

A new kind of quantum field theory of $(n-1)$ -dimensional defects in $2n$ dimensions

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I describe a project to open a new territory of quantum field theory where the fields live not on a space-time manifold but on certain complete metric spaces of $(n-1)$ -dimensional objects (defects) in a $2n$ -dimensional space-time M . These metric spaces are “quasi Riemann surfaces”; they are formally analogous to Riemann surfaces. Every construction of a 2d conformal field theory is to give an analogous construction of a cft on the quasi Riemann surfaces, and thereby a cft on M . The global symmetry group of the 2d cft becomes a local gauge symmetry. Ordinary local quantum fields in space-time are constructed by restricting to small objects. The project is based on writing the free n -form in $2n$ dimensions as the 2d gaussian model on the quasi Riemann surfaces.

This note is a summary of the main points of [1]. References can be found there. A condensed version of this note will appear as [2]. More expositions are collected at [3].

1. Let M be euclidean space-time: an oriented conformal $2n$ -manifold, compact, without boundary. When $n = 1$, M is a Riemann surface. The basic examples are $M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$. The Hodge $*$ -operator acting on n -forms is conformally invariant,

$$(*\omega)_{\nu_1 \dots \nu_n}(x) = \omega_{\mu_1 \dots \mu_n}(x) \frac{1}{n!} \epsilon^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n}(x) \quad *^2 = (-1)^n \quad (1)$$

Nothing else is used of the conformal structure on M .

2. The physical objects are represented mathematically as the integral $(n-1)$ -currents in M , as constructed in Geometric Measure Theory [4]. A k -current

ξ in M is a distribution on the smooth k -forms,

$$\omega \in \Omega_k(M) \mapsto \int_{\xi} \omega = \int_M \frac{1}{k!} \omega_{\mu_1 \dots \mu_k}(x) \xi^{\mu_1 \dots \mu_k}(x) d^{2n}x \quad (2)$$

The boundary operator ∂ on currents is dual to the exterior derivative,

$$\int_{\partial \xi} \omega = \int_{\xi} d\omega \quad \partial^2 = 0 \quad (3)$$

A k -simplex in M , $\sigma: \Delta^k \rightarrow M$, is represented by the k -current $[\sigma]$ which is the delta-function concentrated on $\sigma(\Delta^k) \subset M$,

$$\sigma: \Delta^k \rightarrow M \quad \int_{[\sigma]} \omega = \int_{\Delta^k} \sigma^* \omega \quad (4)$$

A singular k -chain in M is an integer linear combination of k -simplices in M , $\sigma = \sum_i m_i \sigma_i$. The *singular k -currents* $\mathcal{D}_k^{\text{sing}}(M)$ are the currents $[\sigma] = \sum_i m_i [\sigma_i]$ that represent the singular k -chains. Examples are the k -submanifolds. The current $[\sigma]$ represents the physical object in M independent of its expression as a combination of simplices.

The physical difference between two singular k -currents is measured by the *flat metric* $\|\xi_1 - \xi_2\|_{\text{flat}}$,

$$\|\xi\|_{\text{flat}} = \inf \{ \text{vol}_k(\xi - \partial \xi') + \text{vol}_{k+1}(\xi') : \xi' \in \mathcal{D}_{k+1}^{\text{sing}}(M) \} \quad (5)$$

The space of *integral k -currents* $\mathcal{D}_k^{\text{int}}(M)$ is the metric completion of $\mathcal{D}_k^{\text{sing}}(M)$,

$$\mathcal{D}_k^{\text{sing}}(M) \subset \mathcal{D}_k^{\text{int}}(M) \subset \mathcal{D}_k^{\text{distr}}(M) \quad \mathcal{D}_k^{\text{int}}(M) \xrightarrow{\partial} \mathcal{D}_{k-1}^{\text{int}}(M) \quad (6)$$

The boundary of an integral current is an integral current. $\mathcal{D}_k^{\text{int}}(M)$ is a metric abelian group — a complete metric space and an abelian group.

3. Recall the 2d gaussian model, the free 1-form cft in 2d. $j(x)$ is a 1-form on a Riemann surface satisfying

$$dj = 0 \quad d(*j) = 0 \quad (7)$$

The integrals of j and $*j$ are 0-forms ϕ, ϕ^* which take values in dual circles,

$$d\phi = j \quad d\phi^* = *j \quad \phi(x) \in \mathbb{R}/2\pi R\mathbb{Z} \quad \phi^*(x) \in \mathbb{R}/2\pi R^*\mathbb{Z} \quad RR^* = 1 \quad (8)$$

ϕ, ϕ^* are determined up to $U(1) \times U(1)$ global symmetries

$$\phi(x) \rightarrow \phi(x) + a \quad \phi^*(x) \rightarrow \phi^*(x) + a^* \quad (9)$$

