Calculations for *The CGF dark matter fluid*

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Abstract

Details of calculations for *The CGF dark matter fluid*.

Contents

1 Introduction

Numbers from PDG, NIST ($c = 1$, no error bars)

$$\begin{aligned}
\ell_0 &= \frac{\hbar}{m_{\text{Higgs}}} && = 5.2614864632102925 \times 10^{-27}\,\text{s} \\
\rho_0 &= \frac{m_{\text{Higgs}}}{\ell_0^3} && = 5.682491786274899 \times 10^{28}\,\text{kg/m}^3 \\
\ell &= \frac{1}{\sqrt{G\rho_0}} = \frac{\hbar m_{\text{Planck}}}{m_{\text{Higgs}}^2} && = 15.39389190962883\,\text{cm} \\
M_\ell &= \frac{\ell}{G} = \ell^3 \rho_0 && = 1.042510314426683 \times 10^{-4}\,\text{M}_\odot \\
\omega_0 &= \frac{\ell}{\ell_0} = \frac{m_{\text{Planck}}}{m_{\text{Higgs}}} && = 0.9759312573158477 \times 10^{17} \\
g^2 &= \frac{4\pi\alpha}{\sin^2\theta_W} && = 0.4111966909854108 \\
\lambda^2 &= \frac{g^2}{4}\frac{m_{\text{Higgs}}^2}{m_W^2} && = 0.2490112919526795
\end{aligned} \tag{1.1}$$

$$[G] = \frac{\text{Jm}}{\text{kg}^2} \qquad [\rho] = \frac{\text{kg}}{\text{m}^3} \qquad \left[(G\rho)^{-\frac{1}{2}} c\right] = \text{m} \qquad \left[(G\rho)^{-\frac{1}{2}} G^{-1} c^3\right] = \text{kg} \tag{1.2}$$

$$\begin{aligned}
G &= 6.67430 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \\
c &= 2.99792458 \times 10^8\,\text{m/s} \\
\hbar &= 1.054571817 \times 10^{-34}\text{Js} \\
1\,\text{ev} &= 1.602176634 \times 10^{-19}\,\text{J} \\
\text{M}_\odot &= 1.988409870698051 \times 10^{30}\,\text{kg} \\
H_0 &= 67.4\,(\text{km/s})/\text{Mpc} = 2.184 \times 10^{-18}\text{s}^{-1} \\
\rho_c = \tfrac{3H_0^2}{8\pi G} &= 8.53 \times 10^{-27}\text{kg/m}^3 \\
\tfrac{\hbar}{m_{\text{Planck}}} = \sqrt{G\hbar} &= 5.391247 \times 10^{-44}\,\text{s}
\end{aligned} \tag{1.3}$$

$$\alpha = \frac{e^2}{4\pi} = 1/137.035999139(31)$$

$$m_W = 80.379(12)\,\text{GeV} \qquad m_Z = 91.1876(21)\,\text{GeV} \qquad m_{\text{Higgs}} = 125.10(14)\,\text{GeV} \tag{1.4}$$

$$\frac{m_W}{m_Z} = \cos\theta_W = 0.88147(13) \qquad \sin^2\theta_W = 0.2230106391$$

2 Standard Model action

See numbers above.

3 SU(2) structure on space-time spinors

3.1 Equivalence to a 4-velocity field $u^\alpha(x)$

The Dirac algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \qquad \gamma_5 = \frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \qquad \gamma_5^2 = 1 \tag{3.1}$$

has a standard representation on $\mathbb{C}^4$.

$$\begin{gathered}\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\end{gathered} \tag{3.2}$$

where σ_i are the 2×2 Pauli matrices,

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ij}{}^k\sigma_k \qquad \sigma_i^\dagger = \sigma_i \tag{3.3}$$

The chiral spinors are the eigenspace $\gamma_5 = 1$. The generators of Spin(1,3) are

$$L_{\mu\nu} = \frac{1}{4}[\gamma_\mu,\, \gamma_\nu] \qquad L_{0k} = \begin{pmatrix} \frac{1}{2}\sigma_k & 0 \\ 0 & -\frac{1}{2}\sigma_k \end{pmatrix} \qquad L_{jk} = \begin{pmatrix} \frac{1}{4}[\sigma_j,\, \sigma_k] & 0 \\ 0 & -\frac{1}{4}[\sigma_j,\, \sigma_k] \end{pmatrix} \tag{3.4}$$

Adjoints with respect to the standard hermitian form on $\mathbb{C}^4$ are

$$\gamma_0^\dagger = -\gamma_0 \qquad \gamma_i^\dagger = \gamma_i \qquad L_{0i}^\dagger = L_{0i} \qquad L_{ij}^\dagger = -L_{ij} \tag{3.5}$$

The L_{ij} are generators of SU(2), as are the iL_{0k}. Associate this SU(2) structure with the 4-velocity $u_0^\mu = (1,0,0,0)$. The Spin(1,3) action then associates to every 4-velocity u^μ an SU(2) structure on the spinors such that

$$g_{\mu\nu}u^\mu u^\nu = -1 \qquad u^0 > 0 \qquad P^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta \tag{3.6}$$

$$\begin{gathered} u^\alpha\gamma_\alpha^\dagger = -u^\alpha\gamma_\alpha \qquad P^\alpha_\beta\gamma_\alpha^\dagger = P^\alpha_\beta\gamma_\alpha \\ \gamma_\mu^\dagger = \gamma_\mu + 2u_\mu u^\alpha\gamma_\alpha = -\gamma_\mu + 2P^\alpha_\mu\gamma_\alpha \\ u^\alpha L_{\alpha\mu}^\dagger = u^\alpha L_{\alpha\mu} \qquad P^\mu_\alpha P^\nu_\beta L_{\mu\nu}^\dagger = -P^\mu_\alpha P^\nu_\beta L_{\mu\nu} \end{gathered} \tag{3.7}$$

This identifies the space of SU(2) structures with the space of 4-velocities,

$$\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) = \mathrm{SO}(1,3)/\mathrm{SO}(3) = \{u^\alpha\colon u^\alpha u_\alpha = -1,\ u^0 > 0\} \tag{3.8}$$

because the little group SU(2) preserves the SU(2) structure associated to u_0^μ.

3.2 A natural SU(2) spin connection

The local Dirac matrices $\gamma_\mu(x)$ act on the spinors at x, the fiber S_x. The local Lorentz generators are

$$L_{\mu\nu}(x) = \frac{1}{4}[\gamma_\mu,\, \gamma_\nu] \qquad [L_{\mu\nu},\, \gamma_\alpha] = \gamma_\mu g_{\nu\alpha} - \gamma_\nu g_{\mu\alpha} \tag{3.9}$$

The boost operators relative to $u^\mu(x)$ are

$$L_\mu = u^\alpha L_{\alpha\mu} \qquad u^\alpha L_\alpha = 0 \qquad L_\mu^\dagger = L_\mu \tag{3.10}$$

In a spin-rest frame, $u^\mu = (1,0,0,0)$, $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and the $\gamma_\mu(x)$ take the form (3.5) giving

$$L_0 = 0 \qquad L_i = \begin{pmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & -\frac{1}{2}\sigma_i \end{pmatrix} \tag{3.11}$$

The local SU(2) generators are the operators iL_ν or, equivalently, the operators $[L_\mu,\, L_\nu]$. The L_μ satisfy identities that can be derived in a spin-rest frame.

$$\begin{aligned} [L_\nu,\, \gamma_\sigma] &= u^\mu \gamma_\mu g_{\nu\sigma} - u_\sigma \gamma_\nu \\ L_\mu L_\nu &= \frac{1}{4} P_{\mu\nu} + \frac{1}{2} u^\alpha \epsilon_{\alpha\mu\nu}{}^\sigma \gamma_5 i L_\sigma \\ [L_\mu,\, L_\nu] &= u^\rho \epsilon_{\rho\mu\nu}{}^\sigma \gamma_5 i L_\sigma \\ L_{\mu\nu} &= L_\mu u_\nu - L_\nu u_\mu + [L_\mu,\, L_\nu] \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mathrm{tr}(L_\mu) &= 0 \\ \mathrm{tr}(L_\mu L_\nu) &= \frac{1}{2} P_{\mu\nu} \\ \mathrm{tr}\left([L_\mu,\, L_\nu] L_\rho\right) &= \frac{i}{2} u^\alpha \epsilon_{\alpha\mu\nu\rho} \\ \mathrm{tr}\left(-[L_\mu,\, L_\nu][L_{\mu'},\, L_{\nu'}]\right) &= \frac{1}{2}\left(P_{\mu\mu'} P_{\nu\nu'} - P_{\mu\nu'} P_{\nu\mu'}\right) \end{aligned} \tag{3.13}$$

The traces are over the two-dimensional vector space of chiral spinors $(S_+)_x$.

