Integrate the odes to check property P

Imports

In [1]:
```
from mpmath import mp
from mpmath import mpf,mpc
import sage.libs.mpmath.all as a
mp.pretty = True
from timeit import default_timer as timer
# start=timer()
# end=timer()
```

Functions to analyze the odes over a range of parameters

Each ode is parametrized by two numbers, σ and α which are functions of

- j and ϵ for odes 3_j, 2_j, and 1_j
- ϵ for ode 2.

function to analyze ode 3_j, 2_j, or 1_j at $j = \infty$ for list of σ values

The limit $j \to \infty$ is taken with σ fixed.

In [2]:
```
def process_sigma_list():
    global jval,epsilonb,sigmaval,alphaval
    global data_asymp
    set_precision()
    initialize_ode()
    for sigma_arg in sigma_list:
        epsilonb=0.0
        set_Jacobi_parameters()
        sigmaval=myR(sigma_arg)
        jval=Infinity
        alphaval = myR(limit(alphaexp,j=jval))
        #
        analyze_monodromy()
        if GOOD:
            data[ode_name]['asymp'][sigmaval]=data_dict
```

function to analyze ode 3_j, 2_j, or 1_j for list of j values given ϵ

In [3]:
```
def process_j_list():
    global jval,sigmaval,alphaval
    global data_j
    set_precision()
    initialize_ode()
    set_Jacobi_parameters()
    eps_string = epsilonb.str(digits=4,skip_zeroes=True)
    for j_arg in j_list:
        jval = j_arg
        alphaval = myR(alphaexp.subs(j=jval))
        sigmaval = myR(sigmaexp.subs(j=jval,epsb=myR(epsilonb)))
        #
        analyze_monodromy()
        if GOOD:
            if eps_string not in data[ode_name]['finite']:
                data[ode_name]['finite'][eps_string]={}
            data[ode_name]['finite'][eps_string][jval]=data_dict
```

Function to analyze ode 2 for list of ϵ values

In [4]:
```
def process_ode2_epsilonb_list():
    global jval,epsilonb,sigmaval,alphaval,ode_decimal_precision
    global data_small_j
    set_precision()
    initialize_ode()
    jval=None
```

```
    for epsilonb_arg in epsilonb_list:
        epsilonb=epsilonb_arg
        eps_string = epsilonb.str(digits=4,skip_zeroes=True)
        set_Jacobi_parameters()
        alphaval = myR(alphaexp)
        sigmaval = myR(sigmaexp.subs(epsb=myR(epsilonb)))
        #
        analyze_monodromy()
        if GOOD:
            data[ode_name][eps_string]=data_dict
```

Function to analyze imaginary period monodromy matrix

In [5]:

```
def analyze_monodromy():
    global data_dict,GOOD
    start = timer()
    GOOD=True
    if gauged:
        construct_gauge()
    mp.dps=ode_decimal_precision
    create_ode()
    start_ode=timer()
    calculate_Mi()
    end_ode=timer();
    ode_time = end_ode-start_ode
    mp.dps=high_decimal_precision
    check_Mi()
    if gauged:
        check_gauge_invariance()
        construct_Vphys()
    construct_Mi_phys()
    check_property_P()
    if property_P:
        calculate_frequencies()
        end=timer()
        total_time = end-start
        eps_string = epsilonb.str(digits=4,skip_zeroes=True)
        pretty_print(LE('\\epsilon ='), eps_string,LE('\\quad j ='), jval,\
            LE('\\quad\\sigma ='), n(sigmaval,digits=3), LE('\\quad(\\omega/\\sigma)^{2}='), mpstr(fs
                        '    (', ode_decimal_precision,', ', decimal_tol, ')', \
                    '    ode time = ', n(ode_time,digits=5), '    other time = ',  n(total_time-ode_
    if GOOD:
        data_dict = dictify_vars('jval','sigmaval','epsilonb','fs_ratio','Mi','Mi_phys','H','Vmat',\
                                 'ode_decimal_precision','high_decimal_precision',\
                    'N','gauged','H_eigenvalues','min_H_eigenvalue','positive_frequencies','negative_
                    'MiR','Mi_eigenvalues')
```

subroutines

In [6]:

```
def calculate_Mi():
    global Mi4,Mi2,Mi
    Mi4 = cP(Jac_Kp)
    Mi2 = cR*Omega*Mi4.transpose()*Omega*cR*Mi4
    Mi = Omega*Mi2.transpose_conj()*Omega*Mi2
```

In [7]:

```
def check_Mi():
    global GOOD
    if not aeq(mp.det(Mi),1):
        GOOD=False
        print ('FAIL: Mi does not have det 1')
    if not aeq(Mi.transpose()*Omega*Mi,Omega):
        GOOD=False
        print ('FAIL: Mi is not symplectic')
```

In [8]:

```
def check_gauge_invariance():
    global GOOD
    if not aeq(Wgauge,Mi*Wgauge):
        GOOD = False
        print('Mi FAILS gauge invariance check')
```

In [9]:

```
def construct_Vphys():
    global Vmat,GOOD
```

```
    Proj_rem = Proj_phys
    Nphys = N-1
    vlist=[]
    vplist=[]
    for ii in range(Nphys):
        v = mp.matrix(int(Crank),int(1))
        v[ii]=1
        vphys = Proj_rem*v
        vphys = (1/mp.norm(vphys))*vphys
        vpphys = mp.j*Omega*vphys
        vlist.append(vphys)
        vplist.append(vpphys)
        Proj_rem = Proj_rem - vphys*vphys.transpose()- vpphys*vpphys.transpose()
    Vmat = mp.matrix(int(Crank),int(0))
    for ii in range(Nphys):
        Vmat=mp.extend(Vmat,vlist[ii])
    for ii in range(Nphys):
        Vmat=mp.extend(Vmat,vplist[ii])
    if not aeq(Vmat.transpose()*Vmat,mp.eye(int(2*Nphys))):
        GOOD=False
        print('FAIL Vmat^t Vmat not = 1')
```

In [10]:

```
def construct_Mi_phys():
    global Mi_phys, Omega_phys,GOOD
    if gauged:
        Mi_phys = Vmat.transpose()*Mi*Vmat
        Omega_phys = Vmat.transpose()*Omega*Vmat
        if not aeq(mp.det(Mi_phys),1):
            GOOD=False
            print('FAIL Mi_phys not det 1')
        if not aeq(Mi_phys.transpose()*Omega_phys*Mi_phys,Omega_phys):
            GOOD=False
            print('FAIL Mi_phys not symplectic')
    else:
        Mi_phys = Mi
        Omega_phys = Omega
```

In [11]:

```
def check_property_P():
    global H,H_eigenvalues,property_P,min_H_eigenvalue,HR,GOOD
    H=Omega_phys*Mi_phys-Omega_phys
    H_eigenvalues_complex,HR = mp.eig(H)
    property_P = False
    if not aeq(H,H.transpose_conj()):
        GOOD=False
        print('H FAILS to be hermitian')
        return
    H_eigenvalues=[]
    for x in H_eigenvalues_complex:
        if a0(mp.im(x)):
            H_eigenvalues.append(mp.re(x))
        else:
            GOOD=False
            print('H FAILS with non-real eigenvalue')
    H_eigenvalues.sort()
    min_H_eigenvalue = H_eigenvalues[0]
    if min_H_eigenvalue > 0:
        property_P = True
    else:
        GOOD=False
        print('FAILS Property P')
    return
```

In [12]:

```
def calculate_frequencies():
    global Mi_phys,Mi_eigenvalues,positive_frequencies,negative_frequencies,min_frequency,MiR,GOOD,f
    if not aeq(Mi_phys.transpose_conj()*H, H*Mi_phys):
        GOOD=False
        print('FAIL Mi_phys is not H-hermitian')
    Mi_eigenvalues_complex,MiR = mp.eig(Mi_phys)
    Mi_eigenvalues=[]
    for x in Mi_eigenvalues_complex:
        if a0(mp.im(x)):
            Mi_eigenvalues.append(mp.re(x))
        else:
            GOOD=False
            pretty_print('Mi FAILS: non-real eigenvalue')
    positive_frequencies =[]
```

```
    negative_frequencies =[]
    for x in Mi_eigenvalues:
        if x < 0:
            GOOD=False
            print('FAIL: Mi_phys negative eigenvalue')
        else:
            freq = mp.ln(x)/(4*Jac_Kp)
            if freq == 0:
                GOOD=False
                print('FAIL zero frequency')
            if freq < 0:
                negative_frequencies.append(freq)
            else:
                positive_frequencies.append(freq)
    if(len(negative_frequencies) != len(positive_frequencies)):
        GOOD=False
        print('FAIL Mi_phys number of positive and negative frequencies differ')
    negative_frequencies.sort(reverse=True)
    positive_frequencies.sort()
    for neg,pos in zip(negative_frequencies,positive_frequencies):
        if not aeq(-neg,pos):
            GOOD=False
            print('FAIL frequencies not in +- pairs')
    min_frequency = positive_frequencies[0]
    fs_ratio= (min_frequency/sigmaval)^2
    if N == 3:
        if not aeq(positive_frequencies[0],positive_frequencies[1]):
            print('DEGENERACY BROKEN')
```

subroutines for the ode solver

- initial condition at $\tau = 0$ is $y_0 = 1$ the identity matrix
- K2mat, K1mat, K0mat are numerical matrices provided by Sagemath
- y_deriv(tau,y) calculates $\partial_\tau y = -iA(\tau)y$
- for the ode solver, the complex matrix y is converted to a flat list of real numbers
- the solution mp.odefun(y_deriv ,0,y_init) is function of τ that returns a flat list.
- The propagator $P(\tau)$ is extracted from the solution function.

In [13]:
```
def create_ode():
    global cPlist,K0,K1,K2
    K0matval = K0mat.subs(alpha=alphaval)
    K2 = mpify_matrix(K2mat)
    K1 = mpify_matrix(K1mat*sigmaval)
    K0 = mpify_matrix(K0matval*sigmaval^2)
#  create ode
    cPlist = mp.odefun(y_deriv ,0,y_init,tol=ode_tol,degree=ode_degree)
```

In [14]:
```
def construct_gauge():
    global Wgauge, Wpgauge, Proj_gauge, Proj_phys
    w0matval = w0mat.subs(alpha=alphaval)
    w1 = w1mat.change_ring(myC)
    w0 = (w0matval * sigmaval).change_ring(myC)
    zeroNvector = matrix(N,1)
    W0 = mpify_matrix(block_matrix(2,1,[w0,zeroNvector]))
    W1 = mpify_matrix(block_matrix(2,1,[w1,zeroNvector]))
    W1p = mpify_matrix(block_matrix(2,1,[zeroNvector,w1]))
    Wgauge = W0 + W1*F_at_K+ W1p*Fp_at_K
    Wpgauge = mp.j*Omega*Wgauge
    Proj_gauge = mp.eye(int(Crank)) - Wpgauge * mp.inverse(Wpgauge.transpose()*Wpgauge) * Wpgauge.tra
    Proj_phys = Proj_gauge - Wgauge* mp.inverse(Wgauge.transpose()*Wgauge) * Wgauge.transpose()
```

In [15]:
```
def y_deriv(tau,y):
# y is a list of length 4N * Crank
#     representing wr, wr', wi, wi'
#
    Fi = -Jac_k * Jac_kp * mp.ellipfun('sd',u=tau,m=Jac_mp)
    Kr = K2*Fi^2 - K0
    Ny_4= N*Crank
    Ny_2= 2* Ny_4
    y1 = y[:Ny_2]
    y2 = y[Ny_2:]
    wr = matrixify_list(y1[:Ny_4],N,Crank)
    wpr = matrixify_list(y1[Ny_4:],N,Crank)
```

```
    wi = matrixify_list(y2[:Ny_4],N,Crank)
    wpi = matrixify_list(y2[Ny_4:],N,Crank)
    dwr = -wpi
    dwi = wpr
    dwpr = -Kr*wi+ Fi*(K1*wr)
    dwpi = Kr*wr + Fi*(K1*wi)
    dylist = listify_matrix(dwr) + listify_matrix(dwpr) + listify_matrix(dwi) + listify_matrix(dwpi)
    return dylist
```

In [16]:

```
def cP(tau):
    cPl = cPlist(tau)
    cPreal = mp.matrix([cPl[nn:nn+Crank] for nn in range(0,Crank*Crank,Crank)])
    cPimag = mp.matrix([cPl[nn:nn+Crank] for nn in range(Crank*Crank,Crank*Rrank,Crank)])
    cP_mp = cPreal+mp.j*cPimag
    return (cP_mp)
```

Initialization functions

In [17]:

```
def set_ranks():
    global Crank, Rrank, IN, ZN, Z2N, Omega, cR, y_init
    Crank = 2*N
    Rrank = 2*Crank
    IN = identity_matrix(Reals,N)
    ZN = matrix(Reals,N,N)
    I2N = identity_matrix(Reals,2*N)
    Z2N = matrix(Reals,2*N,2*N)
    Omega_sage = I*block_matrix([[ZN,IN],[-IN,ZN]])
    cR_sage= block_matrix([[IN,ZN],[ZN,-IN]])
    Omega = mpify_matrix(Omega_sage)
    cR = mpify_matrix(cR_sage)
    y_init_matrix_sage = block_matrix([[I2N],[Z2N]])
    y_init_matrix = mpify_matrix(y_init_matrix_sage)
    y_init = listify_matrix(y_init_matrix)
```

In [18]:

```
def initialize_ode():
    global ode,K2mat,K1mat,K0mat,K1coeff,K0coeff,alphaexp,sigmaexp,gauged,w1mat,w0mat,w0coeff,N
    ode = odes[ode_name]
    K2mat = ode['K2mat']
    K1mat = ode['K1mat']
    K0mat = ode['K0mat']
    alphaexp = ode['alphaexp']
    sigmaexp = ode['sigmaexp']
    gauged = ode['gauged']
    if gauged:
        w1mat = ode['w1mat']
        w0mat = ode['w0mat']
    N=K2mat.nrows()
    set_ranks()
```

Set precision and tolerances

- mp.dps and decimal_tol are set in the main code
- mp.dps is the decimal precision for mpmath.
- 10^{-decimal_tol} is the tolerance for approximate numerical equality.

