Numerical solution of the TOV equations for CGF stars

This Sagemath notebook does calculations for the paper *Dark matter stars*.

Preamble

In [1]:
```
%display latex
LE = lambda latex_string: LatexExpr(latex_string);
```

In [2]:
```
from timeit import default_timer as timer
# start=timer()
# end=timer()
```

In [3]:
```
from mpmath import mp
from mpmath import mpf,mpc
import sage.libs.mpmath.all as a
mp.pretty = True
```

Code to set floating point precision

mpf(x) converts sage floating point numbers to mp numbers
Reals(x) or **myR(x)** converts mp numbers to sage

setting **mp.dps** tells mpmath to use decimal precision **mp.dps** or a small amount more
Sagemath decimal precision is set to mpmath decimal precision + 3

In [4]:
```
def set_precision(decimal_precision=20):
    global RealNumber,Reals,myR,sage_binary_precision,sage_decimal_precision
    mp.dps = decimal_precision
    binary_precision=mp.prec
    sage_binary_precision=binary_precision+10
    sage_decimal_precision = floor(sage_binary_precision/log(10,2))
    Reals = RealField(sage_binary_precision)
    RealNumber = Reals
    myR = Reals
    pretty_print("mp.dps = ",mp.dps,"   mp decimal precision = ", floor(mp.prec/log(10,2)),\
                "      sage decimal precision = ", sage_decimal_precision )
set_precision(decimal_precision=20)
```

mp.dps =20   mp decimal precision =21   sage decimal precision =24

Numerical parameters

Copied from *A theory of the dark matter*

In [5]:
```
rhohat_EW_val = 7.974153922874089
c_b_val = 0.24289366755876735
#
g2_val   =  0.4262847210445738
lambda_val = 0.5080829235055462
#
lam2_val = lambda_val^2
rhohat_EW_inv_val = 1/rhohat_EW_val
#
rhohat_EW_mpf = mpf(7.974153922874089)
c_a_val = 0.9924810876684255
```

CGF scale

In [6]:
```
m,kg,J,GeV,Mpc,pc,km = var('m,kg,J,GeV,Mpc,pc,km')
s = var('s',domain='positive'); assume(s,'real')
G      = RDF(6.67430e-11)*m^3*kg^(-1)* s^(-2)
c      = RDF(2.99792458e8) * m * s^(-1)
hbar   = RDF(1.054571817e-34)*J*s
```

```
H0    = RDF(6.74e1) *km * s^(-1) * Mpc^(-1)
Msun  = RDF(1.9884e30)*kg
GMsun = RDF(1.3271244e20)*m^3/s^2
m_Higgs = RDF(125.10)*GeV
#
J_per_GeV  = RDF(1e9*1.602176634e-19)*J/GeV
J_per_kg   =   RDF(2.99792458e8^2) *J/kg
m_per_pc = RDF(3.085677581 * 1e16) * m/pc
GeV_per_kg   = J_per_kg/J_per_GeV
pc_per_Mpc = RDF(1e6) *pc/Mpc
m_per_km = RDF(1e3) *m/km
#
m_Higgs_kg = m_Higgs/GeV_per_kg
#
rho_b = m_Higgs_kg^4/(hbar*c/J_per_kg )^3
r_b_s= (4*RDF(pi)*G*rho_b)^(-1/2)
r_b_m = r_b_s*c
m_b_kg = (r_b_m/G)*c^2
m_b_J = m_b_kg*J_per_kg
m_b_solar = m_b_kg/Msun
pretty_print(LE(r"\frac{G M_S}{G} = "),GMsun/G)
pretty_print(LE(r"\rho_b = "),rho_b)
pretty_print(LE(r"r_b = "),r_b_s,LE(r" = "),r_b_m)
pretty_print(LE(r"m_b = "),m_b_kg,LE(r" = "),m_b_solar,LE(r"M_{\odot} = "), m_b_J)
#print("rho_b = ", rho_b)
#print("r_b = ", r_b_s, " = ", r_b)
#print("m_b = ", m_b, " = ", m_b_J)
```

$$\frac{GM_S}{G} = 1.988409870698051 \times 10^{30}\, kg$$

$$\rho_b = \frac{5.682491786274899 \times 10^{28}\, kg}{m^3}$$

$$r_b = \left(1.4485143361532907 \times 10^{-10}\right)\, s = 0.04342536732836333\, m$$

$$m_b = 5.84762054071815 \times 10^{25}\, kg = 2.9408673007031534 \times 10^{-05} M_{\odot} = 5.255579244258227 \times 10^{42}\, J$$