From the above,

$$\begin{aligned} [L^\nu,\, [L_\mu,\, L_\nu]] &= L^\nu L_\mu L_\nu - L^\nu L_\nu L_\mu - L_\mu L_\nu L^\nu + L_\nu L_\mu L^\nu \\ &= 2L^\nu (L_\mu L_\nu + L_\nu L_\mu) - 4L^\nu L_\nu L_\mu \\ &= L^\nu P_{\mu\nu} - 3L_\mu \\ &= -2L_\mu \end{aligned} \tag{3.14}$$

Parallel transport of orthonormal frames lifts to parallel transport of spin frames, since SL(1,3) and Spin(1,3) have the same infinitesimal generators. Therefore the metric covariant derivative ∇_μ acts on spinors such that

$$\nabla_\mu \gamma_\nu = 0 \tag{3.15}$$

Its curvature form

$$F^{\mathrm{metric}}_{\mu\nu} = [\nabla_\mu,\, \nabla_\nu] \tag{3.16}$$

satisfies

$$\begin{gathered} \nabla_\nu (v^\beta \gamma_\beta) = (\nabla_\nu v^\beta)\gamma_\beta \qquad \nabla_\mu \nabla_\nu (v^\beta \gamma_\beta) = (\nabla_\mu \nabla_\nu v^\beta)\gamma_\beta \\ [F^{\mathrm{metric}}_{\mu\nu},\, v^\beta \gamma_\beta] = \nabla_{[\mu} \nabla_{\nu]} (v^\beta \gamma_\beta) = R^\beta{}_{\alpha\mu\nu} v^\alpha \gamma_\beta = R^\alpha{}_{\beta\mu\nu} v^\beta \gamma_\alpha \\ [F^{\mathrm{metric}}_{\mu\nu},\, \gamma_\beta] = R^\alpha{}_{\beta\mu\nu} \gamma_\alpha \end{gathered} \tag{3.17}$$

which has a unique solution since only the identity matrix commutes with all the γ_β and ∇_μ is an SO(1,3) connection on the spinor bundle.

$$F^{\mathrm{metric}}_{\mu\nu} = \frac{1}{2} R^{\alpha\beta}{}_{\mu\nu} L_{\alpha\beta} \tag{3.18}$$

The symplectic structure on S_x depends only on the space-time metric, but the positive hermitian form on S_x depends on both the metric and on the 4-velocity. The metric connection does not preserve the hermitian structure.

$$\nabla_\mu \gamma^\dagger_\nu = \nabla_\mu(\gamma_\nu + 2u_\nu u^\alpha \gamma_\alpha) = 2\nabla_\mu(u_\nu u^\alpha)\gamma_\alpha \neq 0 \tag{3.19}$$

The modified covariant derivative

$$D^0_\mu = \nabla_\mu - \nabla_\mu u^\sigma L_\sigma \tag{3.20}$$

It is a natural SO(1,3) connection.

$$\begin{aligned}
\nabla_\mu L_\nu &= \nabla_\mu u^\alpha L_{\alpha\nu} \\
D^0_\mu L_\nu &= \nabla_\mu L_\nu - \nabla_\mu u^\sigma [L_\sigma,\, L_\nu] \\
&= \nabla_\mu u^\alpha L_{\alpha\beta} - \nabla_\mu u^\sigma (L_{\sigma\nu} - L_\sigma u_\nu + L_\nu u_\sigma) \\
&= \nabla_\mu u^\sigma u_\nu L_\sigma
\end{aligned} \tag{3.21}$$

so D^0_μ preserves the hermitian structure,

$$D^0_\mu \left(L^\dagger_\nu\right) = (D^0_\mu L_\nu)^\dagger \tag{3.22}$$

so D^0_μ is an SU(2) connection. Its curvature form is

$$\begin{aligned}
F^0_{\mu\nu} &= [D^0_\mu,\, D^0_\nu] = [\nabla_\mu - \nabla_\mu u^\alpha L_\alpha,\, \nabla_\nu - \nabla_\nu u^\beta L_\beta] \\
&= F^{\text{metric}}_{\mu\nu} - (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)u^\beta L_\beta - \nabla_\nu u^\beta \nabla_\mu L_\beta + \nabla_\mu u^\beta \nabla_\nu L_\beta \\
&\qquad + \nabla_\mu u^\alpha \nabla_\nu u^\beta [L_\alpha,\, L_\beta] \\
&= \frac{1}{2} R^{\alpha\beta}{}_{\mu\nu} L_{\alpha\beta} - R^\beta{}_{\alpha\mu\nu} u^\alpha L_\beta - \nabla_\nu u^\beta \nabla_\mu u^\alpha L_{\alpha\beta} + \nabla_\mu u^\beta \nabla_\nu u^\alpha L_{\alpha\beta} \\
&\qquad + \nabla_\mu u^\alpha \nabla_\nu u^\beta [L_\alpha,\, L_\beta] \\
&= (\frac{1}{2} R^{\alpha\beta}{}_{\mu\nu} - \nabla_\nu u^\beta \nabla_\mu u^\alpha + \nabla_\mu u^\beta \nabla_\nu u^\alpha)([L_\alpha,\, L_\beta] + L_\alpha u_\beta - L_\beta u_\alpha) \\
&\qquad - R^\beta{}_{\alpha\mu\nu} u^\alpha L_\beta + \nabla_\mu u^\alpha \nabla_\nu u^\beta [L_\alpha,\, L_\beta] \\
&= \left(\frac{1}{2} R^{\alpha\beta}{}_{\mu\nu} - \nabla_\mu u^\alpha \nabla_\nu u^\beta\right) [L_\alpha,\, L_\beta]
\end{aligned} \tag{3.23}$$

4 Form of the CGF

5 Equations of motion

Scalar field equations of motion The scalar field action written in terms of the dimensionless scalar field $\hat{\phi}$,

$$\phi = \frac{v}{\sqrt{2}} \hat{\phi} \tag{5.1}$$

is

$$\begin{aligned}
\frac{1}{\hbar}S_{\text{scalar}} &= \int \left[\ell^{-2}D_\mu\phi^\dagger D^\mu\phi + \frac{1}{2}\lambda^2\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)^2\right]\ell^4\sqrt{-g}\,d^4x \\
&= \int \frac{1}{\lambda^2}\left[\ell^{-2}\frac{\lambda^2 v^2}{2}D_\mu\hat\phi^\dagger D^\mu\hat\phi + \frac{\lambda^4 v^4}{8}\left(\hat\phi^\dagger\hat\phi - 1\right)^2\right]\ell^4\sqrt{-g}\,d^4x \\
&= \int \frac{1}{\lambda^2}\left[\frac{\ell^{-2}\ell_0^{-2}}{2}D_\mu\hat\phi^\dagger D^\mu\hat\phi + \frac{\ell_0^{-4}}{8}\left(\hat\phi^\dagger\hat\phi - 1\right)^2\right]\ell^4\sqrt{-g}\,d^4x \\
&= \int \frac{1}{\lambda^2}\left[\frac{1}{2\omega_0^2}D_\mu\hat\phi^\dagger D^\mu\hat\phi + \frac{1}{8}\left(\hat\phi^\dagger\hat\phi - 1\right)^2\right]\omega_0^4\sqrt{-g}\,d^4x
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
D_\mu\hat\phi^\dagger D^\mu\hat\phi &= (D^0_\mu + B_\mu)\hat\phi^\dagger(D^{0\mu} + B^\mu)\hat\phi \\
&= D^0_\mu\hat\phi^\dagger D^{0\mu}\hat\phi - \hat\phi^\dagger B_\mu D^{0\mu}\hat\phi + D^0_\mu\hat\phi^\dagger B^\mu\hat\phi - \hat\phi^\dagger B_\mu B^\mu\hat\phi
\end{aligned} \tag{5.3}$$

but $B_\mu B^\mu$ is a multiple of the identity,

$$B_\mu B^\mu = g^{\mu\nu}B^\sigma_\mu B^\tau_\nu L_\sigma L_\tau = g^{\mu\nu}B^\sigma_\mu B^\tau_\nu \frac{1}{4}P_{\sigma\tau} \tag{5.4}$$

so

$$B_\mu B^\mu = \frac{1}{2}\text{tr}(B_\mu B^\mu) \tag{5.5}$$

$$D_\mu\hat\phi^\dagger D^\mu\hat\phi = D^0_\mu\hat\phi^\dagger D^{0\mu}\hat\phi + \text{tr}\left[B^\mu(\hat\phi D^0_\mu\hat\phi^\dagger - \hat\phi^\dagger D^0_\mu\hat\phi)\right] - \frac{1}{2}\hat\phi^\dagger\hat\phi\,\text{tr}\,(B_\mu B^\mu) \tag{5.6}$$

Assume $\hat\phi(x)$ is smooth and $B_\mu = O(\omega_0)$. Then to leading order in ω_0

$$\frac{1}{2\omega_0^2}D_\mu\hat\phi^\dagger D^\mu\hat\phi = -\frac{1}{4}\hat\phi^\dagger\hat\phi\,\text{tr}\left(\frac{B_\mu B^\mu}{\omega_0^2}\right) \tag{5.7}$$

$$\frac{1}{\hbar}S_{\text{scalar}} = \int \frac{1}{\lambda^2}\left[\frac{1}{4}\hat\phi^\dagger\hat\phi\,\text{tr}\left(-\frac{B_\mu B^\mu}{\omega_0^2}\right) + \frac{1}{8}\left(\hat\phi^\dagger\hat\phi - 1\right)^2\right]\omega_0^4\sqrt{-g}\,d^4x \tag{5.8}$$