The precision of Sagemath real and complex numbers is set to match the mpmath precision

In [19]:

```
def set_precision():
    global binary_precision,Reals, RealNumber, myR, Complexes, ComplexNumber, myC, I, J
    global mptol, ode_tol,ode_degree
#    mp.dps = decimal_precision
#    binary_precision = mp.prec
#
    binary_precision=mp.prec
    Reals = RealField(binary_precision)
    RealNumber = Reals
    myR = Reals
    Complexes = ComplexField(binary_precision)
    ComplexNumber = Complexes
    myC = Complexes
    I = myC(I)
#    set_tolerances
```

```
    mptol = mpf(10^(-decimal_tol))
    ode_tol = None
    ode_degree = None
#    print ('mp decimal precision =',decimal_precision, '    mp binary precision = sage binary preci
```

Set Jacobi parameters

In [20]:

```
def set_Jacobi_parameters():
    global epsilonb,Jac_m,Jac_mp,Jac_k,Jac_kp,Jac_K,Jac_Kp,F_at_K,Fp_at_K
    Jac_m = mpf(1/2) + mpf(epsilonb)^2
    Jac_mp = mpf(1)-Jac_m
    Jac_k = mp.sqrt(Jac_m)
    Jac_kp = mp.sqrt(Jac_mp)
    Jac_K = mp.ellipk(Jac_m)
    Jac_Kp = mp.ellipk(Jac_mp)
    F_at_K = 0.0
    Fp_at_K = - Jac_k* Jac_kp
```

Utilities

In [21]:

```
%display latex
def LE(str):
    return(LatexExpr(r''+ str))

def flatten_list(L):
    return [val for sublist in L for val in sublist]

def mpify_matrix(matrix_sage):
    return mp.matrix(list(map(list,list(matrix_sage))))

def sagify(x):
    return a.mpmath_to_sage(x,binary_precision)

def sagify_matrix(matrix_mp):
    return matrix([list(map(sagify,sublist)) for sublist in matrix_mp.tolist()])

def matrixify_list(ylist,nrows,ncols):
    return mp.matrix([[ylist[ii+jj] for jj in range(ncols)] for ii in range(0,nrows*ncols,ncols)])

def listify_matrix(mp_matrix):
    return flatten_list(mp_matrix.tolist())

print_digits = 20
def mpstr(x):
    if type(x) is list:
        return list(map(mpstr,x))
    return mp.nstr(x,print_digits)

dictify_vars = lambda *args: {i:eval(i) for i in args}

multby = lambda n,x: [n*item for item in x]
```

Numerical closeness utilities

mptol = approximate equality tolerance $z \approx 0 \equiv |z| < \text{mptol}$

$a0(x) \quad \text{iff} \quad x.\text{norm}() < \text{tol}^2$

aeq(x,y) iff $(x-y).\text{norm}() < \text{tol}^2$

clean(x)

- if x real, if a0(x) set x=0
- if x complex, set real and imaginary parts to 0 if a0
- otherwise assume x is a matrix and clean its entries

In [22]:

```
def almosteq(x,y):
    return(mp.almosteq(x,y,mptol))
#
def aeq(x,y):
    if isinstance(x,mp.mpf) or isinstance(x,mp.mpc):
        return(almosteq(x,y))
```

```
    if isinstance(x,mp.matrix):
        return(almosteq(mp.norm(x-y),0))
def a0(x):
    if isinstance(x,mp.mpf) or isinstance(x,mp.mpc):
        return(almosteq(x,0))
    if isinstance(x,mp.matrix):
        return(almosteq(mp.norm(x),0))
def clean(x):
    if isinstance(x,mpf):
        y=x
        if a0(y):
            y = mpf(0)
        if aeq(y,1.0):
            y = mpf(1.0)
        if aeq(y,-1.0):
            y = mpf(-1.0)
        return y
    if isinstance(x,mpc):
        xRe=clean(mp.re(x))
        xIm=clean(mp.im(x))
        if xIm == mpf(0):
            return(xRe)
        elif xRe == mpf(0):
            return(mp.j*xIm)
        else:
            return (xRe + mp.j* xIm)
    if type(x) is list:
        return(list(map(clean,x)))
    if isinstance(x,mp.matrix):
        xrows=x.rows
        xcols=x.cols
        y = mp.matrix(xrows,xcols)
        for i in range(xrows):
            for j in range(xcols):
                y[i,j]= clean(x[i,j])
        return y
```

Global variables

In [23]:

```
#     global Crank, Rrank, IN, ZN, Z2N, Omega, cR, y_init
#     global ode,K2mat,K1mat,K0mat,K1coeff,K0coeff,alphaexp,sigmaexp,gauged,w1mat,w0mat,w0coeff,N
#     global alpha_asymp, K0mat_asymp, w0mat_asymp
#     global binary_precision,Reals, RealNumber, myR, Complexes, ComplexNumber, myC, I, J
#     global epsilonb,Jac_m,Jac_mp,Jac_k,Jac_kp,Jac_K,Jac_Kp,F_at_t,Fp_at_t
#     global mptol, ode_tol,ode_degree
#     global cPlist,K0,K1,K2, Wgauge, Wpgauge, Proj_gauge, Proj_phys
#     global Mi4,Mi2,Mi
#     global GOOD
#     global Vmat,GOOD
#     global Mi_phys, Omega_phys,GOOD
#     global H,H_eigenvalues,property_P,min_H_eigenvalue,HR,GOOD
#     global Mi_phys,Mi_eigenvalues,positive_frequencies,negative_frequencies,min_frequency,MiR,GOOD
```

Check gauge invariance of $\mathrm{P}(\tau)$

This can be used after a run, i.e. after an ode solution is found.

In [24]:

```
def F(tau):
    return(Jac_k* mp.ellipfun('cn',u=Jac_K+mp.j*tau,m=Jac_m))
def Fp(tau):
    return(- Jac_k* mp.ellipfun('sn',u=Jac_K+mp.j*tau,m=Jac_m)* mp.ellipfun('dn',u=Jac_K+mp.j*tau,m=
def Fi(tau):
    return(-Jac_k * Jac_kp * mp.ellipfun('sd',u=tau,m=Jac_mp))
def Wg(tau):
    w0matval = w0mat.subs(alpha=alphaval)
    w1 = w1mat.change_ring(myC)
    w0 = (w0matval * w0coeff.subs(sigma=sigmaval)).change_ring(myC)
    zeroNvector = matrix(N,1)
    W0 = mpify_matrix(block_matrix(2,1,[w0,zeroNvector]))
    W1 = mpify_matrix(block_matrix(2,1,[w1,zeroNvector]))
    W1p = mpify_matrix(block_matrix(2,1,[zeroNvector,w1]))
    return(W0 + W1*F(tau)+ W1p*Fp(tau))
```

Run the analyses

Integrating the ode requires a certain decimal precision in the ode solver.
For each calculation below the decimal precision has been set by trial and error.
Various consistency checks were made on each calculation.
The decimal precision was raised until all the consistency checks were passed.

load the odes, set precision and tolerance

In [25]:
```
odes=load('odes')
high_decimal_precision=300
mp.dps= high_decimal_precision
decimal_tol = 20
data = {}
```

Ode 2

In [26]:
```
ode_name='2'
data[ode_name]={}
```

In [27]:
```
ode_decimal_precision = 30
epsilonb_list = [0.001,0.002,0.003,0.004,0.005,0.01,0.02,0.03,0.04,0.05,0.1,0.2,0.3,0.4,0.5,0.6,0.70
process_ode2_epsilonb_list()
```

$\epsilon = 0.001$ j = None $\sigma = 0.000707$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.6006 other time =0.76988

$\epsilon = 0.002$ j = None $\sigma = 0.00141$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.4197 other time =0.27256

$\epsilon = 0.003$ j = None $\sigma = 0.00212$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3831 other time =0.27702

$\epsilon = 0.004$ j = None $\sigma = 0.00283$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3885 other time =0.27746

$\epsilon = 0.005$ j = None $\sigma = 0.00354$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3839 other time =0.27714

$\epsilon = 0.01$ j = None $\sigma = 0.00707$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3782 other time =0.27009

$\epsilon = 0.02$ j = None $\sigma = 0.0141$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3872 other time =0.26741

$\epsilon = 0.03$ j = None $\sigma = 0.0212$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3786 other time =0.26892

$\epsilon = 0.04$ j = None $\sigma = 0.0283$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3819 other time =0.26734

$\epsilon = 0.05$ j = None $\sigma = 0.0354$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3894 other time =0.26634

$\epsilon = 0.1$ j = None $\sigma = 0.0707$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =4.3721 other time =0.26822

$\epsilon = 0.2$ j = None $\sigma = 0.141$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =3.9470 other time =0.26340

$\epsilon = 0.3$ j = None $\sigma = 0.212$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =3.8780 other time =0.26140

$\epsilon = 0.4$ j = None $\sigma = 0.283$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =3.8319 other time =0.25747

$\epsilon = 0.5$ j = None $\sigma = 0.354$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =3.3270 other time =0.25333

$\epsilon = 0.6$ j = None $\sigma = 0.424$ $(\omega/\sigma)^2 = 2.0$ (30,20) ode time =2.8495 other time =0.24138

$\epsilon = 0.7071 \quad j = \text{None} \quad \sigma = 0.500 \quad (\omega/\sigma)^2 = 2.0 \quad (30,20) \quad \text{ode time} = 1.0870 \quad \text{other time} = 0.19430$

Ode 3_j

In [28]:
```
ode_name='3j'
data[ode_name]={}
data[ode_name]['asymp']={}
data[ode_name]['finite']={}
```

j finite

In [29]:
```
epsilonb = 0.7071

ode_decimal_precision = 30
j_list = [3/2]
process_j_list()

ode_decimal_precision = 40
j_list = [2,5/2]
process_j_list()

ode_decimal_precision = 50
j_list = [3,7/2,4]
process_j_list()

ode_decimal_precision = 60
j_list = [9/2,5]
process_j_list()
```

$\epsilon = 0.7071 \quad j = \frac{3}{2} \quad \sigma = 0.816 \quad (\omega/\sigma)^2 = 4.4999834016824557022 \quad (30,20) \quad \text{ode time} = 1.3931 \quad \text{other time} =$

$\epsilon = 0.7071 \quad j = 2 \quad \sigma = 1.12 \quad (\omega/\sigma)^2 = 5.1999897704981085583 \quad (40,20) \quad \text{ode time} = 3.3838 \quad \text{other time} = 0$

$\epsilon = 0.7071 \quad j = \frac{5}{2} \quad \sigma = 1.41 \quad (\omega/\sigma)^2 = 5.4999932868850748519 \quad (40,20) \quad \text{ode time} = 3.3771 \quad \text{other time} =$

$\epsilon = 0.7071 \quad j = 3 \quad \sigma = 1.71 \quad (\omega/\sigma)^2 = 5.6571381458966265831 \quad (50,20) \quad \text{ode time} = 7.4176 \quad \text{other time} = 1$

$\epsilon = 0.7071 \quad j = \frac{7}{2} \quad \sigma = 2.00 \quad (\omega/\sigma)^2 = 5.749996519696881626 \quad (50,20) \quad \text{ode time} = 7.4287 \quad \text{other time} = 1$

$\epsilon = 0.7071 \quad j = 4 \quad \sigma = 2.29 \quad (\omega/\sigma)^2 = 5.8095211363058142583 \quad (50,20) \quad \text{ode time} = 7.4138 \quad \text{other time} = 1$

$\epsilon = 0.7071 \quad j = \frac{9}{2} \quad \sigma = 2.58 \quad (\omega/\sigma)^2 = 5.849997883443329219 \quad (60,20) \quad \text{ode time} = 14.721 \quad \text{other time} = 2$

$\epsilon = 0.7071 \quad j = 5 \quad \sigma = 2.87 \quad (\omega/\sigma)^2 = 5.8787861619461414304 \quad (60,20) \quad \text{ode time} = 14.702 \quad \text{other time} = 2$

In [30]:
```
epsilonb = 0.55

ode_decimal_precision = 30
j_list = [3/2,2]
process_j_list()

ode_decimal_precision = 40
j_list = [5/2,3,7/2]
process_j_list()
```

$\epsilon = 0.55 \quad j = \frac{3}{2} \quad \sigma = 0.635 \quad (\omega/\sigma)^2 = 4.0119140651818690656 \quad (30,20) \quad \text{ode time} = 3.5293 \quad \text{other time} = 0$

$\epsilon = 0.55 \quad j = 2 \quad \sigma = 0.870 \quad (\omega/\sigma)^2 = 4.8782922295280631875 \quad (30,20) \quad \text{ode time} = 3.5297 \quad \text{other time} = 0.3$

$\epsilon = 0.55 \quad j = \frac{5}{2} \quad \sigma = 1.10 \quad (\omega/\sigma)^2 = 5.284176568320185352 \quad (40,20) \quad \text{ode time} = 9.1205 \quad \text{other time} = 0.81$

$\epsilon = 0.55 \quad j = 3 \quad \sigma = 1.33 \quad (\omega/\sigma)^2 = 5.5041350396067247139 \quad (40,20) \quad \text{ode time} = 9.1109 \quad \text{other time} = 0.82$

$\epsilon = 0.55 \quad j = \frac{7}{2} \quad \sigma = 1.56 \quad (\omega/\sigma)^2 = 5.6363382785629739775 \quad (40,20) \quad \text{ode time} = 9.1179 \quad \text{other time} = 0.8$

In [31]:

```
epsilonb = 0.4

ode_decimal_precision = 30
j_list = [3/2,2,5/2]
process_j_list()

ode_decimal_precision = 40
j_list = [3,7/2]
process_j_list()
```

$\epsilon = 0.4 \quad j = \frac{3}{2} \quad \sigma = 0.462 \quad (\omega/\sigma)^2 = 3.3043168365709716246 \quad (30,20) \quad \text{ode time} = 4.5847 \quad \text{other time} = 0.3$

$\epsilon = 0.4 \quad j = 2 \quad \sigma = 0.632 \quad (\omega/\sigma)^2 = 4.2926629349963078598 \quad (30,20) \quad \text{ode time} = 4.5846 \quad \text{other time} = 0.34$

$\epsilon = 0.4 \quad j = \frac{5}{2} \quad \sigma = 0.800 \quad (\omega/\sigma)^2 = 4.8595368961520156711 \quad (30,20) \quad \text{ode time} = 4.5844 \quad \text{other time} = 0.3$

$\epsilon = 0.4 \quad j = 3 \quad \sigma = 0.966 \quad (\omega/\sigma)^2 = 5.192260255551817162 \quad (40,20) \quad \text{ode time} = 10.749 \quad \text{other time} = 0.849$

$\epsilon = 0.4 \quad j = \frac{7}{2} \quad \sigma = 1.13 \quad (\omega/\sigma)^2 = 5.4001919850822057902 \quad (40,20) \quad \text{ode time} = 10.755 \quad \text{other time} = 0.85$

In [32]:

```
epsilonb = 0.3

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3]
process_j_list()

ode_decimal_precision = 40
j_list = [7/2]
process_j_list()
```

$\epsilon = 0.3 \quad j = \frac{3}{2} \quad \sigma = 0.346 \quad (\omega/\sigma)^2 = 2.7500646261884110787 \quad (30,20) \quad \text{ode time} = 4.6443 \quad \text{other time} = 0.3$

$\epsilon = 0.3 \quad j = 2 \quad \sigma = 0.474 \quad (\omega/\sigma)^2 = 3.6234231561114523133 \quad (30,20) \quad \text{ode time} = 4.6397 \quad \text{other time} = 0.34$

$\epsilon = 0.3 \quad j = \frac{5}{2} \quad \sigma = 0.600 \quad (\omega/\sigma)^2 = 4.2932101870569633592 \quad (30,20) \quad \text{ode time} = 4.6422 \quad \text{other time} = 0.3$

$\epsilon = 0.3 \quad j = 3 \quad \sigma = 0.725 \quad (\omega/\sigma)^2 = 4.7452772824302272425 \quad (30,20) \quad \text{ode time} = 4.6379 \quad \text{other time} = 0.34$