In [7]:

```
pretty_print(LE(r"c_a= "), RDF(c_a_val),\
             LE(r"\qquad \pi c_a^{1/2} = "), RDF(pi*c_a_val^(1/2)),\
             LE(r"\qquad\pi c_a^{3/2} = "), RDF(pi*c_a_val^(3/2)))
pretty_print(LE(r"\frac{3\pi c_a^{5/2}}{4} = "), RDF(3*pi*c_a_val^(5/2)/4))
```

$$c_a = 0.9924810876684255 \qquad \pi c_a^{1/2} = 3.1297596889937487 \qquad \pi c_a^{3/2} = 3.1062273002733085$$

$$\frac{3\pi c_a^{5/2}}{4} = 2.312153887140458$$

Code

The TOV equations form an ode with independent variable $\hat{r}$ and dependent variables $\hat{p}$, $\hat{m}$, $\hat{e}$.

There are two odes:

- an ode for the core, using the high density equation of state,
- an ode for the outer shell, using the low density equation of state.

TOV equations

In [8]:

```
m, E_tot, rho,p,r,G= var('m,E_tot,rho,p,r,G')
dpdr_sym = var('dpdr_sym',latex_name=r"\frac{dp}{dr}")
dmdr_sym = var('dmdr_sym',latex_name=r"\frac{dm}{dr}")
dEdr_sym = var('dEdr_sym',latex_name=r"\frac{dE_{tot}}{dr}")
#
dpdr = -(rho+p)*(G*m+4*pi*G*r^3*p)/(r*(r-2*G*m)); dpdr
dmdr = 4*pi*r^2*rho
dEdr = 4*pi*rho*r^2*(r/(r-2*G*m))^(1/2)
```

```
#dEdr = 4*pi*rho*r^2*sqrt(r/(r-2*G*m))
pretty_print(LE(r"\text{TOV equations} \qquad"),\
    dpdr_sym," = ",dpdr,LE(r"\qquad"), dmdr_sym," = ",dmdr,LE(r"\qquad"),dEdr_sym," = ",dEdr)
```

TOV equations $\frac{dp}{dr} = \frac{(4\pi G p r^3 + Gm)(p+\rho)}{(2Gm - r)r}$ $\frac{dm}{dr} = 4\pi r^2 \rho$ $\frac{dE_{tot}}{dr} = 4\pi r^2 \rho \sqrt{-\frac{r}{2Gm - r}}$

In [9]:

```
mhat = var('mhat',latex_name=r"\hat m")
phat = var('phat',latex_name=r"\hat p")
rhohat = var('rhohat',latex_name=r"\hat \rho")
rhat = var('rhat',latex_name=r"\hat r")
ehat = var('ehat',latex_name=r"\hat e")
r_b,m_b= var('r_b,m_b')
dmhatdrhat_sym = var('dmhatdrhat_sym',latex_name=r"\frac{d\hat m}{d\hat r}")
dehatdrhat_sym = var('dehatdrhat_sym',latex_name=r"\frac{d\hat e}{d\hat r}")
dphatdrhat_sym = var('dphatdrhat_sym',latex_name=r"\frac{d\hat p}{d\hat r}")
#
rho = rhohat/(4*pi*G*r_b^2)
p   = phat/(4*pi*G*r_b^2)
m   = mhat*r_b/G
E_tot = ehat *r_b/G
r   = rhat*r_b
pretty_print(LE(r"\text{change to dimensionless variables}\qquad"),\
    LE(r"r = "),r,LE(r"\qquad \rho = "),rho,LE(r"\qquad p = "),p,\
    LE(r"\qquad m = "),m,LE(r"\qquad E_{tot} = "),E_tot)
print('\n')
#
dmhatdrhat = dmdr.subs(m=m,p=p,rho=rho,r=r) * diff(r,rhat)/diff(m,mhat)
dphatdrhat = (dpdr.subs(m=m,p=p,rho=rho,r=r) * diff(r,rhat)/diff(p,phat)).simplify_full()
dehatdrhat = (dEdr.subs(m=m,p=p,rho=rho,r=r) * diff(r,rhat)/diff(E_tot,ehat)).simplify_full()
pretty_print(LE(r"\text{dimensionless TOV equations}\qquad"),\
    dphatdrhat_sym," = ",dphatdrhat, LE(r"\qquad"),\
    dmhatdrhat_sym," = ",dmhatdrhat,LE(r"\qquad"),\
    dehatdrhat_sym," = ",dehatdrhat)
```