The vertex operator $V_{p,p^*}(x)$ describes a point defect of charges p, p^* ,

$$V_{p,p^*}(x) = e^{ip\phi(x)+ip^*\phi^*(x)} \quad p, p^* \in \frac{1}{R}\mathbb{Z} \times \frac{1}{R^*}\mathbb{Z} \quad V_{p,p^*} \rightarrow V_{p,p^*} e^{ipa+ip^*a^*} \quad (10)$$

4. Recall the free n -form cft in $2n$ dimensions. $F(x)$ is an n -form on the $2n$ -manifold M satisfying

$$dF = 0 \quad d(*F) = 0 \quad (11)$$

The integrals of F and $*F$ are $(n-1)$ -forms A, A^* on M ,

$$dA = F \quad dA^* = *F \quad (12)$$

which take values in dual circles in the sense that

$$\int_{\xi} A \in \mathbb{R}/2\pi R\mathbb{Z} \quad \int_{\xi} A^* \in \mathbb{R}/2\pi R^*\mathbb{Z} \quad \forall \xi \in \mathcal{D}_{n-1}^{\text{sing}}(M) \quad RR^* = 1 \quad (13)$$

A, A^* are determined up to $U(1) \times U(1)$ local gauge symmetries given by $(n-2)$ -forms f, f^*

$$A \rightarrow A + df \quad A^* \rightarrow A^* + df^* \quad (14)$$

$(n-1)$ -dimensional defects are described by fields $V_{p,p^*}(\xi)$ on $\mathcal{D}_{n-1}^{\text{sing}}(M)$,

$$\begin{aligned} V_{p,p^*}(\xi) &= e^{ip\phi(\xi)+ip^*\phi^*(\xi)} \quad p, p^* \in \frac{1}{R}\mathbb{Z} \times \frac{1}{R^*}\mathbb{Z} \\ \phi(\xi) &= \int_{\xi} A \quad \phi^*(\xi) = \int_{\xi} A^* \quad \xi \in \mathcal{D}_{n-1}^{\text{sing}}(M) \end{aligned} \quad (15)$$

transforming by

$$\begin{aligned} a(\partial\xi) &= \int_{\partial\xi} f \quad a^*(\partial\xi) = \int_{\partial\xi} f^* \\ \phi(\xi) &\rightarrow \phi(\xi) + a(\partial\xi) \quad \phi^*(\xi) \rightarrow \phi^*(\xi) + a^*(\partial\xi) \\ V_{p,p^*}(\xi) &\rightarrow V_{p,p^*}(\xi) e^{ipa(\partial\xi)+ip^*a^*(\partial\xi)} \end{aligned} \quad (16)$$

Fix an $(n-2)$ -boundary $\partial\xi_0$ and consider the abelian subgroup of $\mathcal{D}_{n-1}^{\text{sing}}(M)$

$$\mathcal{D}_{n-1}^{\text{sing}}(M)_{\mathbb{Z}\partial\xi_0} = \{\xi: \partial\xi \in \mathbb{Z}\partial\xi_0\} \subset \mathcal{D}_{n-1}^{\text{sing}}(M) \quad (17)$$

On $\mathcal{D}_{n-1}^{\text{sing}}(M)_{\mathbb{Z}\partial\xi_0}$ the gauge symmetries act as a global $U(1) \times U(1)$ generated by the two numbers $a(\partial\xi_0)$ and $a^*(\partial\xi_0)$.

5. Calculus is needed on $\mathcal{D}_{n-1}^{\text{sing}}(M)_{\mathbb{Z}\partial\xi_0}$ to continue the analogy with the 2d gaussian model. Go to the metric completion, writing it $Q = \mathcal{D}_{n-1}^{\text{int}}(M)_{\mathbb{Z}\partial\xi_0}$. Geometric Measure Theory provides a construction of currents in any such complete metric space [5], providing the spaces $\mathcal{D}_j^{\text{int}}(Q)$ of integral j -currents in Q . Define the j -forms on Q as the real duals of the currents and the exterior derivative as the dual of the boundary operator,

$$\Omega_j(Q) = \text{Hom}(\mathcal{D}_j^{\text{int}}(Q), \mathbb{R}) \quad d\omega(\eta) = \omega(\partial\eta) \quad (18)$$

The infinitesimal j -simplices generate $\mathcal{D}_j^{\text{int}}(Q)$, so the tangent bundle TQ can be defined as the set of infinitesimal 1-simplices in Q . The 1-forms then become the sections of the dual cotangent bundle T^*Q .