$\epsilon = 0.3 \quad j = \frac{7}{2} \quad \sigma = 0.849 \quad (\omega/\sigma)^2 = 5.0484406379829194185 \quad (40,20) \quad \text{ode time} = 12.154 \quad \text{other time} = 0.8$

In [33]:

```
epsilonb = 0.01

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2,5,10,15,20,30,40,50,60,70,80,90]
process_j_list()

ode_decimal_precision = 40
j_list = [100,120,140,160]
process_j_list()

ode_decimal_precision = 50
j_list = [180,200,230]
process_j_list()

ode_decimal_precision = 70
j_list = [260,290,320,350]
process_j_list()
```

$\epsilon = 0.01 \quad j = \frac{3}{2} \quad \sigma = 0.0115 \quad (\omega/\sigma)^2 = 2.0006096917743312498 \quad (30,20) \quad \text{ode time} = 5.2208 \quad \text{other time} =$

$\epsilon = 0.01 \quad j = 2 \quad \sigma = 0.0158 \quad (\omega/\sigma)^2 = 2.001464161862771762 \quad (30,20) \quad \text{ode time} = 5.2136 \quad \text{other time} = 0.3$

$\epsilon = 0.01 \quad j = \frac{5}{2} \quad \sigma = 0.0200 \quad (\omega/\sigma)^2 = 2.0025643102199197761 \quad (30,20) \quad \text{ode time} = 5.2110 \quad \text{other time} =$

$\epsilon = 0.01 \quad j = 3 \quad \sigma = 0.0242 \quad (\omega/\sigma)^2 = 2.0039112915562727295 \quad (30,20) \quad \text{ode time} = 5.2180 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = \frac{7}{2} \quad \sigma = 0.0283 \quad (\omega/\sigma)^2 = 2.005506513527785117 \quad (30,20) \quad \text{ode time} = 5.2142 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 5 \quad \sigma = 0.0406 \quad (\omega/\sigma)^2 = 2.0117994436223473306 \quad (30,20) \quad \text{ode time} = 5.2140 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 10 \quad \sigma = 0.0816 \quad (\omega/\sigma)^2 = 2.049915088956064386 \quad (30,20) \quad \text{ode time} = 5.2188 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 15 \quad \sigma = 0.122 \quad (\omega/\sigma)^2 = 2.1173874742746105617 \quad (30,20) \quad \text{ode time} = 5.2201 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 20 \quad \sigma = 0.163 \quad (\omega/\sigma)^2 = 2.2191564847884813161 \quad (30,20) \quad \text{ode time} = 5.2160 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 30 \quad \sigma = 0.245 \quad (\omega/\sigma)^2 = 2.5408502878760358795 \quad (30,20) \quad \text{ode time} = 5.2151 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 40 \quad \sigma = 0.327 \quad (\omega/\sigma)^2 = 3.0023297892601524571 \quad (30,20) \quad \text{ode time} = 5.2159 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 50 \quad \sigma = 0.408 \quad (\omega/\sigma)^2 = 3.5145618548029226805 \quad (30,20) \quad \text{ode time} = 5.2127 \quad \text{other time} = 0$

$\epsilon = 0.01$ j = 60 $\sigma = 0.490$ $(\omega/\sigma)^2 = 3.9836417374816104468$ (30,20) ode time =5.2192 other time =0

$\epsilon = 0.01$ j = 70 $\sigma = 0.572$ $(\omega/\sigma)^2 = 4.3691189358253653385$ (30,20) ode time =5.2169 other time =0

$\epsilon = 0.01$ j = 80 $\sigma = 0.653$ $(\omega/\sigma)^2 = 4.6713878684154581544$ (30,20) ode time =5.2242 other time =0

$\epsilon = 0.01$ j = 90 $\sigma = 0.735$ $(\omega/\sigma)^2 = 4.905275304219552021$ (30,20) ode time =5.2145 other time =0.3

$\epsilon = 0.01$ j = 100 $\sigma = 0.816$ $(\omega/\sigma)^2 = 5.086722917739205928$ (40,20) ode time =12.503 other time =0

$\epsilon = 0.01$ j = 120 $\sigma = 0.980$ $(\omega/\sigma)^2 = 5.3414664711113240511$ (40,20) ode time =12.492 other time =(

$\epsilon = 0.01$ j = 140 $\sigma = 1.14$ $(\omega/\sigma)^2 = 5.5053424306333022703$ (40,20) ode time =13.734 other time =0

$\epsilon = 0.01$ j = 160 $\sigma = 1.31$ $(\omega/\sigma)^2 = 5.615886790683154156$ (40,20) ode time =13.737 other time =0.8

$\epsilon = 0.01$ j = 180 $\sigma = 1.47$ $(\omega/\sigma)^2 = 5.6935841252232656642$ (50,20) ode time =29.015 other time =2

$\epsilon = 0.01$ j = 200 $\sigma = 1.63$ $(\omega/\sigma)^2 = 5.7501133827543736513$ (50,20) ode time =28.999 other time =2

$\epsilon = 0.01$ j = 230 $\sigma = 1.88$ $(\omega/\sigma)^2 = 5.8097230117170114877$ (50,20) ode time =31.901 other time =2

$\epsilon = 0.01$ j = 260 $\sigma = 2.12$ $(\omega/\sigma)^2 = 5.8504005667678360068$ (70,20) ode time =109.08 other time =8

$\epsilon = 0.01$ j = 290 $\sigma = 2.37$ $(\omega/\sigma)^2 = 5.8793547844855359812$ (70,20) ode time =109.11 other time =8

$\epsilon = 0.01$ j = 320 $\sigma = 2.61$ $(\omega/\sigma)^2 = 5.9006771964032691039$ (70,20) ode time =109.09 other time =8

$\epsilon = 0.01$ j = 350 $\sigma = 2.86$ $(\omega/\sigma)^2 = 5.9168241864476698865$ (70,20) ode time =109.09 other time =8

j = ∞

In [34]:

```
ode_decimal_precision = 30
sigma_list = [0.02,0.04,.06,.08,.10,0.15,0.2,0.25,0.3,0.35,0.4,0.45,0.5,0.55,.6,0.65,0.7,0.75]
process_sigma_list()

ode_decimal_precision = 40
sigma_list = [0.8,0.85,0.9,0.95,1,1.2]
process_sigma_list()

ode_decimal_precision = 50
sigma_list = [1.4,1.6,1.8]
process_sigma_list()

ode_decimal_precision = 60
sigma_list = [2.0,2.2,2.4,2.6]
process_sigma_list()

ode_decimal_precision = 70
```

```
sigma_list = [2.8,3.0]
process_sigma_list()
```

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0200$ $(\omega/\sigma)^2 = 2.0029306396589592143$ (30,20) ode time =5.2236 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0400$ $(\omega/\sigma)^2 = 2.011796370581308 15$ (30,20) ode time =5.2160 other time =0.35

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0600$ $(\omega/\sigma)^2 = 2.026815783726084496$ (30,20) ode time =5.2137 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0800$ $(\omega/\sigma)^2 = 2.0483431301388165799$ (30,20) ode time =5.2211 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.100$ $(\omega/\sigma)^2 = 2.0768512508711160382$ (30,20) ode time =5.2145 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.150$ $(\omega/\sigma)^2 = 2.1824734440870033519$ (30,20) ode time =5.2211 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.200$ $(\omega/\sigma)^2 = 2.3443219375813765233$ (30,20) ode time =5.2149 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.250$ $(\omega/\sigma)^2 = 2.5665348899140685796$ (30,20) ode time =5.2161 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.300$ $(\omega/\sigma)^2 = 2.8416258944066673457$ (30,20) ode time =5.2152 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.350$ $(\omega/\sigma)^2 = 3.1493685608081772441$ (30,20) ode time =5.2148 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.400$ $(\omega/\sigma)^2 = 3.4640786507796314538$ (30,20) ode time =5.2180 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.450$ $(\omega/\sigma)^2 = 3.7641093098968960934$ (30,20) ode time =5.2153 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.500$ $(\omega/\sigma)^2 = 4.0364870439793698411$ (30,20) ode time =5.2225 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.550$ $(\omega/\sigma)^2 = 4.2762245862832203116$ (30,20) ode time =5.2150 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.600$ $(\omega/\sigma)^2 = 4.4835179731049804094$ (30,20) ode time =5.2173 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.650$ $(\omega/\sigma)^2 = 4.6611734256132470388$ (30,20) ode time =5.2189 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.700$ $(\omega/\sigma)^2 = 4.8129437327005195409$ (30,20) ode time =5.2228 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.750$ $(\omega/\sigma)^2 = 4.9426438397638722761$ (30,20) ode time =5.2159 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.800$ $(\omega/\sigma)^2 = 5.0537551161551944156$ (40,20) ode time =12.484 other time =0.8

$\epsilon = 0.$ j = +∞ $\sigma = 0.850$ $(\omega/\sigma)^2 = 5.1492920034437529415$ (40,20) ode time =12.465 other time =0.8

$\epsilon = 0.$ j = +∞ $\sigma = 0.900$ $(\omega/\sigma)^2 = 5.2317943046322504194$ (40,20) ode time =12.489 other time =0.8

$\epsilon = 0.$ j = +∞ $\sigma = 0.950$ $(\omega/\sigma)^2 = 5.3033719781498695124$ (40,20) ode time =12.477 other time =0.8

$\epsilon = 0.$ j = +∞ $\sigma = 1.00$ $(\omega/\sigma)^2 = 5.3657663950167112016$ (40,20) ode time =13.708 other time =0.88

$\epsilon = 0.$ j = +∞ $\sigma = 1.20$ $(\omega/\sigma)^2 = 5.5486814297827637433$ (40,20) ode time =13.738 other time =0.88

$\epsilon = 0.$ j = +∞ $\sigma = 1.40$ $(\omega/\sigma)^2 = 5.6635970315408634098$ (50,20) ode time =29.013 other time =2.07

$\epsilon = 0.$ j = +∞ $\sigma = 1.60$ $(\omega/\sigma)^2 = 5.7400474766166610879$ (50,20) ode time =29.027 other time =2.08

$\epsilon = 0.$ j = +∞ $\sigma = 1.80$ $(\omega/\sigma)^2 = 5.793310029131846546$ (50,20) ode time =31.963 other time =2.075

$\epsilon = 0.$ j = +∞ $\sigma = 2.00$ $(\omega/\sigma)^2 = 5.831831372906549483$ (60,20) ode time =59.779 other time =4.433

$\epsilon = 0.$ j = +∞ $\sigma = 2.20$ $(\omega/\sigma)^2 = 5.8605596139979826243$ (60,20) ode time =59.829 other time =4.44

$\epsilon = 0.$ j = +∞ $\sigma = 2.40$ $(\omega/\sigma)^2 = 5.8825388177663555004$ (60,20) ode time =65.649 other time =4.46

$\epsilon = 0.$ j = +∞ $\sigma = 2.60$ $(\omega/\sigma)^2 = 5.8997208734549874499$ (60,20) ode time =65.545 other time =4.42

$\epsilon = 0.$ j = +∞ $\sigma = 2.80$ $(\omega/\sigma)^2 = 5.9134022698348587444$ (70,20) ode time =108.59 other time =8.31

$\epsilon = 0.$ j = +∞ $\sigma = 3.00$ $(\omega/\sigma)^2 = 5.9244706315060858988$ (70,20) ode time =108.56 other time =8.31

Ode 2_j

In [35]:
```
ode_name='2j'
data[ode_name]={}
data[ode_name]['asymp']={}
data[ode_name]['finite']={}
```

j finite

In [36]:
```
epsilonb = 0.7071

ode_decimal_precision = 30
j_list = [3/2]
process_j_list()

ode_decimal_precision = 40
j_list = [2,5/2]
process_j_list()

ode_decimal_precision = 50
j_list = [3,7/2,4]
```

```
process_j_list()

ode_decimal_precision = 60
j_list = [9/2,5]
process_j_list()

ode_decimal_precision = 70
j_list = [11/2]
process_j_list()
```

$\epsilon = 0.7071 \quad j = \frac{3}{2} \quad \sigma = 1.06 \quad (\omega/\sigma)^2 = 3.111105996386973974 \quad (30,20) \quad \text{ode time} = 1.1663 \quad \text{other time} = 0.$

$\epsilon = 0.7071 \quad j = 2 \quad \sigma = 1.41 \quad (\omega/\sigma)^2 = 3.4999961970124620313 \quad (40,20) \quad \text{ode time} = 3.0246 \quad \text{other time} = 0$

$\epsilon = 0.7071 \quad j = \frac{5}{2} \quad \sigma = 1.77 \quad (\omega/\sigma)^2 = 3.6799973229185163812 \quad (40,20) \quad \text{ode time} = 3.0261 \quad \text{other time} =$

$\epsilon = 0.7071 \quad j = 3 \quad \sigma = 2.12 \quad (\omega/\sigma)^2 = 3.7777758319480891792 \quad (50,20) \quad \text{ode time} = 6.9061 \quad \text{other time} = 1$

$\epsilon = 0.7071 \quad j = \frac{7}{2} \quad \sigma = 2.47 \quad (\omega/\sigma)^2 = 3.8367332270254066081 \quad (50,20) \quad \text{ode time} = 6.9043 \quad \text{other time} =$

$\epsilon = 0.7071 \quad j = 4 \quad \sigma = 2.83 \quad (\omega/\sigma)^2 = 3.8749988587702415691 \quad (50,20) \quad \text{ode time} = 6.9113 \quad \text{other time} = 1$

$\epsilon = 0.7071 \quad j = \frac{9}{2} \quad \sigma = 3.18 \quad (\omega/\sigma)^2 = 3.9012336564667336 \quad (60,20) \quad \text{ode time} = 14.025 \quad \text{other time} = 2.56$

$\epsilon = 0.7071 \quad j = 5 \quad \sigma = 3.54 \quad (\omega/\sigma)^2 = 3.9199992561534346762 \quad (60,20) \quad \text{ode time} = 14.014 \quad \text{other time} = 2$

$\epsilon = 0.7071 \quad j = \frac{11}{2} \quad \sigma = 3.89 \quad (\omega/\sigma)^2 = 3.9338836793777357288 \quad (70,20) \quad \text{ode time} = 25.137 \quad \text{other time} =$

In [37]:

```
epsilonb = 0.6

ode_decimal_precision = 30
j_list = [3/2]
process_j_list()

ode_decimal_precision = 40
j_list = [2,5/2,3]
process_j_list()

ode_decimal_precision = 50
j_list = [7/2]
process_j_list()

ode_decimal_precision = 50
j_list = [4]
process_j_list()
```

$\epsilon = 0.6 \quad j = \frac{3}{2} \quad \sigma = 0.900 \quad (\omega/\sigma)^2 = 3.0219757580714899217 \quad (30,20) \quad \text{ode time} = 2.6289 \quad \text{other time} = 0.2$

$\epsilon = 0.6 \quad j = 2 \quad \sigma = 1.20 \quad (\omega/\sigma)^2 = 3.4289121173565929182 \quad (40,20) \quad \text{ode time} = 7.1452 \quad \text{other time} = 0.70$

$\epsilon = 0.6 \quad j = \frac{5}{2} \quad \sigma = 1.50 \quad (\omega/\sigma)^2 = 3.628680571286386956 \quad (40,20) \quad \text{ode time} = 7.1397 \quad \text{other time} = 0.698$

