.19})$$

$$C_2^{d-1}(V_{\mu_{|s|}-\epsilon_{n-1}}) = C_2^{d-1}(V_{\mu_{|s|}}) - |s| \quad (\text{C.20})$$

Writing \hat{M}^{d-1} as

$$\hat{M}^{d-1} = \mathbf{1} \otimes C_2^{d-1}(V_{\mu_{|s|}}) + C_2^{d-1}(\mathbb{C}^{d-1}) \otimes \mathbf{1} - C_2^{d-1}(\mathbb{C}^{d-1} \otimes V_{\mu_{|s|}}), \quad (\text{C.21})$$

its eigenspaces and eigenvalues are

$$n-1 \quad \text{on } V_{\mu_{|s|}} \quad (\text{C.22})$$

$$-|s| \quad \text{on } V_{\mu_{|s|}+\epsilon_1} \quad (\text{C.23})$$

$$n-1+|s| \quad \text{on } V_{\mu_{|s|}-\epsilon_{n-1}}, \quad (\text{C.24})$$

so that the eigenvalues of \overleftrightarrow{P} are

$$\frac{\Delta}{2\Delta - (n-1)} \quad \text{on } V_{\mu_{|s|}} \quad (\text{C.25})$$

$$\frac{\Delta - |s| - n + 1}{2\Delta - (n-1)} \quad \text{on } V_{\mu_{|s|}+\epsilon_1} \quad (\text{C.26})$$

$$\frac{\Delta + |s|}{2\Delta - (n-1)} \quad \text{on } V_{\mu_{|s|}-\epsilon_{n-1}} \quad (\text{C.27})$$

For the unitary case, $\Delta = |s| + n - 1$, these eigenvalues of \overleftrightarrow{P} are

$$\frac{\Delta}{2\Delta - (n-1)} \quad \text{on } V_{\mu_{|s|}} \quad (\text{C.28})$$

$$0 \quad \text{on } V_{\mu_{|s|}+\epsilon_1} \quad (\text{C.29})$$

$$1 \quad \text{on } V_{\mu_{|s|}-\epsilon_{n-1}} \quad (\text{C.30})$$

For the non-unitary case, $\Delta = -|s|$,

$$\frac{\Delta}{2\Delta - (n-1)} \quad \text{on } V_{\mu|s|} \quad (\text{C.31})$$

$$1 \quad \text{on } V_{\mu|s|+\epsilon_1} \quad (\text{C.32})$$

$$0 \quad \text{on } V_{\mu|s|-\epsilon_{n-1}} \quad (\text{C.33})$$

So, for the unitary case, (8.29) leads to an over-determination condition when $|s| > 1/2$,

$$\text{Proj}_{V_{\mu|s|-\epsilon_{n-1}}}(\vec{\partial}\phi) = 0. \quad (\text{C.34})$$

For the non-unitary case, there is an over-determination condition for all $|s|$,

$$\text{Proj}_{V_{\mu|s|+\epsilon_1}}(\vec{\partial}\phi) = 0. \quad (\text{C.35})$$

In both cases, the over-determination condition can be written

$$\overleftrightarrow{P} \left(\overleftrightarrow{P} - \frac{\Delta}{2\Delta - (n-1)} \right) \vec{\partial}\phi = 0, \quad (\text{C.36})$$

or, equivalently

$$\left(\hat{M}^{d-1} - (n-1) \right) \left(\hat{M}^{d-1} - (n-1) + \Delta \right) \vec{\partial}\phi = 0. \quad (\text{C.37})$$

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