For the CGF,

$$\begin{aligned}
B_\mu &= \omega_0 b_0 i L_\mu \\
\text{tr}\left(-\frac{B_\mu B^\mu}{\omega_0^2}\right) &= \text{tr}\left(L^\mu L_\mu\right)b_0^2 = \frac{3}{2}b_0^2
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\frac{1}{\hbar}S_{\text{scalar}} &= \int \frac{1}{\lambda^2}\left[\frac{1}{4}\hat\phi^\dagger\hat\phi\,\frac{3}{2}b_0^2 + \frac{1}{8}\left(\hat\phi^\dagger\hat\phi - 1\right)^2\right]\omega_0^4\sqrt{-g}\,d^4x \\
&= \int \frac{1}{8\lambda^2}\left[(\hat\phi^\dagger\hat\phi)^2 + 3\left(b_0^2 - \frac{2}{3}\right)\hat\phi^\dagger\hat\phi + 1\right]\omega_0^4\sqrt{-g}\,d^4x
\end{aligned} \tag{5.10}$$

with equation of motion

$$\begin{gathered}
\left[2\hat\phi^\dagger\hat\phi + 3\left(b_0^2 - \frac{2}{3}\right)\right]\phi = 0 \\
\begin{array}{ll}
\phi = 0 & \langle\, b_0^2\,\rangle \geq \frac{2}{3} \\
\hat\phi^\dagger\hat\phi = 1 - \frac{3}{2}\langle\, b_0^2\,\rangle & \langle\, b_0^2\,\rangle \leq \frac{2}{3}
\end{array}
\end{gathered} \tag{5.11}$$

Gauge field equations of motion Vary the gauge action and the leading order scalar action wrt the gauge field.

$$\begin{aligned} \frac{1}{\hbar}S_{\text{gauge}} &= \int \frac{1}{2g^2}\text{tr}\left(-\ell^{-4}F_{\mu\nu}F^{\mu\nu}\right)\,\ell^4\sqrt{-g}\,d^4x \\ \frac{1}{\hbar}S_{\text{scalar}} &= \int \frac{1}{8\lambda^2}\left[2\hat{\phi}^\dagger\hat{\phi}\,\text{tr}\left(-\frac{B_\mu B^\mu}{\omega_0^2}\right) + \left(\hat{\phi}^\dagger\hat{\phi}-1\right)^2\right]\omega_0^4\sqrt{-g}\,d^4x \end{aligned} \tag{5.12}$$

$$\delta F_{\mu\nu} = D_\mu\delta B_\nu - D_\nu\delta B_\mu \tag{5.13}$$

$$\begin{aligned} \delta S_{\text{gauge}} &= \hbar\int\frac{2}{g^2}\text{tr}\left(-\delta B^\mu D^\nu F_{\mu\nu}\right)\sqrt{-g}\,d^4x \\ \delta S_{\text{scalar}} &= \hbar\int\frac{\omega_0^2}{2\lambda^2}\hat{\phi}^\dagger\hat{\phi}\,\text{tr}\left(-\delta B^\mu B_\mu\right)\sqrt{-g}\,d^4x \\ \delta S_{\text{gauge}}+\delta S_{\text{scalar}} &= \hbar\int\frac{2}{g^2}\text{tr}\left[-\frac{\delta B^\mu}{\omega_0}\left(\frac{D^\nu F_{\mu\nu}}{\omega_0^3}+\frac{\omega_0^2 g^2\hat{\phi}^\dagger\hat{\phi}}{4\lambda^2}\frac{B_\mu}{\omega_0}\right)\right]\omega_0^4\sqrt{-g}\,d^4x \end{aligned} \tag{5.14}$$

so the leading order gauge field equation of motion is

$$0 = \frac{D^\nu F_{\mu\nu}}{\omega_0^3} + \frac{\omega_0^2 g^2\hat{\phi}^\dagger\hat{\phi}}{4\lambda^2}\frac{B_\mu}{\omega_0} \tag{5.15}$$

For the CGF,

$$B_\mu = ibL_\mu \qquad b=\omega_0 b_0(\zeta) \qquad \zeta=\omega_0 T(x) \tag{5.16}$$

The curvature form is

$$\begin{aligned} F_{\mu\nu} &= [D_\mu,\,D_\nu] = [D^0_\mu + ibL_\mu,\, D^0_\nu + ibL_\nu] \\ &= F^0_{\mu\nu} + ib\left([D^0_\mu,\,L_\nu]-[D^0_\nu,\,L_\mu]\right) + i\partial_\mu bL_\nu - i\partial_\nu bL_\mu - b^2[L_\mu,\,L_\nu] \\ &= \omega_0^2\left(i\omega_0^{-1}\partial_\mu b_0 L_\nu - i\omega_0^{-1}\partial_\nu b_0 L_\mu - b_0^2[L_\mu,\,L_\nu]\right) + O(\omega_0) \\ &= \omega_0^2\left(i\frac{db_0}{d\zeta}\partial_{[\mu}TL_{\nu]} - b_0^2[L_\mu,\,L_\nu]\right) + O(\omega_0) \\ \frac{1}{\omega_0^2}F_{\mu\nu} &= \frac{db_0}{d\zeta}\partial_{[\mu}TiL_{\nu]} - b_0^2[L_\mu,\,L_\nu] + O(\omega_0^{-1}) \end{aligned} \tag{5.17}$$

$$b_0' = \frac{db_0}{d\zeta} \qquad b_0'' = \frac{d^2b_0}{d\zeta^2} \tag{5.18}$$

$$\begin{aligned} \omega_0^{-3}D^\nu F_{\mu\nu} &= ib_0''\partial^\nu T\partial_{[\mu}TL_{\nu]} \\ &\qquad - 2b_0b_0'\partial^\nu T[L_\mu,\,L_\nu] + ib_0\left[L^\nu,\, ib_0'\partial_{[\mu}TL_{\nu]} - b_0^2[L_\mu,\,L_\nu]\right] \\ &= ib_0''(\partial^\nu T\partial_\mu TL_\nu - \partial^\nu T\partial_\nu TL_\mu) \\ &\qquad - 2b_0b_0'\partial^\nu T[L_\mu,\,L_\nu] + b_0b_0'[L^\nu,\,\partial_\nu TL_\mu] - ib_0^3[L^\nu,\,[L_\mu,\,L_\nu]] \end{aligned} \tag{5.19}$$

Identity (3.14) gives

$$\begin{aligned} \omega_0^{-3}D^\nu F_{\mu\nu} &= -b_0''\partial^\nu T\partial_\nu TiL_\mu + b_0''\partial_\mu T\partial^\nu TiL_\nu - 3b_0b_0'\partial^\nu T[L_\mu,\,L_\nu] + 2b_0^3 iL_\mu \\ &= \left(-b_0''\partial^\nu T\partial_\nu T + 2b_0^3\right)iL_\mu + b_0''\partial_\mu T\partial^\nu TiL_\nu - 3b_0b_0'\partial^\nu T[L_\mu,\,L_\nu] \end{aligned} \tag{5.20}$$

The equation of motion is

$$0 = \left(-\partial^\nu T \partial_\nu T b_0'' + \frac{g^2 \hat{\phi}^\dagger \hat{\phi}}{4\lambda^2} b_0 + 2b_0^3 \right) iL_\mu + b_0'' \partial_\mu T \partial^\nu T iL_\nu - 3b_0 b_0' \partial^\nu T [L_\mu,\, L_\nu] \tag{5.21}$$

Now contract with u^μ.

$$0 = b_0''(u^\mu \partial_\mu T) \partial^\nu T iL_\nu \tag{5.22}$$

But b_0'', $u^\mu \partial_\mu T \neq 0$ so $\partial^\nu T L_\nu = 0$ so

$$u_\mu = -a_\ell \partial_\mu T \qquad u^\mu \partial_\mu T = \frac{1}{a_\ell} \qquad \partial_\nu T \partial^\nu T = -\frac{1}{a_\ell^2} \tag{5.23}$$

The last two summands in (5.21) vanish since $u^\mu L_\mu = 0$. The equation of motion becomes

$$0 = \frac{1}{a_\ell^2} \frac{d^2 b_0}{d\zeta^2} + \left(\frac{g^2 \hat{\phi}^\dagger \hat{\phi}}{4\lambda^2} \right) b_0 + 2b_0^3 \tag{5.24}$$

which is an anharmonic oscillator with conserved energy

$$H = \frac{1}{2} \frac{1}{a_\ell^2} \left(\frac{db_0}{d\zeta} \right)^2 + \frac{1}{2} \left(\frac{g^2 \hat{\phi}^\dagger \hat{\phi}}{4\lambda^2} \right) b_0^2 + \frac{1}{2} b_0^4 \tag{5.25}$$

6 Solution of the equations of motion

References for elliptic function identities are given in [1].

The Jacobi Elliptic function $\mathrm{cn}(z, k)$

$$\mathrm{cn}(z + 4K, k) = \mathrm{cn}(z, k) \qquad \mathrm{cn}'^{\,2} = (1 - \mathrm{cn}^2)(k'^2 + k^2\, \mathrm{cn}^2) \tag{6.1}$$

$$f(z, k) = k\, \mathrm{cn}(z, k)$$

$$\begin{gathered} \left(\frac{df}{dz} \right)^2 = (k^2 - f^2)(k'^2 + f^2) = k^2(1 - k^2) + (2k^2 - 1) f^2 - f^4 \\ \left(\frac{df}{dz} \right)^2 + (1 - 2k^2) f^2 + f^4 = k^2(1 - k^2) \end{gathered} \tag{6.2}$$

The function $F(\zeta, k)$

$$\begin{gathered} F(\zeta, k) = \frac{k\, \mathrm{cn}(z, k)}{\zeta'} = \frac{f(z, k)}{\zeta'} \qquad \zeta = \frac{2\pi}{4K} z \qquad \zeta' = \frac{2\pi}{4K} \\ F(\zeta + 2\pi) = F(\zeta) \end{gathered} \tag{6.3}$$

$$\frac{df}{dz} = \zeta'^2 \frac{dF}{d\zeta}$$

$$\zeta'^4 \left(\frac{dF}{d\zeta} \right)^2 + (1 - 2k^2) \zeta'^2 F^2 + \zeta'^4 F^4 = k^2(1 - k^2) \tag{6.4}$$

$$\frac{1}{2} \left(\frac{dF}{d\zeta} \right)^2 + \frac{(1 - 2k^2)}{2\zeta'^2} F^2 + \frac{1}{2} F^4 = \frac{k^2(1 - k^2)}{2\zeta'^4}$$