$\epsilon = 0.6 \quad j = 3 \quad \sigma = 1.80 \quad (\omega/\sigma)^2 = 3.7400155842771586464 \quad (40,20) \quad \text{ode time} = 7.1452 \quad \text{other time} = 0.699$

$\epsilon = 0.6 \quad j = \frac{7}{2} \quad \sigma = 2.10 \quad (\omega/\sigma)^2 = 3.8080686609021688663 \quad (50,20) \quad \text{ode time} = 16.961 \quad \text{other time} = 1.67$

$\epsilon = 0.6 \quad j = 4 \quad \sigma = 2.40 \quad (\omega/\sigma)^2 = 3.8525999135959532319 \quad (50,20) \quad \text{ode time} = 16.974 \quad \text{other time} = 1.681$

In [38]:

```
epsilonb = 0.5

ode_decimal_precision = 30
j_list = [3/2,2]
process_j_list()

ode_decimal_precision = 40
j_list = [5/2,3,7/2]
process_j_list()

ode_decimal_precision = 50
j_list = [9/2]
process_j_list()

ode_decimal_precision = 60
j_list = [6]
process_j_list()

ode_decimal_precision = 70
j_list = [8]
process_j_list()
```

$\epsilon = 0.5 \quad j = \frac{3}{2} \quad \sigma = 0.750 \quad (\omega/\sigma)^2 = 2.9252106564059074814 \quad (30,20) \quad \text{ode time} = 3.1117 \quad \text{other time} = 0.2$

$\epsilon = 0.5 \quad j = 2 \quad \sigma = 1.00 \quad (\omega/\sigma)^2 = 3.336509692956312181 \quad (30,20) \quad \text{ode time} = 3.1070 \quad \text{other time} = 0.2637$

$\epsilon = 0.5 \quad j = \frac{5}{2} \quad \sigma = 1.25 \quad (\omega/\sigma)^2 = 3.5574871095526227098 \quad (40,20) \quad \text{ode time} = 8.5732 \quad \text{other time} = 0.72$

$\epsilon = 0.5 \quad j = 3 \quad \sigma = 1.50 \quad (\omega/\sigma)^2 = 3.6859375596307213748 \quad (40,20) \quad \text{ode time} = 8.5624 \quad \text{other time} = 0.723$

$\epsilon = 0.5 \quad j = \frac{7}{2} \quad \sigma = 1.75 \quad (\omega/\sigma)^2 = 3.766268941358884203 \quad (40,20) \quad \text{ode time} = 8.5670 \quad \text{other time} = 0.722$

$\epsilon = 0.5 \quad j = \frac{9}{2} \quad \sigma = 2.25 \quad (\omega/\sigma)^2 = 3.8566326173765722934 \quad (50,20) \quad \text{ode time} = 20.328 \quad \text{other time} = 1.77$

$\epsilon = 0.5 \quad j = 6 \quad \sigma = 3.00 \quad (\omega/\sigma)^2 = 3.9186142347692286407 \quad (60,20) \quad \text{ode time} = 42.169 \quad \text{other time} = 3.863$

$\epsilon = 0.5 \quad j = 8 \quad \sigma = 4.00 \quad (\omega/\sigma)^2 = 3.953986459962055927 \quad (70,20) \quad \text{ode time} = 77.547 \quad \text{other time} = 7.3441$

In [39]:

```
epsilonb = 0.4
```

```
ode_decimal_precision = 30
j_list = [3/2,2,5/2]
process_j_list()

ode_decimal_precision = 40
j_list = [3,7/2,9/2]
process_j_list()

ode_decimal_precision = 50
j_list = [11/2]
process_j_list()
```

$\epsilon = 0.4 \quad j = \frac{3}{2} \quad \sigma = 0.600 \quad (\omega/\sigma)^2 = 2.8260860543995007448 \quad (30,20) \quad \text{ode time} = 3.6120 \quad \text{other time} = 0.2$

$\epsilon = 0.4 \quad j = 2 \quad \sigma = 0.800 \quad (\omega/\sigma)^2 = 3.2095538197436441631 \quad (30,20) \quad \text{ode time} = 3.6121 \quad \text{other time} = 0.27$

$\epsilon = 0.4 \quad j = \frac{5}{2} \quad \sigma = 1.00 \quad (\omega/\sigma)^2 = 3.448496411453397044 \quad (30,20) \quad \text{ode time} = 3.6040 \quad \text{other time} = 0.271$

$\epsilon = 0.4 \quad j = 3 \quad \sigma = 1.20 \quad (\omega/\sigma)^2 = 3.5985568104045214463 \quad (40,20) \quad \text{ode time} = 9.9381 \quad \text{other time} = 0.746$

$\epsilon = 0.4 \quad j = \frac{7}{2} \quad \sigma = 1.40 \quad (\omega/\sigma)^2 = 3.6966042108267113169 \quad (40,20) \quad \text{ode time} = 9.9436 \quad \text{other time} = 0.74$

$\epsilon = 0.4 \quad j = \frac{9}{2} \quad \sigma = 1.80 \quad (\omega/\sigma)^2 = 3.8107346628228260028 \quad (40,20) \quad \text{ode time} = 9.9419 \quad \text{other time} = 0.74$

$\epsilon = 0.4 \quad j = \frac{11}{2} \quad \sigma = 2.20 \quad (\omega/\sigma)^2 = 3.8713285468249709094 \quad (50,20) \quad \text{ode time} = 23.727 \quad \text{other time} = 1.8$

In [40]:

```
epsilonb = 0.325

ode_decimal_precision = 30
j_list = [3/2,2,5/2]
process_j_list()

ode_decimal_precision = 40
j_list = [3,7/2]
process_j_list()
```

$\epsilon = 0.325 \quad j = \frac{3}{2} \quad \sigma = 0.487 \quad (\omega/\sigma)^2 = 2.7686172645363766041 \quad (30,20) \quad \text{ode time} = 3.6586 \quad \text{other time} =$

$\epsilon = 0.325 \quad j = 2 \quad \sigma = 0.650 \quad (\omega/\sigma)^2 = 3.0910012237972445732 \quad (30,20) \quad \text{ode time} = 3.6586 \quad \text{other time} = 0$

$\epsilon = 0.325 \quad j = \frac{5}{2} \quad \sigma = 0.812 \quad (\omega/\sigma)^2 = 3.3289994780980640365 \quad (30,20) \quad \text{ode time} = 3.6567 \quad \text{other time} =$

$\epsilon = 0.325 \quad j = 3 \quad \sigma = 0.975 \quad (\omega/\sigma)^2 = 3.4945400530890411016 \quad (40,20) \quad \text{ode time} = 10.106 \quad \text{other time} = 0$

$\epsilon = 0.325 \quad j = \frac{7}{2} \quad \sigma = 1.14 \quad (\omega/\sigma)^2 = 3.6096034560197983957 \quad (40,20) \quad \text{ode time} = 10.097 \quad \text{other time} = 0$

In [41]:

```
epsilonb = 0.25

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]
process_j_list()

ode_decimal_precision = 40
j_list = [9/2,13/2]
process_j_list()

ode_decimal_precision = 50
j_list = [8,10]
process_j_list()

ode_decimal_precision = 70
j_list = [23/2,13,29/2]
process_j_list()

ode_decimal_precision = 70
j_list = [16]
process_j_list()
```

$\epsilon = 0.25 \quad j = \frac{3}{2} \quad \sigma = 0.375 \quad (\omega/\sigma)^2 = 2.7548735658800374599 \quad (30,20) \quad \text{ode time} = 4.2099 \quad \text{other time} = 0$

$\epsilon = 0.25 \quad j = 2 \quad \sigma = 0.500 \quad (\omega/\sigma)^2 = 2.9695797955370071617 \quad (30,20) \quad \text{ode time} = 4.1040 \quad \text{other time} = 0.2$

$\epsilon = 0.25 \quad j = \frac{5}{2} \quad \sigma = 0.625 \quad (\omega/\sigma)^2 = 3.1718000990866994758 \quad (30,20) \quad \text{ode time} = 4.1106 \quad \text{other time} = 0$

$\epsilon = 0.25 \quad j = 3 \quad \sigma = 0.750 \quad (\omega/\sigma)^2 = 3.3384548001964306367 \quad (30,20) \quad \text{ode time} = 4.1044 \quad \text{other time} = 0.2$

$\epsilon = 0.25 \quad j = \frac{7}{2} \quad \sigma = 0.875 \quad (\omega/\sigma)^2 = 3.4682100918407403594 \quad (30,20) \quad \text{ode time} = 4.1099 \quad \text{other time} = 0$

$\epsilon = 0.25 \quad j = \frac{9}{2} \quad \sigma = 1.12 \quad (\omega/\sigma)^2 = 3.6434974763842416896 \quad (40,20) \quad \text{ode time} = 10.253 \quad \text{other time} = 0.7$

$\epsilon = 0.25 \quad j = \frac{13}{2} \quad \sigma = 1.62 \quad (\omega/\sigma)^2 = 3.814084315501186252 \quad (40,20) \quad \text{ode time} = 11.384 \quad \text{other time} = 0.7$

$\epsilon = 0.25 \quad j = 8 \quad \sigma = 2.00 \quad (\omega/\sigma)^2 = 3.8739450433276470651 \quad (50,20) \quad \text{ode time} = 24.435 \quad \text{other time} = 1.87$

$\epsilon = 0.25 \quad j = 10 \quad \sigma = 2.50 \quad (\omega/\sigma)^2 = 3.917799937383960746 \quad (50,20) \quad \text{ode time} = 27.144 \quad \text{other time} = 1.88$

$\epsilon = 0.25 \quad j = \frac{23}{2} \quad \sigma = 2.88 \quad (\omega/\sigma)^2 = 3.937337767536398566 \quad (70,20) \quad \text{ode time} = 93.658 \quad \text{other time} = 7.9$

$\epsilon = 0.25 \quad j = 13 \quad \sigma = 3.25 \quad (\omega/\sigma)^2 = 3.9506944761876452264 \quad (70,20) \quad \text{ode time} = 93.619 \quad \text{other time} = 7.9$

$\epsilon = 0.25 \quad j = \frac{29}{2} \quad \sigma = 3.62 \quad (\omega/\sigma)^2 = 3.96021425385399209 \quad (70,20) \quad \text{ode time} = 104.05 \quad \text{other time} = 7.93$

$\epsilon = 0.25 \quad j = 16 \quad \sigma = 4.00 \quad (\omega/\sigma)^2 = 3.9672316450013068105 \quad (70,20) \quad \text{ode time} = 104.01 \quad \text{other time} = 7.9$

In [42]:

```
epsilonb = 0.2

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]
process_j_list()
```

$\epsilon = 0.2 \quad j = \frac{3}{2} \quad \sigma = 0.300 \quad (\omega/\sigma)^2 = 2.7900223637150635907 \quad (30,20) \quad \text{ode time} = 4.1372 \quad \text{other time} = 0.2$

$\epsilon = 0.2 \quad j = 2 \quad \sigma = 0.400 \quad (\omega/\sigma)^2 = 2.911889776281244591 \quad (30,20) \quad \text{ode time} = 4.1461 \quad \text{other time} = 0.277$

$\epsilon = 0.2 \quad j = \frac{5}{2} \quad \sigma = 0.500 \quad (\omega/\sigma)^2 = 3.0584348766204516843 \quad (30,20) \quad \text{ode time} = 4.1318 \quad \text{other time} = 0.2$

$\epsilon = 0.2 \quad j = 3 \quad \sigma = 0.600 \quad (\omega/\sigma)^2 = 3.2022972174391099247 \quad (30,20) \quad \text{ode time} = 4.1413 \quad \text{other time} = 0.28$

$\epsilon = 0.2 \quad j = \frac{7}{2} \quad \sigma = 0.700 \quad (\omega/\sigma)^2 = 3.3295574084657802527 \quad (30,20) \quad \text{ode time} = 4.1351 \quad \text{other time} = 0.2$

In [43]:

```
epsilonb = 0.15

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]
process_j_list()
```

$\epsilon = 0.15 \quad j = \frac{3}{2} \quad \sigma = 0.225 \quad (\omega/\sigma)^2 = 2.8767082020777589107 \quad (30,20) \quad \text{ode time} = 4.1428 \quad \text{other time} = 0$

$\epsilon = 0.15 \quad j = 2 \quad \sigma = 0.300 \quad (\omega/\sigma)^2 = 2.9060034398802915749 \quad (30,20) \quad \text{ode time} = 4.1402 \quad \text{other time} = 0.2$

$\epsilon = 0.15 \quad j = \frac{5}{2} \quad \sigma = 0.375 \quad (\omega/\sigma)^2 = 2.9730112551102726957 \quad (30,20) \quad \text{ode time} = 4.1373 \quad \text{other time} = 0.$

$\epsilon = 0.15 \quad j = 3 \quad \sigma = 0.450 \quad (\omega/\sigma)^2 = 3.0622401264067358881 \quad (30,20) \quad \text{ode time} = 4.1603 \quad \text{other time} = 0.2$

$\epsilon = 0.15 \quad j = \frac{7}{2} \quad \sigma = 0.525 \quad (\omega/\sigma)^2 = 3.1595334734424986648 \quad (30,20) \quad \text{ode time} = 4.1475 \quad \text{other time} = 0$

In [44]:

```
epsilonb = 0.125

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]
process_j_list()
```

$\epsilon = 0.125 \quad j = \frac{3}{2} \quad \sigma = 0.188 \quad (\omega/\sigma)^2 = 2.9419804646675092648 \quad (30,20) \quad \text{ode time} = 4.1639 \quad \text{other time} =$

$\epsilon = 0.125 \quad j = 2 \quad \sigma = 0.250 \quad (\omega/\sigma)^2 = 2.933976756786076842 \quad (30,20) \quad \text{ode time} = 4.1609 \quad \text{other time} = 0.2$

$\epsilon = 0.125 \quad j = \frac{5}{2} \quad \sigma = 0.312 \quad (\omega/\sigma)^2 = 2.9591701577989723021 \quad (30,20) \quad \text{ode time} = 4.1634 \quad \text{other time} =$

$\epsilon = 0.125 \quad j = 3 \quad \sigma = 0.375 \quad (\omega/\sigma)^2 = 3.0103722063219255875 \quad (30,20) \quad \text{ode time} = 4.1644 \quad \text{other time} = 0$

$\epsilon = 0.125 \quad j = \frac{7}{2} \quad \sigma = 0.438 \quad (\omega/\sigma)^2 = 3.0780823512890894891 \quad (30,20) \quad \text{ode time} = 4.1616 \quad \text{other time} =$

In [45]:
```
epsilonb = 0.1

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2,4]
process_j_list()
```

$\epsilon = 0.1 \quad j = \frac{3}{2} \quad \sigma = 0.150 \quad (\omega/\sigma)^2 = 3.0202372938491234705 \quad (30,20) \quad \text{ode time} = 4.1767 \quad \text{other time} = 0.2$

$\epsilon = 0.1 \quad j = 2 \quad \sigma = 0.200 \quad (\omega/\sigma)^2 = 2.9874597291297524231 \quad (30,20) \quad \text{ode time} = 4.1706 \quad \text{other time} = 0.28$