change to dimensionless variables $r = r_b \hat{r}$ $\rho = \frac{\hat{\rho}}{4\pi G r_b^2}$ $p = \frac{\hat{p}}{4\pi G r_b^2}$ $m = \frac{\hat{m} r_b}{G}$ $E_{tot} = \frac{\hat{e} r_b}{G}$

dimensionless TOV equations $\frac{d\hat{p}}{d\hat{r}} = \frac{\hat{p}^2\hat{r}^3 + \hat{m}\hat{p} + \left(\hat{p}\hat{r}^3 + \hat{m}\right)\hat{\rho}}{2\hat{m}\hat{r} - \hat{r}^2}$ $\frac{d\hat{m}}{d\hat{r}} = \hat{r}^2\hat{\rho}$ $\frac{d\hat{e}}{d\hat{r}} = \hat{r}^2\hat{\rho}\sqrt{-\frac{\hat{r}}{2\hat{m} - \hat{r}}}$

Prepare the core ode

In [10]:

```
c_b = var('c_b')
rhohat_EW = var('rhohat_EW',latex_name=r"\hat \rho_{EW}")
#
phat = (rhohat - c_b*rhohat_EW)/3
pretty_print(LE(r"\text{high density equation of state}\qquad"),LE(r"\hat p = "),phat)
```

high density equation of state $\hat{p} = -\frac{1}{3} c_b \hat{\rho}_{EW} + \frac{1}{3}\hat{\rho}$

In [11]:

```
drhohatdrhat_sym = var('drhohatdrhat_sym',latex_name=r"\frac{d\hat \rho}{d\hat r}")
#
drhohatdrhat = (dphatdrhat.subs(phat=phat)/diff(phat,rhohat)).simplify_rational()
pretty_print(LE(r"\text{substitute for }\hat p\qquad"),\
    drhohatdrhat_sym," = ",drhohatdrhat)
```

substitute for $\hat{p}$ $\frac{d\hat{\rho}}{d\hat{r}} = \frac{c_b^2\hat{r}^3{\hat{\rho}_{EW}}^2 + 4\hat{r}^3\hat{\rho}^2 + 12\hat{m}\hat{\rho} - \left(5c_b\hat{r}^3\hat{\rho} + 3c_b\hat{m}\right)\hat{\rho}_{EW}}{3\left(2\hat{m}\hat{r} - \hat{r}^2\right)}$

change variables

The density $\hat\rho$ decreases monotonically with the radius $\hat r$ so we can take $\hat\rho$ as the independent variable in the ode. This allows integrating the ode from the initial condition $\hat\rho(0)$ down to $\hat\rho = \hat\rho_{EW}$ which is the density at the outer surface of the core.
The ode solver wants the independent variable to increase, so we use

$$u = \frac{\hat\rho_{EW}}{\hat\rho} \qquad u_0 \le u \le 1$$

as independent variable. To simplify the ode, we change dependent variables to

$$w = \frac{\hat m}{\hat\rho_{EW}\hat r^3} \qquad x = \hat r^2 \qquad z = \frac{\hat e}{\hat\rho_{EW}}$$