The solution

$$b_0(\zeta) = \frac{1}{a_\ell} F(\zeta, k) \tag{6.5}$$

$$\frac{k^2(1-k^2)}{2\zeta'^4} = \frac{1}{2}\left(\frac{dF}{d\zeta}\right)^2 + \frac{(1-2k^2)}{2\zeta'^2}F^2 + \frac{1}{2}F^4$$

$$H = \frac{1}{2a_\ell^2}\left(\frac{db_0}{d\zeta}\right)^2 + \frac{g^2\hat{\phi}^\dagger\hat{\phi}}{8\lambda^2}b_0^2 + \frac{1}{2}b_0^4 \tag{6.6}$$

$$a_\ell^4 H = \frac{1}{2}\left(\frac{dF}{d\zeta}\right)^2 + \frac{g^2\hat{\phi}^\dagger\hat{\phi}a_\ell^2}{8\lambda^2}F^2 + \frac{1}{2}F^4$$

$$a_\ell^4 H = \frac{k^2(1-k^2)}{2\zeta'^4} \qquad \frac{g^2\hat{\phi}^\dagger\hat{\phi}a_\ell^2}{8\lambda^2} = \frac{1-2k^2}{2\zeta'^2}$$

$$\hat{\phi}^\dagger\hat{\phi} = \frac{4\lambda^2}{g^2}\frac{1-2k^2}{\zeta'^2 a_\ell^2} \tag{6.7}$$

In the unbroken phase $\hat{\phi} = 0$

$$k^2 = \frac{1}{2} \qquad \frac{1}{8\zeta'^4} = a_\ell^4 H \tag{6.8}$$

Averaging over a period

$$\langle \, \mathrm{cn}^2 \, \rangle = \frac{1}{k^2}\left(k^2 - 1 + \frac{E}{K}\right) \qquad \langle \, F^2 \, \rangle = \frac{1}{\zeta'^2}\left(k^2 - 1 + \frac{E}{K}\right)$$

$$\langle \, b_0^2 \, \rangle = \frac{1}{a_\ell^2}\langle \, F^2 \, \rangle = \frac{1}{\zeta'^2 a_\ell^2}\left(k^2 - 1 + \frac{E}{K}\right) \tag{6.9}$$

In the broken phase

$$\hat{\phi}^\dagger\hat{\phi} = 1 - \frac{3}{2}\langle \, b_0^2 \, \rangle = 1 - \frac{3}{2}\frac{1}{\zeta'^2 a_\ell^2}\left(k^2 - 1 + \frac{E}{K}\right) \tag{6.10}$$

$$1 = \frac{4\lambda^2}{g^2}(1-2k^2)\frac{1}{\zeta'^2 a_\ell^2} + \frac{3}{2}\frac{1}{\zeta'^2 a_\ell^2}\left(k^2 - 1 + \frac{E}{K}\right)$$

$$\zeta'^2 a_\ell^2 = \frac{4\lambda^2}{g^2}(1-2k^2) + \frac{3}{2}\left(k^2 - 1 + \frac{E}{K}\right) \tag{6.11}$$

7 Energy-momentum tensor

scalar field energy-momentum tensor

$$\begin{aligned}
\frac{1}{\hbar}T^{\phi\mu}{}_{\nu} &= \frac{1}{\ell^2}\left(D^\mu\phi^\dagger D_\nu\phi + D_\nu\phi^\dagger D^\mu\phi - \delta^\mu_\nu D_\sigma\phi^\dagger D^\sigma\phi\right) - \delta^\mu_\nu\frac{\lambda^2}{2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 \\
&= \frac{1}{\ell^2}\frac{1}{\ell_0^2}\frac{1}{2\lambda^2}\left(D^\mu\hat{\phi}^\dagger D_\nu\hat{\phi} + D_\nu\hat{\phi}^\dagger D^\mu\hat{\phi} - \delta^\mu_\nu D_\sigma\hat{\phi}^\dagger D^\sigma\hat{\phi}\right) - \delta^\mu_\nu\frac{1}{\ell_0^4}\frac{1}{8\lambda^2}\left(\hat{\phi}^\dagger\hat{\phi} - 1\right)^2 \qquad (7.1) \\
&= \frac{1}{\ell_0^4}\frac{1}{8\lambda^2}\left[\frac{4}{\omega_0^2}\left(D^\mu\hat{\phi}^\dagger D_\nu\hat{\phi} + D_\nu\hat{\phi}^\dagger D^\mu\hat{\phi} - \delta^\mu_\nu D_\sigma\hat{\phi}^\dagger D^\sigma\hat{\phi}\right) - \delta^\mu_\nu\left(\hat{\phi}^\dagger\hat{\phi} - 1\right)^2\right]
\end{aligned}$$

Use

$$B_\mu B_\nu + B_\nu B_\mu = B^\sigma_\mu B^\tau_\nu (L_\sigma L_\tau + L_\tau L_\sigma) = B^\sigma_\mu B^\tau_\nu \frac{1}{2} P_{\sigma\tau} = \mathrm{tr}(B_\mu B_\nu) \tag{7.2}$$

to get

$$\begin{aligned} D_\mu \hat{\phi}^\dagger D_\nu \hat{\phi} &= (D^0_\mu + B_\mu)\hat{\phi}^\dagger (D^0_\nu + B_\nu)\hat{\phi} \\ &= D^0_\mu \hat{\phi}^\dagger D^0_\nu \hat{\phi} - \hat{\phi}^\dagger B_\mu D^0_\nu \hat{\phi} + D^0_\mu \hat{\phi}^\dagger B_\nu \hat{\phi} - \hat{\phi}^\dagger B_\mu B_\nu \hat{\phi} \\ D_\mu \hat{\phi}^\dagger D_\nu \hat{\phi} + D_\nu \hat{\phi}^\dagger D_\mu \hat{\phi} &= \left(D^0_\mu \hat{\phi}^\dagger D^0_\nu \hat{\phi} - \hat{\phi}^\dagger B_\mu D^0_\nu \hat{\phi} + D^0_\mu \hat{\phi}^\dagger B_\nu \hat{\phi} \right) + \mu \leftrightarrow \nu \\ &\quad - \hat{\phi}^\dagger (B_\mu B_\nu + B_\nu B_\mu) \hat{\phi} \\ &= \left(D^0_\mu \hat{\phi}^\dagger D^0_\nu \hat{\phi} - \hat{\phi}^\dagger B_\mu D^0_\nu \hat{\phi} + D^0_\mu \hat{\phi}^\dagger B_\nu \hat{\phi} \right) + \mu \leftrightarrow \nu \\ &\quad - \hat{\phi}^\dagger \hat{\phi} \, \mathrm{tr}\, (B_\mu B_\nu) \end{aligned} \tag{7.3}$$

so, to leading order,

$$\frac{4}{\omega_0^2} \left(D^\mu \hat{\phi}^\dagger D_\nu \hat{\phi} + D_\nu \hat{\phi}^\dagger D^\mu \hat{\phi} - \delta^\mu_\nu D_\sigma \hat{\phi}^\dagger D^\sigma \hat{\phi} \right) = \hat{\phi}^\dagger \hat{\phi} \, \mathrm{tr} \frac{2}{\omega_0^2} \left(-2B^\mu B_\nu + \delta^\mu_\nu B_\sigma B^\sigma \right) \tag{7.4}$$

$$\frac{1}{\hbar} T^{\phi\mu}{}_\nu = \frac{1}{\ell_0^4} \frac{1}{8\lambda^2} \left[\hat{\phi}^\dagger \hat{\phi} \, \mathrm{tr} \frac{2}{\omega_0^2} \left(-2B^\mu B_\nu + \delta^\mu_\nu B_\sigma B^\sigma \right) - \delta^\mu_\nu \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2 \right] \tag{7.5}$$

For the CGF, with $B_\mu = \omega_0 b_0 i L_\mu$

$$\begin{aligned} \mathrm{tr} \frac{2}{\omega_0^2} \left(-2B^\mu B_\nu + \delta^\mu_\nu B_\sigma B^\sigma \right) &= b_0^2 \, \mathrm{tr} \left(4L^\mu L_\nu - 2\delta^\mu_\nu L_\sigma L^\sigma \right) = b_0^2 \left(2P^\mu_\nu - \delta^\mu_\nu P^\sigma_\sigma \right) \\ &= b_0^2 \left(2P^\mu_\nu - 3\delta^\mu_\nu \right) \end{aligned} \tag{7.6}$$

so

$$\begin{aligned} \frac{1}{\hbar} T^{\phi\mu}{}_\nu &= \frac{1}{\ell_0^4} \frac{1}{8\lambda^2} \left[\hat{\phi}^\dagger \hat{\phi} \, b_0^2 \left(2P^\mu_\nu - 3\delta^\mu_\nu \right) - \delta^\mu_\nu \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2 \right] \\ &= \frac{1}{\ell_0^4} \frac{1}{8\lambda^2} \left[\hat{\phi}^\dagger \hat{\phi} \, b_0^2 \left(3u^\mu u_\nu - P^\mu_\nu \right) + \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2 \left(u^\mu u_\nu - P^\mu_\nu \right) \right] \end{aligned} \tag{7.7}$$