$\epsilon = 0.1 \quad j = \frac{5}{2} \quad \sigma = 0.250 \quad (\omega/\sigma)^2 = 2.9771606541773666915 \quad (30,20) \quad \text{ode time} = 4.1698 \quad \text{other time} = 0.2$

$\epsilon = 0.1 \quad j = 3 \quad \sigma = 0.300 \quad (\omega/\sigma)^2 = 2.9892814480213505723 \quad (30,20) \quad \text{ode time} = 4.1694 \quad \text{other time} = 0.28$

$\epsilon = 0.1 \quad j = \frac{7}{2} \quad \sigma = 0.350 \quad (\omega/\sigma)^2 = 3.0200780860474762668 \quad (30,20) \quad \text{ode time} = 4.1726 \quad \text{other time} = 0.2$

$\epsilon = 0.1 \quad j = 4 \quad \sigma = 0.400 \quad (\omega/\sigma)^2 = 3.0645544106568858882 \quad (30,20) \quad \text{ode time} = 4.1752 \quad \text{other time} = 0.28$

In [46]:
```
epsilonb = 0.08

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2,4]
process_j_list()
```

$\epsilon = 0.08 \quad j = \frac{3}{2} \quad \sigma = 0.120 \quad (\omega/\sigma)^2 = 3.0885824003150966159 \quad (30,20) \quad \text{ode time} = 4.1877 \quad \text{other time} = 0$

$\epsilon = 0.08 \quad j = 2 \quad \sigma = 0.160 \quad (\omega/\sigma)^2 = 3.0482652232348396391 \quad (30,20) \quad \text{ode time} = 4.1732 \quad \text{other time} = 0.2$

$\epsilon = 0.08 \quad j = \frac{5}{2} \quad \sigma = 0.200 \quad (\omega/\sigma)^2 = 3.0195927817201708685 \quad (30,20) \quad \text{ode time} = 4.1777 \quad \text{other time} = 0$

$\epsilon = 0.08 \quad j = 3 \quad \sigma = 0.240 \quad (\omega/\sigma)^2 = 3.0059180001580421244 \quad (30,20) \quad \text{ode time} = 4.1736 \quad \text{other time} = 0.2$

$\epsilon = 0.08 \quad j = \frac{7}{2} \quad \sigma = 0.280 \quad (\omega/\sigma)^2 = 3.0073934028957272796 \quad (30,20) \quad \text{ode time} = 4.1771 \quad \text{other time} = 0$

$\epsilon = 0.08 \quad j = 4 \quad \sigma = 0.320 \quad (\omega/\sigma)^2 = 3.022377590477805206 \quad (30,20) \quad \text{ode time} = 4.1732 \quad \text{other time} = 0.28$

In [47]:

```
epsilonb = 0.01

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]+[m/2 for m in range(12,32,5)]+[m/2 for m in range(32,122,10)]
process_j_list()

j_list = [m/2 for m in range(122,182,20)]
process_j_list()

ode_decimal_precision = 40
j_list = [m/2 for m in range(182,342,20)]
process_j_list()

ode_decimal_precision = 50
j_list = [m/2 for m in range(342,502,40)]
process_j_list()

ode_decimal_precision = 60
j_list = [m/2 for m in range(500,660,80)]
process_j_list()

ode_decimal_precision = 80
j_list = [m/2 for m in range(660,820,80)]
process_j_list()
```

$\epsilon = 0.01 \quad j = \frac{3}{2} \quad \sigma = 0.0150 \quad (\omega/\sigma)^2 = 3.2672043864311006399 \quad (30,20) \quad \text{ode time} = 4.1767 \quad \text{other time} =$

$\epsilon = 0.01 \quad j = 2 \quad \sigma = 0.0200 \quad (\omega/\sigma)^2 = 3.2654202519516805471 \quad (30,20) \quad \text{ode time} = 4.1784 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = \frac{5}{2} \quad \sigma = 0.0250 \quad (\omega/\sigma)^2 = 3.263148361992893767 \quad (30,20) \quad \text{ode time} = 4.1756 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 3 \quad \sigma = 0.0300 \quad (\omega/\sigma)^2 = 3.2604047479759726416 \quad (30,20) \quad \text{ode time} = 4.1721 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = \frac{7}{2} \quad \sigma = 0.0350 \quad (\omega/\sigma)^2 = 3.2572085030256455228 \quad (30,20) \quad \text{ode time} = 4.1723 \quad \text{other time} =$

$\epsilon = 0.01 \quad j = 6 \quad \sigma = 0.0600 \quad (\omega/\sigma)^2 = 3.2352832046354860316 \quad (30,20) \quad \text{ode time} = 4.1708 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = \frac{17}{2} \quad \sigma = 0.0850 \quad (\omega/\sigma)^2 = 3.206121949782006774 \quad (30,20) \quad \text{ode time} = 4.1733 \quad \text{other time} =$

$\epsilon = 0.01 \quad j = 11 \quad \sigma = 0.110 \quad (\omega/\sigma)^2 = 3.1736417351061720374 \quad (30,20) \quad \text{ode time} = 4.1783 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = \frac{27}{2} \quad \sigma = 0.135 \quad (\omega/\sigma)^2 = 3.1414236275692625523 \quad (30,20) \quad \text{ode time} = 4.1814 \quad \text{other time} =$

$\epsilon = 0.01 \quad j = 16 \quad \sigma = 0.160 \quad (\omega/\sigma)^2 = 3.1122487572175275056 \quad (30,20) \quad \text{ode time} = 4.1751 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 21 \quad \sigma = 0.210 \quad (\omega/\sigma)^2 = 3.0696875586792866563 \quad (30,20) \quad \text{ode time} = 4.1778 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 26 \quad \sigma = 0.260 \quad (\omega/\sigma)^2 = 3.0521943360032247625 \quad (30,20) \quad \text{ode time} = 4.1736 \quad \text{other time} = 0$

$\epsilon = 0.01 \quad j = 31 \quad \sigma = 0.310 \quad (\omega/\sigma)^2 = 3.0584974343907985133 \quad (30,20) \quad \text{ode time} = 4.1752 \quad \text{other time} = 0$

$\epsilon = 0.01$ j = 36 $\sigma = 0.360$ $(\omega/\sigma)^2 = 3.084063345537294108$ (30,20) ode time =4.1753 other time =0.2

$\epsilon = 0.01$ j = 41 $\sigma = 0.410$ $(\omega/\sigma)^2 = 3.1235085639426417268$ (30,20) ode time =4.1712 other time =0

$\epsilon = 0.01$ j = 46 $\sigma = 0.460$ $(\omega/\sigma)^2 = 3.1717877497744656857$ (30,20) ode time =4.1730 other time =0

$\epsilon = 0.01$ j = 51 $\sigma = 0.510$ $(\omega/\sigma)^2 = 3.2247105184655556095$ (30,20) ode time =4.1740 other time =0

$\epsilon = 0.01$ j = 56 $\sigma = 0.560$ $(\omega/\sigma)^2 = 3.2790876124698933971$ (30,20) ode time =4.1738 other time =0

$\epsilon = 0.01$ j = 61 $\sigma = 0.610$ $(\omega/\sigma)^2 = 3.3326732310851522941$ (30,20) ode time =4.1720 other time =0

$\epsilon = 0.01$ j = 71 $\sigma = 0.710$ $(\omega/\sigma)^2 = 3.4322462691108951995$ (30,20) ode time =4.1708 other time =0

$\epsilon = 0.01$ j = 81 $\sigma = 0.810$ $(\omega/\sigma)^2 = 3.5180450475770610144$ (30,20) ode time =4.1753 other time =0

$\epsilon = 0.01$ j = 91 $\sigma = 0.910$ $(\omega/\sigma)^2 = 3.5896614240309616849$ (40,20) ode time =11.596 other time =0

$\epsilon = 0.01$ j = 101 $\sigma = 1.01$ $(\omega/\sigma)^2 = 3.6486648274752725845$ (40,20) ode time =11.601 other time =0

$\epsilon = 0.01$ j = 111 $\sigma = 1.11$ $(\omega/\sigma)^2 = 3.6971296260447738647$ (40,20) ode time =11.588 other time =0.

$\epsilon = 0.01$ j = 121 $\sigma = 1.21$ $(\omega/\sigma)^2 = 3.7370310384526100958$ (40,20) ode time =11.585 other time =0

$\epsilon = 0.01$ j = 131 $\sigma = 1.31$ $(\omega/\sigma)^2 = 3.7700511363266407975$ (40,20) ode time =11.586 other time =0

$\epsilon = 0.01$ j = 141 $\sigma = 1.41$ $(\omega/\sigma)^2 = 3.7975548447115829331$ (40,20) ode time =11.595 other time =0

$\epsilon = 0.01$ j = 151 $\sigma = 1.51$ $(\omega/\sigma)^2 = 3.820626139695773359$ (40,20) ode time =11.589 other time =0.7

$\epsilon = 0.01$ j = 161 $\sigma = 1.61$ $(\omega/\sigma)^2 = 3.8401187078847102487$ (40,20) ode time =11.588 other time =0

$\epsilon = 0.01$ j = 171 $\sigma = 1.71$ $(\omega/\sigma)^2 = 3.8567039682251523216$ (50,20) ode time =27.688 other time =1

$\epsilon = 0.01$ j = 191 $\sigma = 1.91$ $(\omega/\sigma)^2 = 3.8831601923182423238$ (50,20) ode time =27.697 other time =1

$\epsilon = 0.01$ j = 211 $\sigma = 2.11$ $(\omega/\sigma)^2 = 3.9030531012025266795$ (50,20) ode time =27.699 other time =1

$\epsilon = 0.01$ j = 231 $\sigma = 2.31$ $(\omega/\sigma)^2 = 3.9183462616756493358$ (50,20) ode time =27.693 other time =1

$\epsilon = 0.01$ j = 250 $\sigma = 2.50$ $(\omega/\sigma)^2 = 3.929798561076493345$ (60,20) ode time =57.953 other time =4.2

$\epsilon = 0.01$ j = 290 $\sigma = 2.90$ $(\omega/\sigma)^2 = 3.9472798590403610717$ (60,20) ode time =57.990 other time =4

$\epsilon = 0.01$ j = 330 $\sigma = 3.30$ $(\omega/\sigma)^2 = 3.9590035760635018684$ (80,20) ode time =190.91 other time =1

$\epsilon = 0.01$ j = 370 $\sigma = 3.70$ $(\omega/\sigma)^2 = 3.9672316604535058991$ (80,20) ode time =190.33 other time =1

$j \to \infty$

In [48]:

```
ode_decimal_precision = 30
sigma_list = [0.02,0.04,.06,.08,.10,0.15,0.2,0.25,0.3,0.35,0.4,0.45,0.5,0.55,.6,0.65,0.7,0.75,0.8,0.8
process_sigma_list()

ode_decimal_precision = 40
sigma_list = [0.95,1,1.15,1.3,1.45,1.6]
process_sigma_list()

ode_decimal_precision = 50
sigma_list = [1.85,2.1]
process_sigma_list()

ode_decimal_precision = 60
sigma_list = [2.35,2.65,3.0]
process_sigma_list()

ode_decimal_precision = 70
sigma_list = [3.35,3.7]
process_sigma_list()

ode_decimal_precision = 80
sigma_list = [4.0]
process_sigma_list()
```

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0200$ $(\omega/\sigma)^2 = 3.2668296476156010782$ (30,20) ode time =4.1716 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0400$ $(\omega/\sigma)^2 = 3.2549624194185527516$ (30,20) ode time =4.1589 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0600$ $(\omega/\sigma)^2 = 3.2366200733195958741$ (30,20) ode time =4.1549 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0800$ $(\omega/\sigma)^2 = 3.2136259717282729338$ (30,20) ode time =4.1632 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.100$ $(\omega/\sigma)^2 = 3.1880069613841409911$ (30,20) ode time =4.1581 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.150$ $(\omega/\sigma)^2 = 3.1244685863857124095$ (30,20) ode time =4.1562 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.200$ $(\omega/\sigma)^2 = 3.0771560571495292472$ (30,20) ode time =4.1551 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.250$ $(\omega/\sigma)^2 = 3.0544570413357088867$ (30,20) ode time =4.1496 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.300$ $(\omega/\sigma)^2 = 3.0561983284001380199$ (30,20) ode time =4.1643 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.350$ $(\omega/\sigma)^2 = 3.0782494102199292586$ (30,20) ode time =4.1518 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.400$ $(\omega/\sigma)^2 = 3.115268631675504591$ (30,20) ode time =4.1571 other time =0.28

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.450$ $(\omega/\sigma)^2 = 3.1620831000589164024$ (30,20) ode time =4.1531 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.500$ $(\omega/\sigma)^2 = 3.2143090210928620561$ (30,20) ode time =4.1573 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.550$ $(\omega/\sigma)^2 = 3.2685555230236133845$ (30,20) ode time =4.1517 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.600$ $(\omega/\sigma)^2 = 3.322397590023560884$ (30,20) ode time =4.1528 other time =0.28

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.650$ $(\omega/\sigma)^2 = 3.3742337839221701029$ (30,20) ode time =4.1558 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.700$ $(\omega/\sigma)^2 = 3.4231048786692838874$ (30,20) ode time =4.1519 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.750$ $(\omega/\sigma)^2 = 3.4685188101731642574$ (30,20) ode time =4.1529 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.800$ $(\omega/\sigma)^2 = 3.5103035433550040559$ (30,20) ode time =4.1540 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.850$ $(\omega/\sigma)^2 = 3.5484937597379243632$ (30,20) ode time =4.1566 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.900$ $(\omega/\sigma)^2 = 3.5832488584938412583$ (30,20) ode time =4.1546 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.950$ $(\omega/\sigma)^2 = 3.6147964345447534 21$ (40,20) ode time =11.568 other time =0.78

$\epsilon = 0.$ $j = +\infty$ $\sigma = 1.00$ $(\omega/\sigma)^2 = 3.6433949315513066236$ (40,20) ode time =11.550 other time =0.77

$\epsilon = 0.$ $j = +\infty$ $\sigma = 1.15$ $(\omega/\sigma)^2 = 3.7141090034650305173$ (40,20) ode time =11.561 other time =0.77

$\epsilon = 0.$ $j = +\infty$ $\sigma = 1.30$ $(\omega/\sigma)^2 = 3.7670902207074880688$ (40,20) ode time =11.556 other time =0.78

$\epsilon = 0.$ $j = +\infty$ $\sigma = 1.45$ $(\omega/\sigma)^2 = 3.8073212784009466351$ (40,20) ode time =11.559 other time =0.77

$\epsilon = 0.$ $j = +\infty$ $\sigma = 1.60$ $(\omega/\sigma)^2 = 3.8383580870573511167$ (40,20) ode time =11.565 other time =0.78

$\epsilon = 0.$ $j = +\infty$ $\sigma = 1.85$ $(\omega/\sigma)^2 = 3.8760613017121425409$ (50,20) ode time =27.643 other time =1.92

$\epsilon = 0.$ $j = +\infty$ $\sigma = 2.10$ $(\omega/\sigma)^2 = 3.9022085114187509892$ (50,20) ode time =27.642 other time =1.92