In [12]:

```
rhohat_EW_inv = var('rhohat_EW_inv',latex_name=r"\hat \rho_{EW}^{-1}")
u = var('u')
w = var('w')
x = var('x')
z = var('z')
#
dudrhat_sym = var('dudrhat_sym',latex_name=r"\frac{du}{d\hat r}")
dwdrhat_sym = var('dwdrhat_sym',latex_name=r"\frac{dw}{d\hat r}")
dzdrhat_sym = var('dzdrhat_sym',latex_name=r"\frac{dz}{d\hat r}")
dwdx_sym = var('dwdx_sym',latex_name=r"\frac{dw}{dx}")
dzdx_sym = var('dzdx_sym',latex_name=r"\frac{dz}{dx}")
dxdu_sym = var('dxdu_sym',latex_name=r"\frac{dx}{du}")
dwdu_sym = var('dwdu_sym',latex_name=r"\frac{dw}{du}")
dzdu_sym = var('dzdu_sym',latex_name=r"\frac{dz}{du}")
#
mhat = w*rhat^3*rhohat_EW
ehat = z*rhohat_EW
rhohat = rhohat_EW/u
#
dwdrhat=((dmhatdrhat.subs(mhat=mhat,rhohat=rhohat)- diff(mhat,rhat))/diff(mhat,w)).simplify_full()
dudrhat= (drhohatdrhat.subs(mhat=mhat,rhohat=rhohat)/diff(rhohat,u)).simplify_full()
dzdrhat = (dehatdrhat.subs(mhat=mhat,rhohat=rhohat)/rhohat_EW).simplify_full()
rhohat_EW = 1/rhohat_EW_inv
dudrhat = (dudrhat.subs(rhohat_EW=rhohat_EW)).simplify_rational()
#pretty_print(dwdrhat_sym," = ",dwdrhat,LE(r"\qquad"),dudrhat_sym," = ",dudrhat)
#
rhat = sqrt(x)
#
dwdx = ((dwdrhat.subs(rhat=rhat)*diff(rhat,x)).simplify_full()).factor()
dzdx = ((dzdrhat.subs(rhat=rhat)*diff(rhat,x)).simplify_full()).factor()
dxdu = ((1/(dudrhat.subs(rhat=rhat)*diff(rhat,x))).simplify_full()).factor()
pretty_print(LE(r"\hat \rho = "),rhohat, LE(r"\qquad\hat r = "),rhat,LE(r"\qquad\hat m = "),mhat)
print('\n')
pretty_print(dwdx_sym," = ",dwdx,LE(r"\qquad"),dzdx_sym," = ",dzdx)
print('\n')
pretty_print(dxdu_sym," = ",dxdu,LE(r"\qquad"),dwdu_sym," = ",dwdx_sym *dxdu_sym,\
             LE(r"\qquad"),dzdu_sym," = ",dzdx_sym *dxdu_sym)
```

$$\hat\rho = \frac{\hat\rho_{EW}}{u} \qquad \hat r = \sqrt{x} \qquad \hat m = \hat r^3 \hat\rho_{EW} w$$

$$\frac{dw}{dx} = -\frac{3\,uw - 1}{2\,ux} \qquad \frac{dz}{dx} = \frac{\sqrt{x}\sqrt{-\frac{1}{2\,\hat\rho_{EW} wx - 1}}}{2\,u}$$

$$\frac{dx}{du} = -\frac{6\left(2\,wx - \hat\rho_{EW}^{-1}\right)}{(c_b u - 3\,uw - 1)(c_b u - 4)} \qquad \frac{dw}{du} = \frac{dw}{dx}\frac{dx}{du} \qquad \frac{dz}{du} = \frac{dx}{du}\frac{dz}{dx}$$

make ode derivatives into functions callable by the ode solver

$\frac{dw}{du}, \frac{dx}{du}, \frac{dz}{du}$

In [13]:

```
dwdu = ((dwdx*dxdu).simplify_full()).factor()
dwdu_num = dwdu.subs(rhohat_EW_inv=rhohat_EW_inv_val,c_b=c_b_val)
```

```
dxdu_num = dxdu.subs(rhohat_EW_inv=rhohat_EW_inv_val,c_b=c_b_val)
#
dwdu_fn=fast_callable(dwdu_num,vars=[u,w,x])
dxdu_fn=fast_callable(dxdu_num,vars=[u,w,x])
#
dzdu_fn = lambda u,w,x: (mp.sqrt(x/(1-2*rhohat_EW_mpf*w*x))/(2*u)) *dxdu_fn(u,w,x)
```

prepare the initial condition

The ode is singular at $\hat{r} = 0$ so start the ode at $u = u_0(1+\epsilon)$.