gauge field energy-momentum tensor

$$\begin{aligned} \frac{1}{\hbar} T^{\mathrm{gauge}\mu}{}_\nu &= \frac{1}{\ell^4 g^2} \mathrm{tr} \left(-2F^\mu{}_\sigma F_\nu{}^\sigma + \frac{1}{2} \delta^\mu_\nu F_{\rho\sigma} F^{\rho\sigma} \right) \\ &= \frac{1}{\ell_0^4} \frac{2}{g^2} \frac{1}{\omega_0^4} \mathrm{tr} \left(-F^\mu{}_\sigma F_\nu{}^\sigma + \frac{1}{4} \delta^\mu_\nu F_{\rho\sigma} F^{\rho\sigma} \right) \end{aligned} \tag{7.8}$$

$$\frac{1}{\omega_0^2} F_{\mu\nu} = \frac{db_0}{d\zeta} \partial_{[\mu} T i L_{\nu]} - b_0^2 [L_\mu,\, L_\nu] \tag{7.9}$$

$$
\begin{aligned}
\frac{1}{\omega_0^4}\mathrm{tr}\left(-F_{\mu\sigma}F_\nu{}^\sigma\right) = \mathrm{tr}\Bigg[-\bigg(&\frac{db_0}{d\zeta}\partial_{[\mu}TiL_{\sigma]} - b_0^2[L_\mu,\, L_\sigma]\bigg) \\
&\bigg(\frac{db_0}{d\zeta}\partial_{[\nu}TiL^{\sigma]} - b_0^2[L_\nu,\, L^\sigma]\bigg)\Bigg] \\
= &\left(\frac{db_0}{d\zeta}\right)^2 \mathrm{tr}\left(\partial_{[\mu}TL_{\sigma]}\partial_{[\nu}TL^{\sigma]}\right) - b_0^4\mathrm{tr}\left([L_\mu,\, L_\sigma][L_\nu,\, L^\sigma]\right) \\
= &\left(\frac{db_0}{d\zeta}\right)^2 \mathrm{tr}\left[(\partial_\mu TL_\sigma - \partial_\sigma TL_\mu)(\partial_\nu TL^\sigma - \partial^\sigma TL_\nu)\right] \\
&+ b_0^4\frac{1}{2}\left(P_{\mu\nu}g^{\sigma\sigma'}P_{\sigma\sigma'} - g^{\sigma\sigma'}P_{\mu\sigma}P_{\nu\sigma'}\right) \\
= &\left(\frac{db_0}{d\zeta}\right)^2 \frac{1}{2}\left[\partial_\mu T\partial_\nu TP^\sigma_\sigma - \partial_\sigma T\partial_\nu TP^\sigma_\mu - \partial_\mu T\partial_\sigma TP^\sigma_\nu\right. \\
&\left. + \partial_\sigma T\partial^\sigma TP_{\mu\nu}\right] + b_0^4P_{\mu\nu} \\
= &\left(\frac{db_0}{d\zeta}\right)^2 \frac{1}{2}\left(3\partial_\mu T\partial_\nu T + \partial_\sigma T\partial^\sigma TP_{\mu\nu}\right) + b_0^4P_{\mu\nu} \\
= &\frac{1}{2a_\ell^2}\left(\frac{db_0}{d\zeta}\right)^2 (3u_\mu u_\nu - P_{\mu\nu}) + b_0^4P_{\mu\nu}
\end{aligned}
\tag{7.10}
$$

$$
\begin{aligned}
\frac{1}{\omega_0^4}\mathrm{tr}\left(-F_{\mu\sigma}F_\nu{}^\sigma + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) = &\frac{1}{2a_\ell^2}\left(\frac{db_0}{d\zeta}\right)^2\left(3u_\mu u_\nu + \frac{3}{4}g_{\mu\nu} - P_{\mu\nu} + \frac{3}{4}g_{\mu\nu}\right) \\
&+ b_0^4\left(P_{\mu\nu} - \frac{3}{4}g_{\mu\nu}\right) \\
= &\frac{1}{2a_\ell^2}\left(\frac{db_0}{d\zeta}\right)^2\left(3u_\mu u_\nu + \frac{3}{4}g_{\mu\nu} - P_{\mu\nu} + \frac{3}{4}g_{\mu\nu}\right) \\
&+ b_0^4\left(P_{\mu\nu} - \frac{3}{4}g_{\mu\nu}\right) \\
= &\frac{1}{4a_\ell^2}\left(\frac{db_0}{d\zeta}\right)^2 (6u_\mu u_\nu + 3(-u_\mu u_\nu + P_{\mu\nu}) - 2P_{\mu\nu}) \\
&+ b_0^4\left(P_{\mu\nu} - \frac{3}{4}(-u_\mu u_\nu + P_{\mu\nu})\right) \\
= &\frac{1}{4}\left[\frac{1}{a_\ell^2}\left(\frac{db_0}{d\zeta}\right)^2 + b_0^4\right](3u_\mu u_\nu + P_{\mu\nu})
\end{aligned}
\tag{7.11}
$$

$$\frac{1}{\hbar} T^{\text{gauge}\mu}{}_{\nu} = \frac{1}{\ell_0^4} \frac{1}{2g^2} \left[\frac{1}{a_\ell^2} \left(\frac{db_0}{d\zeta} \right)^2 + b_0^4 \right] (3u^\mu u_\nu + P^\mu_\nu)$$
$$= \frac{1}{\ell_0^4} \frac{1}{g^2} \left[H - \frac{g^2 \hat{\phi}^\dagger \hat{\phi}}{8\lambda^2} b_0^2 \right] (3u_\mu u_\nu + P_{\mu\nu}) \tag{7.12}$$

combined energy-momentum tensor

$$\rho_0 = \frac{\hbar}{\ell_0^4} \tag{7.13}$$

$$T^\mu_\nu = T^{\text{gauge}\mu}{}_\nu + T^{\phi\mu}{}_\nu$$
$$= \rho_0 \left[\left(\frac{1}{g^2} H - \frac{\hat{\phi}^\dagger \hat{\phi}}{8\lambda^2} b_0^2 \right) (3u_\mu u_\nu + P_{\mu\nu}) \right.$$
$$\left. + \frac{\hat{\phi}^\dagger \hat{\phi}\, b_0^2}{8\lambda^2} (3u^\mu u_\nu - P^\mu_\nu) + \frac{1}{8\lambda^2} \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2 (u^\mu u_\nu - P^\mu_\nu) \right] \tag{7.14}$$
$$= \rho_0 \left[\frac{1}{g^2} H (3u_\mu u_\nu + P_{\mu\nu}) - \frac{\hat{\phi}^\dagger \hat{\phi}\, b_0^2}{4\lambda^2} P^\mu_\nu + \frac{1}{8\lambda^2} \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2 (u^\mu u_\nu - P^\mu_\nu) \right]$$

density and pressure

$$T^\mu_\nu = \rho u^\mu u_\nu + p P^\mu_\nu \tag{7.15}$$

$$\frac{\rho}{\rho_0} = \frac{3}{g^2} H + \frac{1}{8\lambda^2} \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2$$
$$\frac{p}{\rho_0} = \frac{1}{g^2} H - \frac{\hat{\phi}^\dagger \hat{\phi}\, b_0^2}{4\lambda^2} - \frac{1}{8\lambda^2} \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2 \tag{7.16}$$
$$= \frac{1}{g^2} H - \frac{\hat{\phi}^\dagger \hat{\phi} \langle b_0^2 \rangle}{4\lambda^2} - \frac{1}{8\lambda^2} \left(\hat{\phi}^\dagger \hat{\phi} - 1 \right)^2$$

in the unbroken phase

$$\frac{\rho}{\rho_0} = \frac{3}{g^2} H + \frac{1}{8\lambda^2} \qquad \frac{p}{\rho_0} = \frac{1}{g^2} H - \frac{1}{8\lambda^2} \tag{7.17}$$

in the broken phase

$$\hat{\phi}^\dagger \hat{\phi} = 1 - \frac{3}{2} \langle b_0^2 \rangle \tag{7.18}$$

$$\frac{\rho}{\rho_0} = \frac{3}{g^2} H + \frac{9}{32\lambda^2} \langle b_0^2 \rangle^2 \qquad \frac{p}{\rho_0} = \frac{1}{g^2} H - \frac{\hat{\phi}^\dagger \hat{\phi}}{4\lambda^2} \langle b_0^2 \rangle - \frac{9}{32\lambda^2} \langle b_0^2 \rangle^2 \tag{7.19}$$

equivalence with the cosmological equation of state

See section 10 below.