$\epsilon = 0.$ $j = +\infty$ $\sigma = 2.35$ $(\omega/\sigma)^2 = 3.920997493443859379$ (60,20) ode time =57.962 other time =4.229

$\epsilon = 0.$ $j = +\infty$ $\sigma = 2.65$ $(\omega/\sigma)^2 = 3.9372577317750071412$ (60,20) ode time =57.927 other time =4.24

$\epsilon = 0.$ $j = +\infty$ $\sigma = 3.00$ $(\omega/\sigma)^2 = 3.9506514252909105507$ (60,20) ode time =57.910 other time =4.23

$\epsilon = 0.$ $j = +\infty$ $\sigma = 3.35$ $(\omega/\sigma)^2 = 3.9602013952518077806$ (70,20) ode time =106.41 other time =8.17

$\epsilon = 0.$ $j = +\infty$ $\sigma = 3.70$ $(\omega/\sigma)^2 = 3.9672405307184597691$ (70,20) ode time =106.42 other time =8.09

$\epsilon = 0.$ $j = +\infty$ $\sigma = 4.00$ $(\omega/\sigma)^2 = 3.9718944720574996035$ (80,20) ode time =190.40 other time =14.5

Ode 1_j

In [49]:
```
ode_name='1j'
data[ode_name]={}
data[ode_name]['asymp']={}
data[ode_name]['finite']={}
```

j finite

In [50]:
```
data['1j']['finite']={}
```

In [51]:
```
epsilonb = 0.7071

ode_decimal_precision = 40
j_list = [3/2]
process_j_list()

ode_decimal_precision = 40
j_list = [2,5/2]
process_j_list()

ode_decimal_precision = 50
j_list = [3,7/2,4]
process_j_list()
```

$\epsilon = 0.7071$ $j = \frac{3}{2}$ $\sigma = 1.06$ $(\omega/\sigma)^2 = 4.4444480978367223919$ (40,20) ode time =2.5378 other time =

$\epsilon = 0.7071$ $j = 2$ $\sigma = 1.41$ $(\omega/\sigma)^2 = 4.2500021920368275213$ (40,20) ode time =2.5364 other time =0

$\epsilon = 0.7071$ $j = \frac{5}{2}$ $\sigma = 1.77$ $(\omega/\sigma)^2 = 4.1600014475720327977$ (40,20) ode time =2.5336 other time =

$\epsilon = 0.7071$ $j = 3$ $\sigma = 2.12$ $(\omega/\sigma)^2 = 4.1111121340622449649$ (50,20) ode time =6.0651 other time =1.

$\epsilon = 0.7071$ $j = \frac{7}{2}$ $\sigma = 2.47$ $(\omega/\sigma)^2 = 4.0816334126785126723$ (50,20) ode time =6.0601 other time =

$\epsilon = 0.7071$ $j = 4$ $\sigma = 2.83$ $(\omega/\sigma)^2 = 4.0625005856591875058$ (50,20) ode time =6.0614 other time =1

In [52]:

```
epsilonb = 0.5

ode_decimal_precision = 30
j_list = [3/2]
process_j_list()

ode_decimal_precision = 40
j_list = [2,5/2,3,7/2]
process_j_list()
```

$\epsilon = 0.5 \quad j = \frac{3}{2} \quad \sigma = 0.750 \quad (\omega/\sigma)^2 = 4.6133268717304487007 \quad (30,20) \quad \text{ode time} = 2.6206 \quad \text{other time} = 0.2$

$\epsilon = 0.5 \quad j = 2 \quad \sigma = 1.00 \quad (\omega/\sigma)^2 = 4.3542164025695802042 \quad (40,20) \quad \text{ode time} = 7.6163 \quad \text{other time} = 0.613$

$\epsilon = 0.5 \quad j = \frac{5}{2} \quad \sigma = 1.25 \quad (\omega/\sigma)^2 = 4.2299812085082966298 \quad (40,20) \quad \text{ode time} = 7.6104 \quad \text{other time} = 0.61$

$\epsilon = 0.5 \quad j = 3 \quad \sigma = 1.50 \quad (\omega/\sigma)^2 = 4.1610829215236342351 \quad (40,20) \quad \text{ode time} = 7.6011 \quad \text{other time} = 0.611$

$\epsilon = 0.5 \quad j = \frac{7}{2} \quad \sigma = 1.75 \quad (\omega/\sigma)^2 = 4.1189953689480166178 \quad (40,20) \quad \text{ode time} = 7.6033 \quad \text{other time} = 0.61$

In [53]:

```
epsilonb = 0.35

ode_decimal_precision = 30
j_list = [3/2,2,5/2]
process_j_list()

ode_decimal_precision = 40
j_list = [3,7/2]
process_j_list()
```

$\epsilon = 0.35 \quad j = \frac{3}{2} \quad \sigma = 0.525 \quad (\omega/\sigma)^2 = 4.8909684915731806421 \quad (30,20) \quad \text{ode time} = 2.7455 \quad \text{other time} = 0$

$\epsilon = 0.35 \quad j = 2 \quad \sigma = 0.700 \quad (\omega/\sigma)^2 = 4.5358855153537569218 \quad (30,20) \quad \text{ode time} = 3.0874 \quad \text{other time} = 0.2$

$\epsilon = 0.35 \quad j = \frac{5}{2} \quad \sigma = 0.875 \quad (\omega/\sigma)^2 = 4.356661399443839756 \quad (30,20) \quad \text{ode time} = 3.0889 \quad \text{other time} = 0.2$

$\epsilon = 0.35 \quad j = 3 \quad \sigma = 1.05 \quad (\omega/\sigma)^2 = 4.2538454972933859456 \quad (40,20) \quad \text{ode time} = 8.9799 \quad \text{other time} = 0.64$

$\epsilon = 0.35 \quad j = \frac{7}{2} \quad \sigma = 1.22 \quad (\omega/\sigma)^2 = 4.1895656797028750557 \quad (40,20) \quad \text{ode time} = 8.9825 \quad \text{other time} = 0.6$

In [54]:

```
epsilonb = 0.25

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]
process_j_list()
```

$\epsilon = 0.25 \quad j = \frac{3}{2} \quad \sigma = 0.375 \quad (\omega/\sigma)^2 = 5.2882102889748398006 \quad (30,20) \quad \text{ode time} = 3.1448 \quad \text{other time} = 0$

$\epsilon = 0.25 \quad j = 2 \quad \sigma = 0.500 \quad (\omega/\sigma)^2 = 4.8130021114066852972 \quad (30,20) \quad \text{ode time} = 3.1451 \quad \text{other time} = 0.2$

$\epsilon = 0.25 \quad j = \frac{5}{2} \quad \sigma = 0.625 \quad (\omega/\sigma)^2 = 4.5586215534658206707 \quad (30,20) \quad \text{ode time} = 3.1446 \quad \text{other time} = 0$

$\epsilon = 0.25 \quad j = 3 \quad \sigma = 0.750 \quad (\omega/\sigma)^2 = 4.406506287046373958 \quad (30,20) \quad \text{ode time} = 3.1437 \quad \text{other time} = 0.21$

$\epsilon = 0.25 \quad j = \frac{7}{2} \quad \sigma = 0.875 \quad (\omega/\sigma)^2 = 4.3084606397713180209 \quad (30,20) \quad \text{ode time} = 3.1405 \quad \text{other time} = 0$

In [55]:

```
epsilonb = 0.15

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2,4]
process_j_list()
```

$\epsilon = 0.15 \quad j = \frac{3}{2} \quad \sigma = 0.225 \quad (\omega/\sigma)^2 = 6.1901268599450224521 \quad (30,20) \quad \text{ode time} = 3.1731 \quad \text{other time} = 0$

$\epsilon = 0.15 \quad j = 2 \quad \sigma = 0.300 \quad (\omega/\sigma)^2 = 5.5073555959279322088 \quad (30,20) \quad \text{ode time} = 3.1742 \quad \text{other time} = 0.2$

$\epsilon = 0.15 \quad j = \frac{5}{2} \quad \sigma = 0.375 \quad (\omega/\sigma)^2 = 5.0997668285842154872 \quad (30,20) \quad \text{ode time} = 3.1732 \quad \text{other time} = 0$

$\epsilon = 0.15 \quad j = 3 \quad \sigma = 0.450 \quad (\omega/\sigma)^2 = 4.8363596374252251442 \quad (30,20) \quad \text{ode time} = 3.1694 \quad \text{other time} = 0.2$

$\epsilon = 0.15 \quad j = \frac{7}{2} \quad \sigma = 0.525 \quad (\omega/\sigma)^2 = 4.6563688999228258291 \quad (30,20) \quad \text{ode time} = 3.1696 \quad \text{other time} = 0$

$\epsilon = 0.15 \quad j = 4 \quad \sigma = 0.600 \quad (\omega/\sigma)^2 = 4.5280529556201950957 \quad (30,20) \quad \text{ode time} = 3.1775 \quad \text{other time} = 0.2$

In [56]:

```
epsilonb = 0.01

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]+[m/2 for m in range(12,32,5)]+[m/2 for m in range(32,122,10)]
process_j_list()

j_list = [m/2 for m in range(122,182,20)]
process_j_list()

ode_decimal_precision = 40
j_list = [m/2 for m in range(182,342,20)]
process_j_list()

ode_decimal_precision = 50
j_list = [m/2 for m in range(342,502,40)]
process_j_list()

ode_decimal_precision = 60
j_list = [m/2 for m in range(500,660,80)]
process_j_list()
```

$\epsilon = 0.01 \quad j = \frac{3}{2} \quad \sigma = 0.0150 \quad (\omega/\sigma)^2 = 9.4738902559517118445 \quad (30,20) \quad \text{ode time} = 2.8419 \quad \text{other time} = 0$

$\epsilon = 0.01$ j = 2 $\sigma = 0.0200$ $(\omega/\sigma)^2 = 9.2413505144763454294$ (30,20) ode time =2.8459 other time =0

$\epsilon = 0.01$ $j = \frac{5}{2}$ $\sigma = 0.0250$ $(\omega/\sigma)^2 = 9.1034872690655768504$ (30,20) ode time =2.8428 other time =

$\epsilon = 0.01$ j = 3 $\sigma = 0.0300$ $(\omega/\sigma)^2 = 8.9977300265488129387$ (30,20) ode time =2.8428 other time =0

$\epsilon = 0.01$ $j = \frac{7}{2}$ $\sigma = 0.0350$ $(\omega/\sigma)^2 = 8.9032365967418369482$ (30,20) ode time =2.8485 other time =

$\epsilon = 0.01$ j = 6 $\sigma = 0.0600$ $(\omega/\sigma)^2 = 8.4348049879824903449$ (30,20) ode time =3.1984 other time =0

$\epsilon = 0.01$ $j = \frac{17}{2}$ $\sigma = 0.0850$ $(\omega/\sigma)^2 = 7.9360626017987203662$ (30,20) ode time =3.1994 other time =

$\epsilon = 0.01$ j = 11 $\sigma = 0.110$ $(\omega/\sigma)^2 = 7.4521637269881348836$ (30,20) ode time =3.2116 other time =0

$\epsilon = 0.01$ $j = \frac{27}{2}$ $\sigma = 0.135$ $(\omega/\sigma)^2 = 7.0148984974119558545$ (30,20) ode time =3.2009 other time =

$\epsilon = 0.01$ j = 16 $\sigma = 0.160$ $(\omega/\sigma)^2 = 6.634432658562039674$ (30,20) ode time =3.2008 other time =0.2

$\epsilon = 0.01$ j = 21 $\sigma = 0.210$ $(\omega/\sigma)^2 = 6.0335279543213022425$ (30,20) ode time =3.2021 other time =0

$\epsilon = 0.01$ j = 26 $\sigma = 0.260$ $(\omega/\sigma)^2 = 5.60129266793853753$ (30,20) ode time =3.2002 other time =0.22

$\epsilon = 0.01$ j = 31 $\sigma = 0.310$ $(\omega/\sigma)^2 = 5.2869324427377732675$ (30,20) ode time =3.1996 other time =0

$\epsilon = 0.01$ j = 36 $\sigma = 0.360$ $(\omega/\sigma)^2 = 5.0536364818784515874$ (30,20) ode time =3.2048 other time =0

$\epsilon = 0.01$ j = 41 $\sigma = 0.410$ $(\omega/\sigma)^2 = 4.8767263549800293421$ (30,20) ode time =3.2008 other time =0

$\epsilon = 0.01$ j = 46 $\sigma = 0.460$ $(\omega/\sigma)^2 = 4.7398231694763618516$ (30,20) ode time =3.1984 other time =0

$\epsilon = 0.01$ j = 51 $\sigma = 0.510$ $(\omega/\sigma)^2 = 4.6319265908440108073$ (30,20) ode time =3.2012 other time =0

$\epsilon = 0.01$ j = 56 $\sigma = 0.560$ $(\omega/\sigma)^2 = 4.5455044361387006678$ (30,20) ode time =3.2017 other time =0

$\epsilon = 0.01$ j = 61 $\sigma = 0.610$ $(\omega/\sigma)^2 = 4.4752885985560094966$ (30,20) ode time =3.2008 other time =0

$\epsilon = 0.01$ j = 71 $\sigma = 0.710$ $(\omega/\sigma)^2 = 4.3694500400463820245$ (30,20) ode time =3.2033 other time =0

$\epsilon = 0.01$ j = 81 $\sigma = 0.810$ $(\omega/\sigma)^2 = 4.2948079405334089646$ (30,20) ode time =3.5615 other time =0

$\epsilon = 0.01$ j = 91 $\sigma = 0.910$ $(\omega/\sigma)^2 = 4.2403263394586815659$ (40,20) ode time =9.3336 other time =0

$\epsilon = 0.01$ j = 101 $\sigma = 1.01$ $(\omega/\sigma)^2 = 4.1994225594625710714$ (40,20) ode time =9.3294 other time =0

$\epsilon = 0.01$ j = 111 $\sigma = 1.11$ $(\omega/\sigma)^2 = 4.167979915550613492$ (40,20) ode time =9.3277 other time =0.6

$\epsilon = 0.01$ j = 121 $\sigma = 1.21$ $(\omega/\sigma)^2 = 4.143320917670859077$ (40,20) ode time =10.360 other time =0.6

$\epsilon = 0.01$ j = 131 $\sigma = 1.31$ $(\omega/\sigma)^2 = 4.1236456759962382082$ (40,20) ode time =10.353 other time =0

$\epsilon = 0.01$ j = 141 $\sigma = 1.41$ $(\omega/\sigma)^2 = 4.1077096497099901994$ (40,20) ode time =10.365 other time =0

$\epsilon = 0.01$ j = 151 $\sigma = 1.51$ $(\omega/\sigma)^2 = 4.0946310478733501747$ (40,20) ode time =10.357 other time =0

$\epsilon = 0.01$ j = 161 $\sigma = 1.61$ $(\omega/\sigma)^2 = 4.0837716248010645245$ (40,20) ode time =10.359 other time =0

$\epsilon = 0.01$ j = 171 $\sigma = 1.71$ $(\omega/\sigma)^2 = 4.0746606294551170864$ (50,20) ode time =25.544 other time =1