In [14]:

```
uu0,xx,bb_1,aa_1 = var('uu0,xx,bb_1,aa_1')
usubs = uu0*(1+xx)
wsubs = (1+aa_1*xx)/(3*uu0)
xsubs = bb_1*xx
rhs_w =( ( dwdu.subs(u==usubs,w==wsubs,x== xsubs) ).simplify_rational() ).subs(xx==0)
rhs_x =( ( dxdu.subs(u==usubs,w==wsubs,x== xsubs) ).simplify_rational() ).subs(xx==0)
sol = solve([diff(wsubs,xx) == rhs_w, diff(xsubs,xx) == rhs_x],aa_1,bb_1,solution_dict=True)
a_1 = sol[0][aa_1]
b_1 = sol[0][bb_1]
pretty_print(LE(r"u = u_0(1+\epsilon)"),LE(r"\qquad w = \frac{1+a_1\epsilon}{3 u_0}"),\
    LE(r"\qquad x = b_1 \epsilon \qquad z = \frac13 \hat\rho_0\hat r^3 = \frac{x^{3/2}}{3u_0} "))
print('\n')
pretty_print(LE(r"a_1 = "),a_1,LE(r"\qquad b_1 = "),b_1)
a_1_fn = fast_callable(a_1.subs(c_b == c_b_val,rhohat_EW_inv == rhohat_EW_inv_val),vars=[uu0])
b_1_fn = fast_callable(b_1.subs(c_b == c_b_val,rhohat_EW_inv == rhohat_EW_inv_val),vars=[uu0])
```

$$u = u_0(1+\epsilon) \qquad w = \frac{1+a_1\epsilon}{3u_0} \qquad x = b_1\epsilon \qquad z = \frac{1}{3}\hat{\rho}_0\hat{r}^3 = \frac{x^{3/2}}{3u_0}$$

$$a_1 = -\frac{3}{5} \qquad b_1 = \frac{6\,\hat{\rho}_{EW}^{-1}}{c_b^2 uu_0^2 - 6\,c_b uu_0 + 8}$$

define a function to convert ode-solver output to star data

The ode-solver outputs **y=[w,x,z]**.
The star data is $(\hat{r}, \hat{m}, \hat{e}, \hat{\rho}, \hat{p})$

In [15]:

```
mhat_wx= mhat.subs(rhat=rhat);mhat_wx
mhat_wx_num= mhat_wx.subs(rhohat_EW = rhohat_EW_val)
mhat_wx_fn = fast_callable(mhat_wx_num,vars=[w,x])
#
rhat_x_fn= fast_callable(rhat,vars=[x])
#
rhohat_num = rhohat.subs(rhohat_EW = rhohat_EW_val); rhohat_num
rhohat_u_fn = fast_callable(rhohat_num,vars=[u])
#
phat1 = phat.subs(rhohat=rhohat)
phat_num= phat1.subs(rhohat_EW=rhohat_EW_val,c_b=c_b_val); phat_num
phat_u_fn = fast_callable(phat_num,vars=[u])
#
pretty_print(LE(r"\hat r = "),rhat,LE(r"\qquad\hat m = "),mhat_wx,\
             LE(r"\qquad\hat e = z \hat \rho_{EW}"),LE(r"\qquad\hat \rho = "),\
            rhohat, LE(r"\qquad\hat p = "),phat)
#
def rhat_mhat_ehat_rhohat_phat(u,y):
    w=y[0]
    x=y[1]
    z=y[2]
    ehat = z*rhohat_EW_mpf
    return (rhat_x_fn(x),mhat_wx_fn(w,x),ehat,rhohat_u_fn(u),phat_u_fn(u))
```

$$\hat{r} = \sqrt{x} \qquad \hat{m} = \hat{\rho}_{EW} w x^{\frac{3}{2}} \qquad \hat{e} = z\hat{\rho}_{EW} \qquad \hat{\rho} = \frac{\hat{\rho}_{EW}}{u} \qquad \hat{p} = -\frac{1}{3}\,c_b\hat{\rho}_{EW} + \frac{1}{3}\,\hat{\rho}$$

(rhohat_norm_0_actual, core_ode) = make_core_ode(rhohat_norm_0)

rhohat_norm_0 $= \hat{\rho}_{norm}(0)$ is the initial condition at $\hat{r} = 0$.

$$\hat\rho_{norm} = \frac{\hat\rho}{\hat\rho_{EW}} = \frac{1}{u}$$

rhohat_norm_0_actual = $(1-\epsilon)\hat\rho_{norm}(0)$ is the actual initial value given to the ode-solver.