8 Irrotationality

Consider the first order differential operators

$$S_\mu = P_\mu^\rho \partial_\rho = \partial_\mu + u_\mu u^\rho \partial_\rho \tag{8.1}$$

Calculate the commutators

$$\begin{aligned} [S_\mu,\, S_\nu] &= [P_\mu^\rho \partial_\rho,\, P_\nu^\sigma \partial_\sigma] = (P_\mu^\rho \partial_\rho P_\nu^\sigma - P_\nu^\rho \partial_\rho P_\mu^\sigma)\partial_\sigma \\ &= (P_\mu^\rho \partial_\rho P_\nu^\sigma - P_\nu^\rho \partial_\rho P_\mu^\sigma)(P_\sigma^\tau - u^\tau u_\sigma)\partial_\tau \\ &= (P_\mu^\rho \partial_\rho P_\nu^\sigma - P_\nu^\rho \partial_\rho P_\mu^\sigma) S_\sigma + (P_\mu^\rho \partial_\rho P_\nu^\sigma - P_\nu^\rho \partial_\rho P_\mu^\sigma)(-u^\tau u_\sigma)\partial_\tau \end{aligned} \tag{8.2}$$

$$[S_\mu,\, S_\nu] - T^\sigma_{\mu\nu} S_\sigma = A_{\mu\nu} u^\tau \partial_\tau \qquad T^\sigma_{\mu\nu} = (P_\mu^\rho \partial_\rho P_\nu^\sigma - P_\nu^\rho \partial_\rho P_\mu^\sigma) \tag{8.3}$$

$$A_{\mu\nu} = -(P_\mu^\rho \partial_\rho P_\nu^\sigma - P_\nu^\rho \partial_\rho P_\mu^\sigma) u_\sigma = P^\rho_{[\mu} P^\sigma_{\nu]} \partial_\rho u_\sigma = P^\rho_\mu P^\sigma_\nu \partial_{[\rho} u_{\sigma]} \tag{8.4}$$

By taking a linear frame at x in which $u^i = 0$ and $u_i = 0$ we see that

$$A_{\mu\nu} = 0 \quad \Leftrightarrow \quad u_{[\mu} \partial_\nu u_{\rho]} = 0 \tag{8.5}$$

So $u^\mu(x)$ irrotational implies that the operators S_μ are holonomic,

$$[S_\mu,\, S_\nu] - T^\sigma_{\mu\nu} S_\sigma = 0 \tag{8.6}$$

which in turn implies that there exists a nonzero holonomic function, i.e. a solution of

$$S_\mu T = \partial_\mu T + u_\mu u^\sigma \partial_\sigma T = 0 \tag{8.7}$$

So irrotationality implies the existence of a time coordinate $T(x)$.

9 Adiabatic time evolution implies a continuity equation

$$\begin{aligned} \frac{1}{\omega_0^4} \mathrm{tr}\,(-F_{\mu\sigma} F_\nu{}^\sigma) &= \frac{1}{2a_\ell^2} \left(\frac{db_0}{d\zeta}\right)^2 (3u_\mu u_\nu - P_{\mu\nu}) + b_0^4 P_{\mu\nu} + O(\omega_0^{-1}) \\ \frac{1}{\omega_0^4} \mathrm{tr}\,(-F_{\mu\sigma} F^{\mu\sigma}) &= -\frac{3}{a_\ell^2} \left(\frac{db_0}{d\zeta}\right)^2 + 3b_0^4 + O(\omega_0^{-1}) \end{aligned} \tag{9.1}$$

$$\begin{aligned} \frac{1}{\hbar} S_{\mathrm{gauge}} &= \int \frac{1}{2g^2} \mathrm{tr}\,(-F_{\mu\nu} F^{\mu\nu})\, \sqrt{-g}\, d^4x \\ &= \int \frac{3\omega_0^4}{2g^2} \left[-\frac{1}{a_\ell^2} \left(\frac{db_0}{d\zeta}\right)^2 + b_0^4 \right] \sqrt{-g}\, d^4x \; + O(\omega_0^3) \end{aligned} \tag{9.2}$$

$$\frac{1}{\hbar} S_{\mathrm{scalar}} = \int \frac{\omega_0^4}{8\lambda^2} \left[3\hat\phi^\dagger \hat\phi b_0^2 + (\hat\phi^\dagger \hat\phi - 1)^2 \right] \sqrt{-g}\, d^4x + O(\omega_0^3) \tag{9.3}$$

In the CGF rest frame,

$$\begin{aligned} &ds^2 = \ell^2 g_{\mu\nu} dx^\mu dx^\nu \qquad g_{\mu\nu} dx^\mu dx^\nu = a_\ell(x)^2 \left[-(dT)^2 + g^{(3)}_{ij}(x) dx^i dx^j \right] \\ &\sqrt{-g} = a_\ell^4 \sqrt{\det g^{(3)}_{ij}} \qquad d^4x = dT\, d^3x = \frac{d\zeta\, d^3x}{\omega_0} \qquad \zeta = \omega_0 T \end{aligned} \tag{9.4}$$

and

$$b_0(\zeta) = \frac{1}{a_\ell} F(\zeta, k) \tag{9.5}$$

$$\begin{aligned}
\frac{1}{\hbar} S_{\text{CGF}} &= \int \frac{3\omega_0^4}{g^2} \left[-\frac{1}{2a_\ell^2} \left(\frac{db_0}{d\zeta} \right)^2 + \frac{g^2 \hat{\phi}^\dagger \hat{\phi}}{8\lambda^2} b_0^2 + \frac{b_0^4}{2} + \frac{g^2 (\hat{\phi}^\dagger \hat{\phi} - 1)^2}{24\lambda^2} \right] \sqrt{-g}\, d^4x \\
&= \int \frac{3\omega_0^4}{g^2} \left[-\frac{1}{2} \left(\frac{dF}{d\zeta} \right)^2 + \frac{g^2 \hat{\phi}^\dagger \hat{\phi} a_\ell^2}{8\lambda^2} F^2 + \frac{1}{2} F^4 + \frac{g^2 (\hat{\phi}^\dagger \hat{\phi} - 1)^2 a_\ell^4}{24\lambda^2} \right] \frac{\sqrt{-g}}{a_\ell^4}\, d^4x
\end{aligned} \tag{9.6}$$

$$\frac{1}{\hbar} S_{\text{CGF}} =$$

$$\int d\zeta \int d^3x \sqrt{\det g_{ij}^{(3)}}\, \frac{3\omega_0^3}{g^2} \left[-\frac{1}{2} \left(\frac{dF}{d\zeta} \right)^2 + \frac{g^2 \hat{\phi}^\dagger \hat{\phi} a_\ell^2}{8\lambda^2} F^2 + \frac{1}{2} F^4 + \frac{g^2 (\hat{\phi}^\dagger \hat{\phi} - 1)^2 a_\ell^4}{24\lambda^2} \right] \tag{9.7}$$

At leading order in ω_0 each small volume element d^3x of the fluid is an independent anharmonic oscillator decoupled from the rest of the fluid (ultralocality). The coupling constants of each anharmonic oscillator vary slowly in time compared to the period of oscillation. In such an adiabatic time evolution, the adiabatic invariant $\oint pdq$ stays constant in time, where q is the oscillator degree of freedom, p its canonical conjugate, and the integral is over one period of oscillation. Measured in quanta, the adiabatic invariant is

$$\frac{1}{2\pi\hbar} \oint pdq = \frac{1}{2\pi\hbar} \int_0^{2\pi} p \frac{dq}{d\zeta}\, d\zeta \tag{9.8}$$

Taking $q = F$,

$$\frac{1}{\hbar} p = d^3x \sqrt{\det g_{ij}^{(3)}}\, \frac{3\omega_0^3}{g^2} \frac{dF}{d\zeta} \tag{9.9}$$

$$\begin{aligned}
\frac{1}{2\pi\hbar} \oint pdq &= d^3x \sqrt{\det g_{ij}^{(3)}}\, N(x) \\
N(x) &= \frac{3\omega_0^3}{g^2} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{dF}{d\zeta} \right)^2 d\zeta
\end{aligned} \tag{9.10}$$

and

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{dF}{d\zeta} \right)^2 d\zeta &= \frac{1}{2\pi} \int_0^{4K} \frac{1}{\zeta'^2} \frac{k^2}{\zeta'^2} \left(\frac{d\,\text{cn}}{dz} \right)^2 \zeta' dz \\
&= \frac{1}{2\pi \zeta'^3} \int_0^{4K} k^2 \left(\frac{d\,\text{cn}}{dz} \right)^2 dz \\
&= \frac{1}{K\zeta'^2} \frac{1}{3} \left[(1-k^2)K + (2k^2-1)E \right] \\
&= \frac{1}{3\zeta'^2} \left[1 - k^2 + (2k^2-1) \frac{E}{K} \right]
\end{aligned} \tag{9.11}$$

so

$$\begin{aligned}
N(x) &= \frac{3\omega_0^3}{g^2} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{dF}{d\zeta} \right)^2 d\zeta \\
&= \frac{\omega_0^3}{g^2 \zeta'^2} \left[1 - k^2 + (2k^2-1) \frac{E}{K} \right]
\end{aligned} \tag{9.12}$$

and the adiabatic invariance condition is

$$0 = \frac{\partial}{\partial T}\left[\sqrt{\det g^{(3)}_{ij}}\; N(x)\right] = \frac{\partial}{\partial T}\left[\frac{\sqrt{-g}}{a_\ell^4} a_\ell u^0 N(x)\right] = \partial_\mu \left[\frac{N(x)}{\ell^3 a_\ell^3}\ell^3 u^\mu \sqrt{-g}\right] \tag{9.13}$$

which is the continuity equation for the number density

$$n(x) = \frac{N(x)}{(\ell a_\ell)^3} \tag{9.14}$$

$$\partial_\mu J^\mu = 0 \qquad J^\mu(x) = n(x)\,\ell^3\, u^\mu(x)\,\sqrt{-g}\,d^4x \tag{9.15}$$