$\epsilon = 0.01$ j = 191 $\sigma = 1.91$ $(\omega/\sigma)^2 = 4.0603558372035246335$ (50,20) ode time =25.599 other time =1

$\epsilon = 0.01$ j = 211 $\sigma = 2.11$ $(\omega/\sigma)^2 = 4.0497725139562661289$ (50,20) ode time =25.563 other time =1

$\epsilon = 0.01$ j = 231 $\sigma = 2.31$ $(\omega/\sigma)^2 = 4.0417303300317987103$ (50,20) ode time =25.579 other time =1

$\epsilon = 0.01$ j = 250 $\sigma = 2.50$ $(\omega/\sigma)^2 = 4.0357586472445873108$ (60,20) ode time =54.514 other time =3

$\epsilon = 0.01$ j = 290 $\sigma = 2.90$ $(\omega/\sigma)^2 = 4.026722737877334427$ (60,20) ode time =54.536 other time =3.9

In [57]:

```
ode_name='1j'
epsilonb = 0.001

ode_decimal_precision = 30
j_list = [3/2,2,5/2,3,7/2]+[m/2 for m in range(12,32,5)]+[m/2 for m in range(32,122,10)]
process_j_list()
```

$\epsilon = 0.001$ $j = \frac{3}{2}$ $\sigma = 0.00150$ $(\omega/\sigma)^2 = 9.5276530065894383767$ (30,20) ode time =2.8454 other time

$\epsilon = 0.001$ j = 2 $\sigma = 0.00200$ $(\omega/\sigma)^2 = 9.3328127565368508927$ (30,20) ode time =2.8473 other time

$\epsilon = 0.001$ $j = \frac{5}{2}$ $\sigma = 0.00250$ $(\omega/\sigma)^2 = 9.2423039854204874794$ (30,20) ode time =2.8417 other time

$\epsilon = 0.001 \quad j = 3 \quad \sigma = 0.00300 \quad (\omega/\sigma)^2 = 9.1927934511757878821 \quad (30,20) \quad \text{ode time} = 2.8401 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = \frac{7}{2} \quad \sigma = 0.00350 \quad (\omega/\sigma)^2 = 9.1625805849842362445 \quad (30,20) \quad \text{ode time} = 2.8437 \quad \text{other time}$

$\epsilon = 0.001 \quad j = 6 \quad \sigma = 0.00600 \quad (\omega/\sigma)^2 = 9.1033675328226328681 \quad (30,20) \quad \text{ode time} = 2.8393 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = \frac{17}{2} \quad \sigma = 0.00850 \quad (\omega/\sigma)^2 = 9.0812815323141515761 \quad (30,20) \quad \text{ode time} = 2.8400 \quad \text{other tim}$

$\epsilon = 0.001 \quad j = 11 \quad \sigma = 0.0110 \quad (\omega/\sigma)^2 = 9.0648013382893590722 \quad (30,20) \quad \text{ode time} = 2.8414 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = \frac{27}{2} \quad \sigma = 0.0135 \quad (\omega/\sigma)^2 = 9.0484130879589703181 \quad (30,20) \quad \text{ode time} = 2.8438 \quad \text{other time}$

$\epsilon = 0.001 \quad j = 16 \quad \sigma = 0.0160 \quad (\omega/\sigma)^2 = 9.0305736661629697545 \quad (30,20) \quad \text{ode time} = 2.8391 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 21 \quad \sigma = 0.0210 \quad (\omega/\sigma)^2 = 8.9887608911766755323 \quad (30,20) \quad \text{ode time} = 2.8449 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 26 \quad \sigma = 0.0260 \quad (\omega/\sigma)^2 = 8.9381583304940496491 \quad (30,20) \quad \text{ode time} = 2.8475 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 31 \quad \sigma = 0.0310 \quad (\omega/\sigma)^2 = 8.8790518944857559469 \quad (30,20) \quad \text{ode time} = 2.8438 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 36 \quad \sigma = 0.0360 \quad (\omega/\sigma)^2 = 8.8121375510479175313 \quad (30,20) \quad \text{ode time} = 2.8420 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 41 \quad \sigma = 0.0410 \quad (\omega/\sigma)^2 = 8.7382692060644393831 \quad (30,20) \quad \text{ode time} = 2.8434 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 46 \quad \sigma = 0.0460 \quad (\omega/\sigma)^2 = 8.6583643093475697946 \quad (30,20) \quad \text{ode time} = 2.8423 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 51 \quad \sigma = 0.0510 \quad (\omega/\sigma)^2 = 8.5733536534673498756 \quad (30,20) \quad \text{ode time} = 2.8476 \quad \text{other time} =$

$\epsilon = 0.001 \quad j = 56 \quad \sigma = 0.0560 \quad (\omega/\sigma)^2 = 8.4841488567930561233 \quad (30,20) \quad \text{ode time} = 3.1985 \quad \text{other time} =$

$j \to \infty$

In [58]:

```
ode_decimal_precision = 30
sigma_list = [0.001,0.01,0.02,0.04,.06,.08,.10,0.15,0.2,0.25,0.3,0.35,0.4,0.45,0.5,.6,0.7,0.8]
process_sigma_list()

ode_decimal_precision = 40
sigma_list = [0.9,1.00,1.15,1.30,1.45,1.6]
process_sigma_list()

ode_decimal_precision = 50
sigma_list = [1.85,2.10,2.35]
process_sigma_list()
```

```
ode_decimal_precision = 60
sigma_list = [2.65,3.0]
process_sigma_list()
```

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.00100$ $(\omega/\sigma)^2 = 9.0835330455416439274$ (30,20) ode time =2.8446 other time =(

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0100$ $(\omega/\sigma)^2 = 9.0612674877696019662$ (30,20) ode time =2.8430 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0200$ $(\omega/\sigma)^2 = 8.9953627256328547181$ (30,20) ode time =2.8435 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0400$ $(\omega/\sigma)^2 = 8.7529666561250607944$ (30,20) ode time =2.8420 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0600$ $(\omega/\sigma)^2 = 8.4101046362495019905$ (30,20) ode time =3.1951 other time =0.

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.0800$ $(\omega/\sigma)^2 = 8.023039607122564902$ (30,20) ode time =3.1970 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.100$ $(\omega/\sigma)^2 = 7.6333997588370682807$ (30,20) ode time =3.1977 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.150$ $(\omega/\sigma)^2 = 6.7765482096795657426$ (30,20) ode time =3.1939 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.200$ $(\omega/\sigma)^2 = 6.1366900976602906555$ (30,20) ode time =3.1971 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.250$ $(\omega/\sigma)^2 = 5.6758107936233344452$ (30,20) ode time =3.1979 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.300$ $(\omega/\sigma)^2 = 5.3416107840948261744$ (30,20) ode time =3.2002 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.350$ $(\omega/\sigma)^2 = 5.0946036902475815238$ (30,20) ode time =3.2015 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.400$ $(\omega/\sigma)^2 = 4.9080698932375542945$ (30,20) ode time =3.1974 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.450$ $(\omega/\sigma)^2 = 4.7642709897178321575$ (30,20) ode time =3.1954 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.500$ $(\omega/\sigma)^2 = 4.6513274620791411213$ (30,20) ode time =3.1941 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.600$ $(\omega/\sigma)^2 = 4.4880550633673341101$ (30,20) ode time =3.1988 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.700$ $(\omega/\sigma)^2 = 4.3782629343472021188$ (30,20) ode time =3.3357 other time =0.3

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.800$ $(\omega/\sigma)^2 = 4.3011278478124393871$ (30,20) ode time =3.7687 other time =0.2

$\epsilon = 0.$ $j = +\infty$ $\sigma = 0.900$ $(\omega/\sigma)^2 = 4.2450010384491445038$ (40,20) ode time =9.3235 other time =0.6

$\epsilon = 0.$ j = +∞ $\sigma = 1.00$ $(\omega/\sigma)^2 = 4.2029703221284508332$ (40,20) ode time =9.3210 other time =0.67

$\epsilon = 0.$ j = +∞ $\sigma = 1.15$ $(\omega/\sigma)^2 = 4.1573713863219759644$ (40,20) ode time =10.358 other time =0.67

$\epsilon = 0.$ j = +∞ $\sigma = 1.30$ $(\omega/\sigma)^2 = 4.12539105593133652$ (40,20) ode time =10.365 other time =0.6724

$\epsilon = 0.$ j = +∞ $\sigma = 1.45$ $(\omega/\sigma)^2 = 4.1021461737375348904$ (40,20) ode time =10.355 other time =0.67

$\epsilon = 0.$ j = +∞ $\sigma = 1.60$ $(\omega/\sigma)^2 = 4.0847481269085376936$ (40,20) ode time =10.358 other time =0.67

$\epsilon = 0.$ j = +∞ $\sigma = 1.85$ $(\omega/\sigma)^2 = 4.0641690184776466222$ (50,20) ode time =25.565 other time =1.74

$\epsilon = 0.$ j = +∞ $\sigma = 2.10$ $(\omega/\sigma)^2 = 4.0502197825876173506$ (50,20) ode time =25.575 other time =1.74

$\epsilon = 0.$ j = +∞ $\sigma = 2.35$ $(\omega/\sigma)^2 = 4.0403445880660037356$ (50,20) ode time =25.599 other time =1.73

$\epsilon = 0.$ j = +∞ $\sigma = 2.65$ $(\omega/\sigma)^2 = 4.0318918859941641937$ (60,20) ode time =54.594 other time =3.93

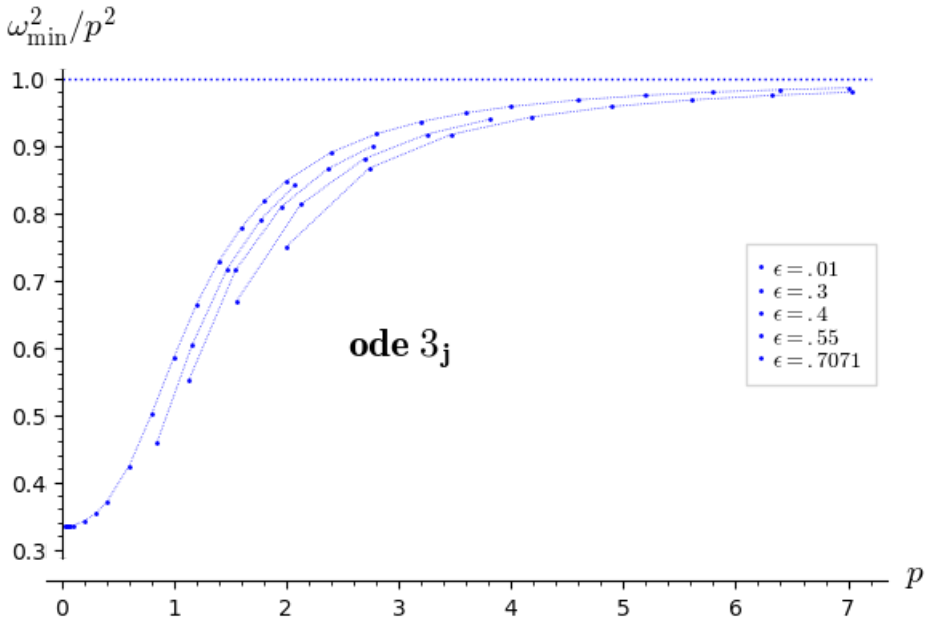
$\epsilon = 0.$ j = +∞ $\sigma = 3.00$ $(\omega/\sigma)^2 = 4.0249908291237605178$ (60,20) ode time =54.584 other time =3.94

Graphics

In [59]:
```
ode_name='3j'
plot_data_j ={}
for eps_string,eps_data in data[ode_name]['finite'].items():
    if eps_string not in plot_data_j:
        plot_data_j[eps_string]=[]
    for jval,item in eps_data.items():
        sigmaval = item['sigmaval']
        pval = sqrt(6)*sigmaval
        p = RDF(pval)
        fs_ratio = RDF(item['fs_ratio']*sigmaval^2/pval^2)
        plot_data_j[eps_string].append((p,fs_ratio))
epsvals= list(plot_data_j.keys())
epsvals.sort()
print(epsvals)
```

```
['0.01', '0.3', '0.4', '0.55', '0.7071']
```

In [60]:
```
p=\
list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='.',markersize=2,plotjoined=True
+list_plot(plot_data_j['0.3'],legend_label=r'$\epsilon=.3$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.4'],legend_label=r'$\epsilon=.4$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.55'],legend_label=r'$\epsilon=.55$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.7071'],legend_label=r'$\epsilon=.7071$',marker='.',markersize=2,plotjoined
+text(r"${\bf ode~3_j}$",(3,.6),fontsize=18,color='black')\
+line([(0,1),(7.2,1)],linestyle=":")
p.set_legend_options(loc='center right',markerscale=1,numpoints=1,shadow=False)
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
p.axes_labels_size(1.5)
p.show(ticks_integer=True,ymin=0.3)
p.save('3j_finite.pdf',dpi=600,ticks_integer=True,ymin=0.3)
```

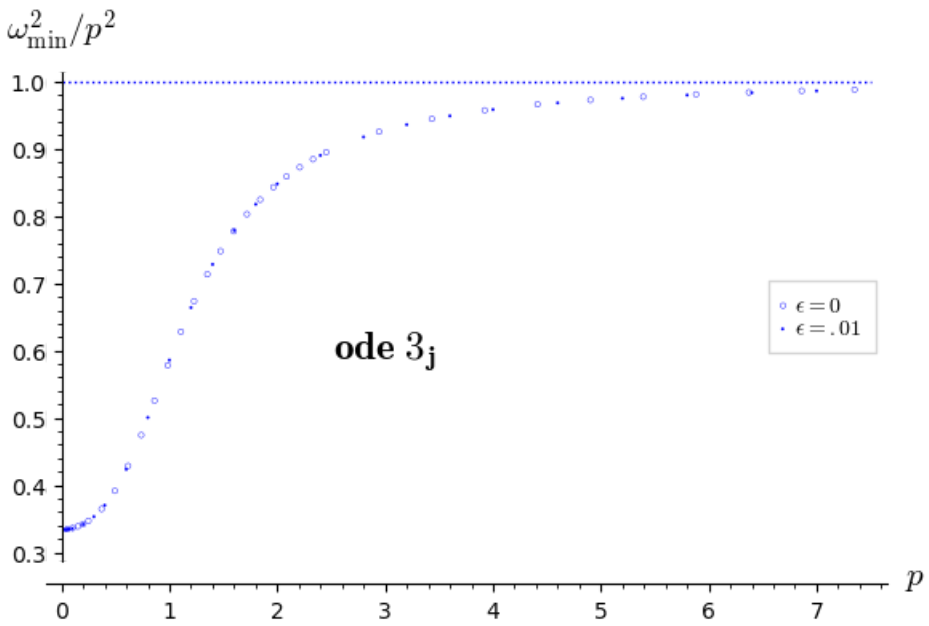


In [61]:

```
ode_name='3j'
plot_data_asymp =[]
for key,item in data[ode_name]['asymp'].items():
    sigmaval = item['sigmaval']
    pval = sqrt(6)*sigmaval
    p = RDF(pval)
    fs_ratio = RDF(item['fs_ratio']*sigmaval^2/pval^2)
    plot_data_asymp.append((p,fs_ratio))
```