The equivalences of simplices $\Delta^j \times \Delta^{n-1} \cong \Delta^{j+n-1}$ give natural maps

$$\Pi_{j,n-1}: \mathcal{D}_j^{\text{int}}(Q) \rightarrow \mathcal{D}_{j+n-1}^{\text{int}}(M) \quad \partial\Pi_{j,n-1} = \Pi_{j-1,n-1}\partial \quad (19)$$

The map $\Pi_{1,n-1}$ identifies each tangent space $T_\xi Q$ with a certain subspace $\mathcal{V}_n \subset \mathcal{D}_n^{\text{distr}}(M)$. That $*\mathcal{V}_n = \mathcal{V}_n$ is a crucial technical point whose demonstration uses the flat metric completion. Then $*$ acts on each tangent space $T_\xi Q$. The forms $F, *F, A, A^*$ on M pull back to $j, *j, \phi, \phi^*$ on Q ,

$$\begin{aligned} j &= \Pi_{1,n-1}^* F & *j &= \Pi_{1,n-1}^* (*F) & \phi &= \Pi_{0,n-1}^* A & \phi^* &= \Pi_{0,n-1}^* A^* \\ d\phi &= j & d\phi^* &= *j \end{aligned} \quad (20)$$

So there is the classical 2d gaussian model on each of the spaces Q , except that $*^2 = 1$ for n even, while $*^2 = -1$ in 2d. Define

$$J = \epsilon_n * \quad \epsilon_n^2 = (-1)^{n-1} \quad J^2 = -1 \quad (21)$$

J is imaginary when n is even, so the currents have to be complexified in order that J act on the tangent spaces $T_\xi Q$,

$$Q = \mathcal{D}_{n-1}^{\text{int}}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}_n^{\text{int}}(M) \quad (22)$$

Now, for all n , on each of the spaces Q there is a 2d gaussian model written in terms of the chiral fields

$$d\phi_\pm = j_\pm \quad j_\pm = \frac{1}{2}(1 \pm i^{-1}J)j \quad (23)$$

6. Quantization of a free field theory is expressed by the Schwinger-Dyson equation on the 2-point functions. In the 2d gaussian model, the chiral fields

are (anti-)holomorphic. The S-D equation on $\langle \bar{\phi}_\pm j_\pm \rangle$ is the Cauchy-Riemann equation

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z - z'} = \pi \delta^2(z - z') \quad (24)$$

which is the foundation for complex analysis on Riemann surfaces. The 2d gaussian model would have led to complex analysis on Riemann surfaces had that not existed already. For the free n -form in $2n$ dimensions, the S-D equation has an expression containing no explicit mention of n ,

$$\langle \int_{\bar{\xi}_1} \bar{A}_\alpha \int_{\xi_2} dF_\beta \rangle = 2\pi i c_{\alpha\beta} I_M \langle \bar{\xi}_1, \xi_2 \rangle \quad c_{\alpha\beta} = -c_{\beta\alpha} \quad c_{+-} = 1 \quad (25)$$

The lhs is the 2-pt function $\langle \bar{A}_\pm(x) dF_\pm(x') \rangle$ smeared against the $(n-1)$ -current $\bar{\xi}_1$ and the $(n+1)$ -current ξ_2 . The rhs is a slight modification of the intersection number, which is nonzero only if $k_1 + k_2 = 2n$,

$$I_M(\xi_1, \xi_2) = \int_M \frac{1}{k_1! k_2!} \xi_1^{\mu_1 \dots} (x) \xi_2^{\nu_1 \dots \mu_{k_1}} (x) \epsilon_{\mu_1 \dots \mu_{k_1} \nu_1 \dots \nu_{k_2}} (x) d^{2n}x \quad (26)$$

The modification is such that $I_M \langle \bar{\xi}_1, \xi_2 \rangle$ has properties independent of n ,

$$I_M \langle \bar{\xi}_1, \xi_2 \rangle = \epsilon_{n, k_2 - n} I_M(\bar{\xi}_1, \xi_2) \quad \epsilon_{n, m} = (-1)^{nm + m(m+1)/2} \epsilon_n^{-1} \quad (27)$$

$$I_M \langle \bar{\xi}_1, \xi_2 \rangle \text{ is skew-hermitian.} \quad (28)$$

$$I_M \langle \bar{\partial} \bar{\xi}_1, \xi_2 \rangle = -I_M \langle \bar{\xi}_1, \partial \xi_2 \rangle \quad (29)$$

$$I_M \langle \bar{\xi}_1, J \xi_2 \rangle \text{ on } n\text{-currents is hermitian and positive definite} \quad (30)$$

Pulled back to Q , the S-D equation of the free n -form cft is

$$\langle \int_{\bar{\eta}_1} \bar{\phi}_\alpha \int_{\eta_2} dj_\beta \rangle = 2\pi i c_{\alpha\beta} I_Q \langle \bar{\eta}_1, \eta_2 \rangle \quad (31)$$

The rhs is $I_M \langle \bar{\xi}_1, \xi_2 \rangle$ pulled back to a skew-hermitian form on currents in Q

$$I_Q \langle \bar{\eta}_1, \eta_2 \rangle = I_M \langle \Pi_{j_1, n-1} \bar{\eta}_1, \Pi_{j_2, n-1} \eta_2 \rangle \quad (32)$$

which is nonzero only if $(j_1 + n - 1) + (j_2 + n - 1) = 2n$, which is $j_1 + j_2 = 2$, just like the intersection number of currents in a 2-manifold. The S-D equation (31) on Q is formally analogous to the S-D equation (25) of the 2d gaussian model on a Riemann surface, which is the Cauchy-Riemann equation.