10 Summary of the parametrization

$$\begin{gathered} C_1 = \frac{1}{2}k^2(1-k^2) \qquad C_2 = 1-2k^2 \\ C_3 = k^2 - 1 + \frac{E}{K} \qquad C_4 = \zeta'\left[1-k^2+(2k^2-1)\frac{E}{K}\right] \qquad \zeta' = \frac{2\pi}{4K} \end{gathered} \tag{10.1}$$

$$\begin{aligned} \hat{a} &= \zeta' a_\ell \\ H &= \frac{k^2(1-k^2)}{2\zeta'^4 a_\ell^4} = \frac{C_1}{\hat{a}^4} \\ \frac{g^2\hat{\phi}^\dagger\hat{\phi}}{4\lambda^2} &= \frac{1-2k^2}{\zeta'^2 a_\ell^2} = \frac{C_2}{\hat{a}^2} \\ \hat{\phi}^\dagger\hat{\phi} &= \frac{4\lambda^2}{g^2}\frac{C_2}{\hat{a}^2} \\ \langle\, b_0^2 \,\rangle &= \left(k^2 - 1 + \frac{E}{K}\right)\frac{1}{\zeta'^2 a_\ell^2} = \frac{C_3}{\hat{a}^2} \end{aligned} \tag{10.2}$$

$$\begin{aligned} \frac{\rho}{\rho_0} &= \frac{3}{g^2}H + \frac{1}{8\lambda^2}\left(\hat{\phi}^\dagger\hat{\phi}-1\right)^2 \\ &= \frac{3}{g^2}\frac{C_1}{\hat{a}^4} + \frac{1}{8\lambda^2}\left(\frac{4\lambda^2}{g^2}\frac{C_2}{\hat{a}^2}-1\right)^2 \\ \frac{p}{\rho_0} &= \frac{1}{g^2}H - \frac{1}{4\lambda^2}\hat{\phi}^\dagger\hat{\phi}\,\langle\, b_0^2\,\rangle - \frac{1}{8\lambda^2}\left(\hat{\phi}^\dagger\hat{\phi}-1\right)^2 \\ &= \frac{1}{g^2}\frac{C_1}{\hat{a}^4} - \frac{1}{4\lambda^2}\frac{4\lambda^2}{g^2}\frac{C_2}{\hat{a}^2}\frac{C_3}{\hat{a}^2} - \frac{1}{8\lambda^2}\left(\frac{4\lambda^2}{g^2}\frac{C_2}{\hat{a}^2}-1\right)^2 \\ &= \frac{1}{g^2}\frac{1}{\hat{a}^4}(C_1 - C_2C_3) - \frac{1}{8\lambda^2}\left(\frac{4\lambda^2}{g^2}\frac{C_2}{\hat{a}^2}-1\right)^2 \\ \ell_0^3 n &= \frac{1}{a_\ell^3}\frac{1}{g^2\zeta'^2}\left[1-k^2+(2k^2-1)\frac{E}{K}\right] \\ &= \frac{1}{\hat{a}^3}\frac{1}{g^2}C_4 \end{aligned} \tag{10.3}$$

unbroken phase

$$k^2 = \frac{1}{2} \qquad C_1 = \frac{1}{8} \qquad C_2 = 0 \qquad C_3 = -\frac{1}{2} + \frac{E}{K} \qquad C_4 = \frac{1}{2}\zeta' \qquad \zeta' = \frac{2\pi}{4K} \tag{10.4}$$

$$\langle\, b_0^2 \,\rangle = \frac{C_3}{\hat{a}^2} \geq \frac{2}{3} \qquad \hat{a}^2 \leq \frac{3}{2} C_3 \tag{10.5}$$

$$\frac{\rho}{\rho_0} = \frac{1}{\hat{a}^4}\frac{3}{8g^2} + \frac{1}{8\lambda^2}$$

$$\frac{p}{\rho_0} = \frac{1}{\hat{a}^4}\frac{1}{8g^2} - \frac{1}{8\lambda^2} \tag{10.6}$$

$$\ell_0^3 n = \frac{1}{\hat{a}^3}\frac{1}{g^2} C_4$$

broken phase

$$\hat{\phi}^\dagger \hat{\phi} = 1 - \frac{3}{2}\langle\, b_0^2 \,\rangle \qquad \hat{\phi}^\dagger \hat{\phi} + \frac{3}{2}\langle\, b_0^2 \,\rangle = 1 \qquad \frac{4\lambda^2}{g^2}\frac{C_2}{\hat{a}^2} + \frac{3}{2}\frac{C_3}{\hat{a}^2} = 1$$

$$\hat{a}^2 = \frac{4\lambda^2}{g^2} C_2 + \frac{3}{2} C_3 \tag{10.7}$$

$$\frac{\rho}{\rho_0} = \frac{1}{\hat{a}^4}\left(\frac{3}{g^2} C_1 + \frac{9}{32\lambda^2} C_3^2\right)$$

$$\frac{p}{\rho_0} = \frac{1}{\hat{a}^4}\left[\frac{1}{g^2}(C_1 - C_2 C_3) - \frac{9}{32\lambda^2} C_3^2\right] \tag{10.8}$$

$$\ell_0^3 n = \frac{1}{\hat{a}^3}\frac{1}{g^2} C_4$$

Comparison with CGF cosmological construction

unbroken phase (cosmology on left, fluid on right)

$$t_{\text{Higgs}} = \ell_0 \tag{10.9}$$

$\rho_{\text{CGF}} = \frac{\hbar}{t_{\text{Higgs}}^4}\left(\frac{3}{8g^2}\frac{1}{\hat{a}^4} + \frac{1}{8\lambda^2}\right)$	$\rho = \frac{\hbar}{\ell_0^4}\left(\frac{3}{8g^2}\frac{1}{\hat{a}^4} + \frac{1}{8\lambda^2}\right)$
$p_{\text{CGF}} = \frac{\hbar}{t_{\text{Higgs}}^4}\left(\frac{1}{8g^2}\frac{1}{\hat{a}^4} - \frac{1}{8\lambda^2}\right)$	$p = \frac{\hbar}{\ell_0^4}\left(\frac{1}{8g^2}\frac{1}{\hat{a}^4} - \frac{1}{8\lambda^2}\right)$

broken phase

$\alpha^2\langle\, b^2\,\rangle \ = k^2 - 1 + \dfrac{E}{K}$	$\hat{a}^2\langle\, b_0^2\,\rangle \ = k^2 - 1 + \dfrac{E}{K}$
$\alpha^2\mu^2 = 1 - 2k^2$	$\hat{a}^2\dfrac{g^2\hat{\phi}^\dagger\hat{\phi}}{4\lambda^2} = 1 - 2k^2$
$\alpha^4 E_{\rm CGF} = \dfrac{k^2(1-k^2)}{2}$	$\hat{a}^4 H = \dfrac{k^2(1-k^2)}{2}$
$\alpha^2\hat{a}^2 = \dfrac{3}{2}\alpha^2\langle\, b^2\,\rangle + \dfrac{4\lambda^2}{g^2}(1-2k^2)$	$\hat{a}^2 = \dfrac{3}{2}\hat{a}^2\langle\, b_0^2\,\rangle + \dfrac{4\lambda^2}{g^2}(1-2k^2)$
$\dfrac{2(\phi^\dagger\phi)_0}{v^2} = 1 - \dfrac{3}{2}\dfrac{\alpha^2\langle\, b^2\,\rangle}{\alpha^2\hat{a}^2}$	$\hat{\phi}^\dagger\hat{\phi} = 1 - \dfrac{3}{2}\dfrac{\hat{a}^2\langle\, b_0^2\,\rangle}{\hat{a}^2}$

$\hat{\rho}_{\rm CGF} = \dfrac{3}{g^2}\dfrac{E_{\rm CGF}}{\hat{a}^4} + \dfrac{9}{32\lambda^2}\dfrac{\langle\, b^2\,\rangle^2}{\hat{a}^4}$	$\dfrac{\rho}{\rho_0} = \dfrac{3}{g^2}H + \dfrac{9}{32\lambda^2}\langle\, b_0^2\,\rangle^2$
$\hat{p}_{\rm CGF} = \dfrac{1}{g^2}\dfrac{E_{\rm CGF}}{\hat{a}^4} - \dfrac{1}{g^2}\dfrac{\mu^2\langle\, b^2\,\rangle}{\hat{a}^4} - \dfrac{9}{32\lambda^2}\dfrac{\langle\, b^2\,\rangle^2}{\hat{a}^4}$	$\dfrac{p}{\rho_0} = \dfrac{1}{g^2}H - \dfrac{1}{g^2}\dfrac{g^2\hat{\phi}^\dagger\hat{\phi}}{4\lambda^2}\langle\, b_0^2\,\rangle - \dfrac{9}{32\lambda^2}\langle\, b_0^2\,\rangle^2$

$\dfrac{K_{\rm EW}}{2K}\alpha^3 \ = 1 - k^2 + (2k^2-1)\dfrac{E}{K}$	$\dfrac{g^2}{\zeta'}\hat{a}^3\ell_0^3 n \ = 1 - k^2 + (2k^2-1)\dfrac{E}{K}$

$z \sim z + 4K\alpha \qquad \dfrac{\zeta' z}{\alpha\omega_0} \sim \dfrac{\zeta' z}{\alpha\omega_0} + \dfrac{2\pi}{\omega_0}$	$T \sim T + \dfrac{2\pi}{\omega_0}$
$\begin{aligned} ds^2 &= t_{\rm Higgs}^2\hat{a}^2\left(-dz^2 + \frac{1}{\epsilon^2}\hat{g}_{ij}dx^i dx^j\right) \\ &= t_{\rm Higgs}^2\omega_0^2\frac{\alpha^2\hat{a}^2}{\zeta'^2}\left[-\left(\frac{\zeta' dz}{\alpha\omega_0}\right)^2 + \frac{\zeta'^2\hat{g}_{ij}}{\alpha^2\omega_0^2\epsilon^2}dx^i dx^j\right] \end{aligned}$	$\begin{aligned} ds^2 &= \ell^2 a_\ell^2\left(-dT^2 + g_{ij}^{(3)}dx^i dx^j\right) \\ &= \ell_0^2\omega_0^2\frac{\hat{a}^2}{\zeta'^2}\left(-dT^2 + g_{ij}^{(3)}dx^i dx^j\right) \end{aligned}$