In [62]:

```
p=\
list_plot(plot_data_asymp,legend_label=r'$\epsilon=0$',marker=r'$\circ$',size=10)\
+list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='.',size=8)\
+text(r"${\bf ode~3_j}$",(3,.6),fontsize=18,color='black')\
+line([(0,1),(7.5,1)],linestyle=":")
p.set_legend_options(loc='center right',markerscale=1,numpoints=1,shadow=False)
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
p.axes_labels_size(1.5)
p.show(ticks_integer=True,ymin=0.3)
p.save('3j_asymp.pdf',dpi=600,ticks_integer=True,ymin=0.3)
```



In [63]:

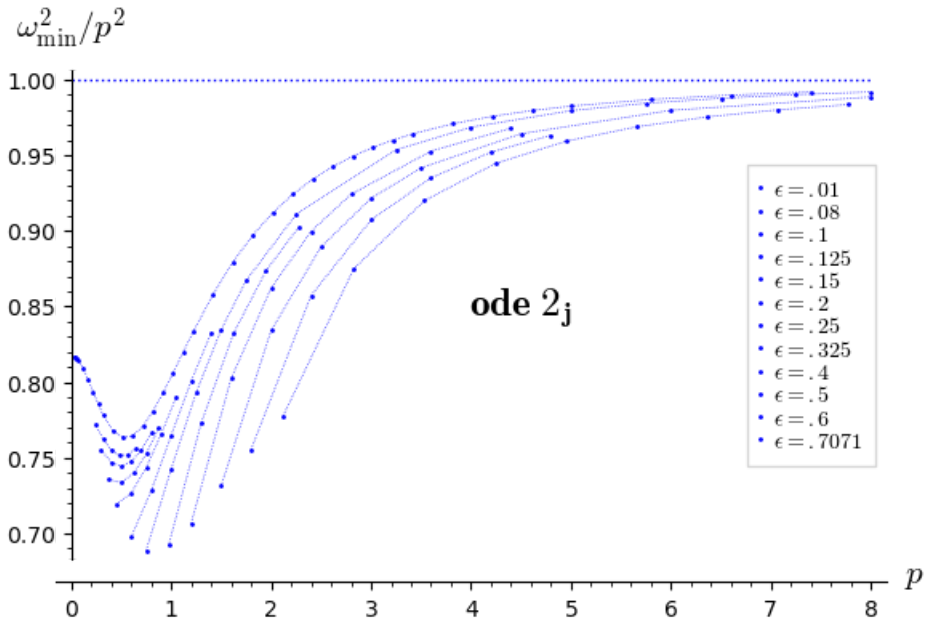
```
ode_name='2j'
plot_data_j ={}
for eps_string,eps_data in data[ode_name]['finite'].items():
    if eps_string not in plot_data_j:
        plot_data_j[eps_string]=[]
    for jval,item in eps_data.items():
```

```
        sigmaval = item['sigmaval']
        pval = 2*sigmaval
        p = RDF(pval)
        fs_ratio = RDF(item['fs_ratio']*sigmaval^2/pval^2)
        plot_data_j[eps_string].append((p,fs_ratio))
epsvals= list(plot_data_j.keys())
epsvals.sort()
print(epsvals)
```

```
['0.01', '0.08', '0.1', '0.125', '0.15', '0.2', '0.25', '0.325', '0.4', '0.5', '0.6', '0.7071']
```

In [64]:

```
p=\
list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='.',markersize=2,plotjoined=True
+list_plot(plot_data_j['0.08'],legend_label=r'$\epsilon=.08$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.1'],legend_label=r'$\epsilon=.1$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.125'],legend_label=r'$\epsilon=.125$',marker='.',markersize=2,plotjoined=T
+list_plot(plot_data_j['0.15'],legend_label=r'$\epsilon=.15$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.2'],legend_label=r'$\epsilon=.2$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.25'],legend_label=r'$\epsilon=.25$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.325'],legend_label=r'$\epsilon=.325$',marker='.',markersize=2,plotjoined=T
+list_plot(plot_data_j['0.4'],legend_label=r'$\epsilon=.4$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.5'],legend_label=r'$\epsilon=.5$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.6'],legend_label=r'$\epsilon=.6$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.7071'],legend_label=r'$\epsilon=.7071$',marker='.',markersize=2,plotjoined
+text(r"${\bf ode~2_j}$",(4.5,.85),fontsize=18,color='black')\
+line([(0,1),(8,1)],linestyle=":")
p.set_legend_options(loc='center right',markerscale=1,numpoints=1,shadow=False)
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
p.axes_labels_size(1.5)
p.show(ticks_integer=True,xmin=0)
p.save('2j_finite.pdf',dpi=600,ticks_integer=True,xmin=0)
```

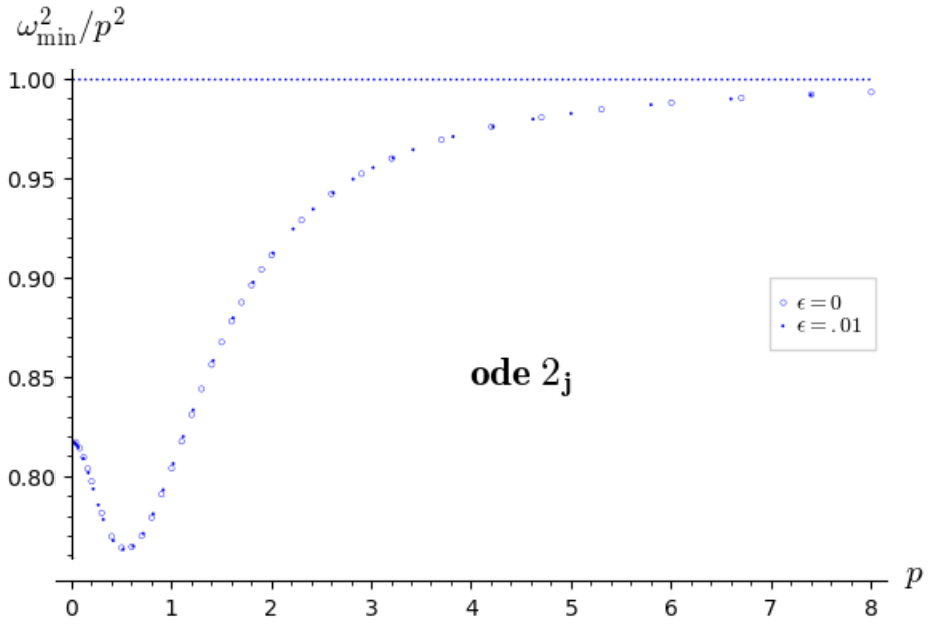


In [65]:

```
ode_name='2j'
plot_data_asymp =[]
for key,item in data[ode_name]['asymp'].items():
    sigmaval = item['sigmaval']
    pval = 2*sigmaval
    p = RDF(pval)
    fs_ratio = RDF(item['fs_ratio']*sigmaval^2/pval^2)
    plot_data_asymp.append((p,fs_ratio))
```

In [66]:

```
p=list_plot(plot_data_asymp,legend_label=r'$\epsilon=0$',marker=r'$\circ$',size=10)\
+list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='.',size=8)\
+text(r"${\bf ode~2_j}$",(4.5,.85),fontsize=18,color='black')\
+line([(0,1),(8.0,1)],linestyle=":")
p.set_legend_options(loc='center right',markerscale=1,numpoints=1,shadow=False)
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
p.axes_labels_size(1.5)
p.show(ticks_integer=True)
p.save('2j_asymp.pdf',dpi=600,ticks_integer=True)
```



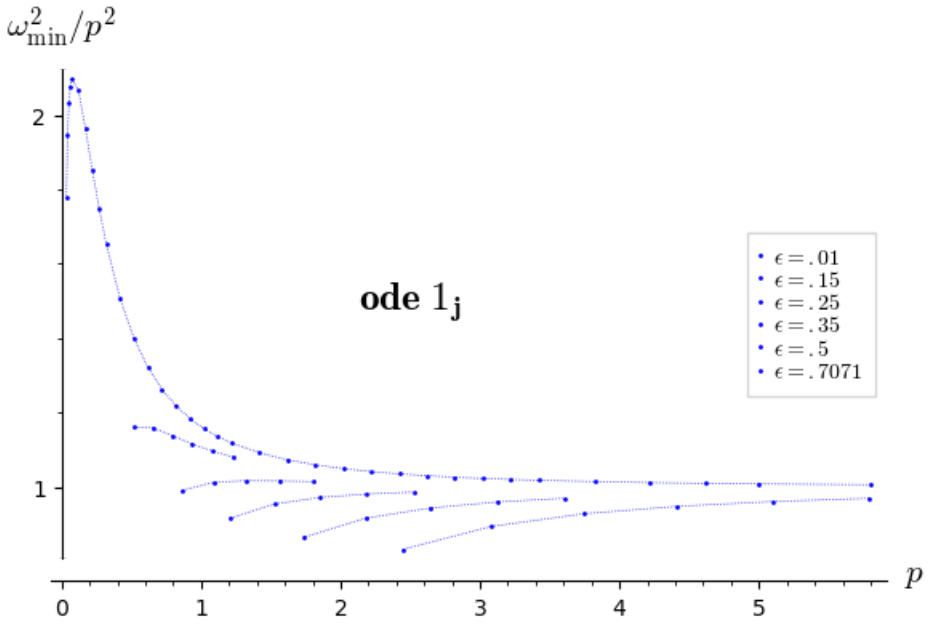
In [67]:

```
ode_name='1j'
plot_data_j ={}
for eps_string,eps_data in data[ode_name]['finite'].items():
    if eps_string not in plot_data_j:
        plot_data_j[eps_string]=[]
    for jval,item in eps_data.items():
        sigmaval = item['sigmaval']
        epsval = item['epsilonb']
        pval = sqrt(4*sigmaval^2+3*epsval^2)
        p = RDF(pval)
        fs_ratio = RDF(item['fs_ratio']*sigmaval^2/pval^2)
        plot_data_j[eps_string].append((p,fs_ratio))
epsvals= list(plot_data_j.keys())
epsvals.sort()
print(epsvals)
```

```
['0.001', '0.01', '0.15', '0.25', '0.35', '0.5', '0.7071']
```

In [68]:

```
p=\
list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='.',markersize=2,plotjoined=True
+list_plot(plot_data_j['0.15'],legend_label=r'$\epsilon=.15$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.25'],legend_label=r'$\epsilon=.25$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.35'],legend_label=r'$\epsilon=.35$',marker='.',markersize=2,plotjoined=Tru
+list_plot(plot_data_j['0.5'],legend_label=r'$\epsilon=.5$',marker='.',markersize=2,plotjoined=True,
+list_plot(plot_data_j['0.7071'],legend_label=r'$\epsilon=.7071$',marker='.',markersize=2,plotjoined
+text(r"${\bf ode~1_j}$",(2.5,1.5),fontsize=18,color='black')
p.set_legend_options(loc='center right',markerscale=1,numpoints=1,shadow=False)
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
p.axes_labels_size(1.5)
p.show(ticks_integer=True)
p.save('1j_finite.pdf',dpi=600,ticks_integer=True)
```

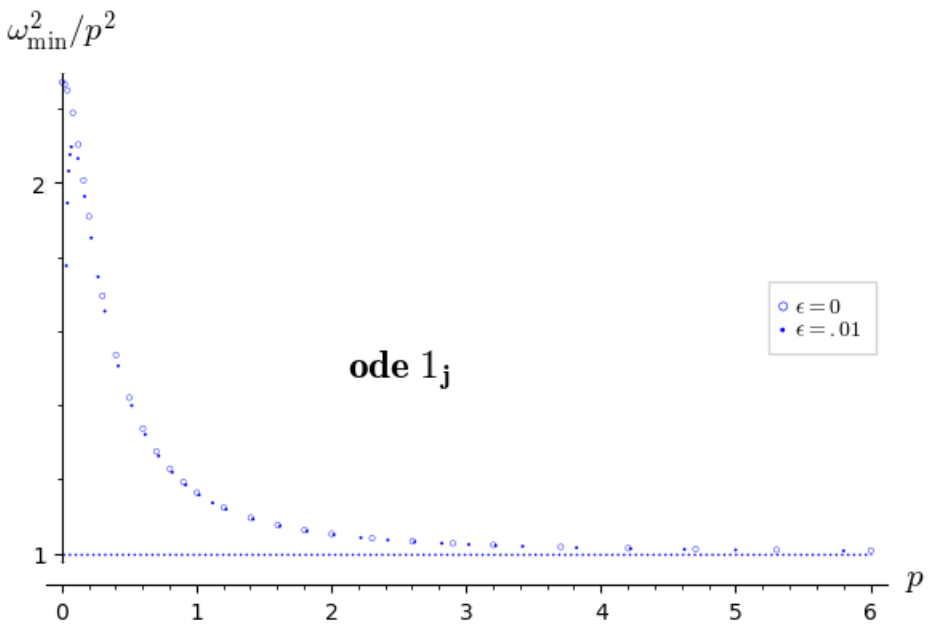


In [69]:

```
ode_name='1j'
plot_data_asymp =[]
for key,item in data[ode_name]['asymp'].items():
    sigmaval = item['sigmaval']
    pval = 2*sigmaval
#    pval = sqrt(4*sigmaval^2+3*epsval^2)
    p = RDF(pval)
    fs_ratio = RDF(item['fs_ratio']*sigmaval^2/pval^2)
    plot_data_asymp.append((p,fs_ratio))
```

In [70]:

```
p=list_plot(plot_data_asymp,legend_label=r'$\epsilon=0$',marker=r'$\circ$',size=10)\
+list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='.',size=8)\
+text(r"${\bf ode~1_j}$",(2.5,1.5),fontsize=18,color='black')\
+line([(0,1),(6.0,1)],linestyle=":")
#p.set_legend_options(loc='center right',markerscale=1,numpoints=1,shadow=False)
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
p.set_legend_options(loc='center right',markerscale=1.5,shadow=False)
p.axes_labels_size(1.5)
p.show(ticks_integer=True,ymin=1)
p.save('1j_asymp.pdf',dpi=600,ticks_integer=True,ymin=1)
```



In [78]:

```
p=list_plot(plot_data_asymp,legend_label=r'$\epsilon=0$',marker=r'$\circ$',size=24)\
+list_plot(plot_data_j['0.001'],legend_label=r'$\epsilon=.001$',marker='.',markersize=3,plotjoined=T
+text(r"${\bf ode~1_j}$",(0.1,1.8),fontsize=18,color='black')\
+list_plot(plot_data_j['0.01'],legend_label=r'$\epsilon=.01$',marker='x',markersize=3,plotjoined=Tru
p.axes_labels(['$p$','$\\omega_{\\min}^2/p^2$'])
```

```
p.set_legend_options(shadow=False)
#p.set_legend_options(loc='center right',markerscale=1.5,shadow=False)
p.axes_labels_size(1.5)
p.show(xmax=.2,ymin=1.7)
p.save('1j_smallp.pdf',dpi=600,xmax=.2,ymin=1.7)
```