7. The free n -form cft on M has now become the 2d gaussian model on each of the metric spaces Q . Moreover, each Q has the structure needed to write

the Cauchy-Riemann equation. This is taken to be the defining structure of a *quasi Riemann surface*. The $Q = \mathcal{D}_{\mathbb{Z}\partial\xi_0}^{\text{int}}$ are the quasi Riemann surfaces. They are the fibers of a bundle of quasi Riemann surfaces

$$\begin{aligned} \mathcal{Q}(M) &\rightarrow \mathcal{B}(M) \\ \mathcal{B}(M) &= \{\text{maximal infinite cyclic subgroups } \mathbb{Z}\partial\xi_0 \subset \partial\mathcal{D}_{n-1}^{\text{int}}(M)\} \end{aligned} \tag{33}$$

On each fiber Q there is a 2d gaussian model with its global $U(1) \times U(1)$ symmetry group, collectively comprising a local gauge symmetry over $\mathcal{B}(M)$.

8. All of the constructions of 2d cft are based on the Cauchy-Riemann equation and on the 2d gaussian model. So there is the prospect of carrying out those constructions on each of the fibers Q to obtain, for every 2d cft, a new cft of defects in M . The 2d cft on each fiber Q will be ambiguous up to its global 2d symmetry group. The collection of global symmetry groups on the fibers will form a local gauge symmetry group over $\mathcal{B}(M)$.

9. There are many basic problems to be worked on: opportunities to leverage 2d qft to develop a new technology of qft in $2n$ dimensions. Some are the following.

Complex analysis on quasi Riemann surfaces needs to be developed in analogy with ordinary Riemann surfaces.

Conjecturally, every quasi Riemann surface Q is isomorphic to $\mathcal{D}_0^{\text{int}}(\Sigma)$ for Σ the 2d conformal space with the same Jacobian as Q , the Jacobian being the complex torus made from the homology in the middle dimension.

The conjectured isomorphism would allow constructing a 2d cft on each Q by lifting an ordinary 2d cft from Σ to $\mathcal{D}_0^{\text{int}}(\Sigma)$, which is a purely 2d problem, then to each $Q = \mathcal{D}^{\text{int}}(M)_{\mathbb{Z}\partial\xi_0}$ via one of the conjectured isomorphisms. As an example of the first step, the 2d gaussian model is to be lifted by extending the renormalization of the vertex operators $V_{p,\bar{p}}(\xi)$ from singular 0-currents ξ to integral 0-currents.

If the conjecture is true, there will be some universal objects to study. The automorphism group $\mathbf{Aut}(Q)$ will encode information about all the conformal groups of the conformal manifolds M and about the global symmetry groups of all 2d cfts. There will be a universal homogeneous bundle of quasi Riemann surfaces with structure group $\mathbf{Aut}(Q)$ in which all the bundles $\mathcal{Q}(M) \rightarrow \mathcal{B}(M)$ are embedded.

There should be a large collection of structure preserving maps from the complex disk $\mathbb{D}_1$ into Q . Meromorphic functions on Q will pull back to ordinary meromorphic functions on $\mathbb{D}_1$. The local structure of a cft on Q will be expressed as a collection of ordinary 2d cfts on each of these *local quasi holomorphic curves*, each with its radial quantization, pair of Virasoro

algebras, and operator product expansion. Explicit constructions of local quasi holomorphic curves are needed, say for $M = S^{2n}$.

The local gauge symmetry in the bundle of quasi Riemann surfaces needs a space-time interpretation. What, for example, is the space-time interpretation of the local $SU(2) \times SU(2)$ symmetry over $\mathcal{B}(M)$ corresponding to the global $SU(2) \times SU(2)$ at the self-dual point $R = 1$ of the 2d gaussian model?

It should be possible to imitate on the quasi Riemann surfaces the usual constructions of 2d cft such as orbifolding and perturbation theory.

Tiny defects look like points in M , so fields $\Phi(\xi)$ when restricted to the small ξ in Q will give ordinary local quantum fields on M . Will these form new local qfts in $2n$ -dimensions?

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