11 Low density

G&R 8.113-4, DLMF 19.5.1-2, 19.12.1

$$\begin{aligned} k \to 0 \qquad K(k) &= \frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}\right)+O(k^6) \\ E(k) &= \frac{\pi}{2}\left(1-\frac{k^2}{4}-\frac{3k^4}{64}\right)+O(k^6) \\ \frac{E}{K} &= 1-\frac{k^2}{2}-\frac{k^4}{16}+O(k^6) \end{aligned} \tag{11.1}$$

$$\begin{aligned} C_1 &= \frac{1}{2}k^2(1-k^2) \\ C_2 &= 1-2k^2 \\ C_3 &= k^2-1+\frac{E}{K}=\frac{1}{2}k^2\left(1-\frac{k^2}{8}\right)+O(k^6) \\ \zeta' &= \frac{2\pi}{4K}=1-\frac{k^2}{4}+O(k^4) \\ 1-k^2+(2k^2-1)\frac{E}{K} &= 1-k^2+(2k^2-1)\left(1-\frac{k^2}{2}\right)+O(k^4)=\frac{3}{2}k^2+O(k^4) \\ C_4 &= \zeta'\left[1-k^2+(2k^2-1)\frac{E}{K}\right]=\frac{3}{2}k^2+O(k^4) \\ \hat{a}^2 &= \frac{4\lambda^2}{g^2}C_2+\frac{3}{2}C_3=\frac{4\lambda^2}{g^2}+O(k^2) \\ \frac{\rho}{\rho_0} &= \frac{1}{\hat{a}^4}\left(\frac{3}{g^2}C_1+\frac{9}{32\lambda^2}C_3^2\right)=\left(\frac{g^2}{4\lambda^2}\right)^2\frac{3}{g^2}\frac{1}{2}k^2+O(k^4) \\ &= \frac{3g^2}{32\lambda^4}k^2+O(k^4)=0.600\,k^2+O(k^4) \\ \frac{p}{\rho_0} &= \frac{1}{\hat{a}^4}\left[\frac{1}{g^2}(C_1-C_2C_3)-\frac{9}{32\lambda^2}C_3^2\right] \\ C_1-C_2C_3 &= \frac{1}{2}k^2-\frac{1}{2}k^4-(1-2k^2)\frac{1}{2}k^2\left(1-\frac{k^2}{8}\right)=\frac{9}{16}k^4 \\ \frac{p}{\rho_0} &= \left(\frac{g^2}{4\lambda^2}\right)^2\left[\frac{1}{g^2}\frac{9}{16}k^4-\frac{9}{32\lambda^2}\frac{1}{4}k^4\right] \\ &= \frac{9g^4}{256\lambda^4}\left[\frac{1}{g^2}-\frac{1}{8\lambda^2}\right]k^4 \\ n &= \frac{1}{\ell_0^3}\frac{1}{\hat{a}^3}\frac{1}{g^2}C_4=\frac{1}{\ell_0^3}\frac{g^3}{8\lambda^3}\frac{1}{g^2}\frac{3}{2}k^2=\frac{1}{\ell_0^3}\frac{3g}{16\lambda^3}k^2 \end{aligned} \tag{11.2}$$

equation of state

$$\frac{p}{\rho_0} = \frac{9g^4}{256\lambda^4}\left[\frac{1}{g^2} - \frac{1}{8\lambda^2}\right] k^4 = \frac{9g^4}{256\lambda^4}\left[\frac{1}{g^2} - \frac{1}{8\lambda^2}\right]\left(\frac{32\lambda^4}{3g^2}\frac{\rho}{\rho_0}\right)^2$$
$$= 4\lambda^4\left(\frac{1}{g^2} - \frac{1}{8\lambda^2}\right)\frac{\rho^2}{\rho_0^2} = \frac{1}{2}\frac{\lambda^2\,(8\lambda^2 - g^2)}{g^2}\frac{\rho^2}{\rho_0^2} \tag{11.3}$$

$$p = \frac{c_a}{2}\frac{\rho^2}{\rho_0} \qquad c_a = \frac{\lambda^2\,(8\lambda^2 - g^2)}{g^2} = 0.992 \tag{11.4}$$

density of quanta

$$n = \frac{1}{\ell_0^3}\frac{3g}{16\lambda^3}k^2 = \frac{1}{\ell_0^3}\frac{3g}{16\lambda^3}\frac{32\lambda^4}{3g^2}\frac{\rho}{\rho_0} = \frac{2\lambda}{g}\frac{\rho}{m_{\text{Higgs}}} = \frac{\rho}{m_W} \tag{11.5}$$

present dark matter density and $\langle\, k^2 \,\rangle_{\textbf{now}}$

$$\rho_c = \frac{3H_0^2}{8\pi G} = 8.53\times 10^{-27}\text{kg/m}^3$$
$$\langle\, \rho_{\text{CDM}} \,\rangle_{\text{now}} = \Omega_{\text{CDM}}\rho_c = 0.27\cdot 8.53\times 10^{-27}\text{kg/m}^3 = 2.3\times 10^{-27}\text{kg/m}^3$$
$$\frac{\langle\, \rho_{\text{CDM}} \,\rangle_{\text{now}}}{\rho_0} = 4.1\times 10^{-56} \tag{11.6}$$
$$\langle\, k^2 \,\rangle_{\text{now}} = \frac{32\lambda^4}{3g^2}\frac{\langle\, \rho_{\text{CDM}} \,\rangle_{\text{now}}}{\rho_0} = 1.67\cdot 4.1\times 10^{-56} = 7\times 10^{-56}$$

TOV stability and central density

From *Dark matter stars* [2], the maximum CGF star mass is

$$M_{\text{max}} = 9.145\times 10^{-6}\text{M}_\odot \qquad \rho_{\text{max}} = 0.799\rho_0 \qquad k^2_{\text{max}} = 0.341 \tag{11.7}$$

The last two numbers are from the numerical solutions in Sagemath. The entire star is in the broken phase. The core is not in the low density regime.

The unit of mass in [2] is

$$m_b = (4\pi)^{-\frac{1}{2}}\frac{\ell}{G} = 2.94\times 10^{-5}\text{M}_\odot \tag{11.8}$$

Here the unit of mass is

$$M_\ell = \frac{\ell}{G} = 1.04\times 10^{-4}\,\text{M}_\odot \tag{11.9}$$

The TOV equation is exactly solvable with the low density equation of state

$$p = \frac{c_a}{2}\frac{\rho^2}{\rho_0} \qquad c_a = 0.99248 \tag{11.10}$$

In the low density regime, [2, (5.1), (2.2)] give

$$\frac{M}{m_b} = \pi c_a^{3/2}\frac{\rho_{\text{central}}}{\rho_0} \tag{11.11}$$

which is

$$\frac{M}{M_\ell} = \frac{\pi^{1/2} c_a^{3/2}}{2} \frac{\rho_{\mathrm{central}}}{\rho_0} = 0.87625 \frac{\rho_{\mathrm{central}}}{\rho_0} \tag{11.12}$$

The parametrization by k^2 is

$$\frac{\rho}{\rho_0} = \frac{3}{32} \frac{g^2}{\lambda^4} k^2 = 0.599698\, k^2 \tag{11.13}$$

so

$$M = \frac{\pi^{1/2} c_a^{3/2}}{2} \frac{3}{32} \frac{g^2}{\lambda^4} M_\ell k^2_{\mathrm{central}} = 5.4782 \times 10^{-5} \mathrm{M}_\odot k^2_{\mathrm{central}} \tag{11.14}$$

The microlensing limit $M = 10^{-11}\mathrm{M}_\odot$ corresponds to

$$k^2_{\mathrm{central}} = \frac{10^{-11}\mathrm{M}_\odot}{5.4782 \times 10^{-5}\mathrm{M}_\odot} = 1.825 \times 10^{-7} \tag{11.15}$$

12 Possibilities of detection and verification?

The metric in the rest frame is

$$ds^2 = \ell^2 g_{\mu\nu} dx^\mu dx^\nu = \ell^2 a_\ell(x)^2 \left[-(dT)^2 + g^{(3)}_{ij}(x) dx^i dx^j \right] \tag{12.1}$$

In the low density regime $a_\ell = 4\lambda^2/g^2$.

$$ds^2 = \ell^2 \frac{4\lambda^2}{g^2} \left[-(dT)^2 + g^{(3)}_{ij}(x) dx^i dx^j \right] \tag{12.2}$$

so proper time t is given by

$$t = \ell \frac{2\lambda}{g} T \tag{12.3}$$

The CGF oscillation period is

$$T \sim T + \frac{2\pi}{\omega_0} \tag{12.4}$$

so the period in proper time is

$$t \sim t + \frac{2\pi}{\omega_0} \ell \frac{2\lambda}{g} = t + 2\pi \ell_0 \frac{2\lambda}{g} = t + 2\pi \frac{\hbar}{m_{\mathrm{Higgs}}} \frac{2\lambda}{g} = t + \frac{2\pi\hbar}{m_W} \tag{12.5}$$

The proper time frequency is

$$\omega_{\mathrm{proper}} = \frac{m_W}{\hbar} \tag{12.6}$$

References

[1] D. Friedan, "A theory of the dark matter," arXiv:2203.12405 [astro-ph.CO]. March, 2022.

[2] D. Friedan, "Dark matter stars," arXiv:2203.12181 [astro-ph.CO]. March, 2022.