core_ode : $\hat\rho_{norm} \mapsto (\hat r, \hat m, \hat e, \hat\rho, \hat p)$ is the solution of the ode for

$$\hat\rho_{norm} \le (1-\epsilon)\hat\rho_{norm}(0)$$

The actual ode-solver is **[w,x,z] = the_odefun(u)** for

$$(1+\epsilon)u_0 \le u$$

In [16]:

```
ode_tol = None
ode_degree = None
epsilon=mpf(1e-20)
F_core = lambda u,y: [dwdu_fn(u,y[0],y[1]),dxdu_fn(u,y[0],y[1]),dzdu_fn(u,y[0],y[1])]
#
def make_core_ode(rhohat_norm_0):
    rhohat_norm_0_actual = mpf(rhohat_norm_0) - epsilon
    u0 = mpf(1/rhohat_norm_0_actual)
    a_1_val = a_1_fn(u0)
    b_1_val = b_1_fn(u0)
    w0 = mpf((1-a_1_val*epsilon)/(3*u0))
    x0= mpf(b_1_val*epsilon)
    z0 = mp.power(x0,3/2)/(3*u0)
    the_odefun = mp.odefun(F_core,u0,[w0,x0,z0],tol=ode_tol,degree=ode_degree)
    core_ode = lambda rhohat_norm: rhat_mhat_ehat_rhohat_phat(1/rhohat_norm,\
                the_odefun(1/rhohat_norm))
    return (rhohat_norm_0_actual,core_ode)
```

Prepare the shell ode

The low density equation of state is expressed implicitly

$$\hat\rho, \hat p = \hat\rho(k^2), \hat p(k^2) \qquad \frac{1}{2} \ge k^2 \ge 0 \qquad \hat\rho(1/2) = \hat\rho_{EW} \qquad \hat\rho(0) = 0$$

The functions $\hat\rho(k^2)$, $\hat p(k^2)$ are expressed by a series of formulas listed in Table 2 of *A theory of the dark matter*.
The formulas involve the the ratio K/E where $K(k)$ and $E(k)$ are the complete elliptic integrals. k^2 is the *elliptic parameter*.
The usual notation is $m = k^2$.

The TOV equations are solved with

$$\text{independent variable} \quad \sigma = 1-2m \qquad \text{dependent variables} \quad \hat r,\ \hat m,\ \hat e$$

During the preparation of the ode it is convenient to use

$$s = -\ln m \qquad \frac{d}{ds} = -m\frac{d}{dm} = \frac{1-\sigma}{2}\frac{d}{d\sigma}$$

code to define functions of m

The following code enters the Table 2 formulas as SageMath symbolic expressions.

In [17]:

```
latex_name={}
latex_name['alpha2_b2']      = r"\alpha^2\langle b^2\rangle"
latex_name['alpha2_mu2']    = r"\alpha^2\mu^2"
latex_name['alpha4_ECGF']    = r"\alpha^{4} E_{\mathrm{EW}}"
latex_name['alpha2_ahat2']    = r"\alpha^2 \hat a^2"
latex_name['rhohat']    = r"\hat \rho"
latex_name['phat'] = r"\hat p"
latex_name['lam2'] = r"\lambda^2"
latex_name['g2'] = r"g^2"
latex_name['EoverK'] = r"\frac{E}{K}"
latex_name['m'] = r"m"
for vstr,form in latex_name.items():
    var(vstr,latex_name=latex_name[vstr])
#
formula={}
formula['alpha2_b2']          = m-1+EoverK
```

```
formula['alpha2_mu2']       = 1-2*m
formula['alpha4_ECGF']       = m*(1-m)/2
formula['alpha2_ahat2']     = 3*alpha2_b2/2 + (4*lam2/g2)*alpha2_mu2
formula['rhohat']     = ((3/g2)*alpha4_ECGF+(9/(32*lam2))*alpha2_b2^2)/alpha2_ahat2^2
formula['phat']     = ( (1/g2)*(alpha4_ECGF-alpha2_mu2*alpha2_b2)\
                    - (9/(32*lam2))*alpha2_b2^2 )/alpha2_ahat2^2
pretty_print("formulas from Table 2")
print('\n')
for vstr,form in formula.items():
    pretty_print(LE(latex_name[vstr]),' = ', formula[vstr], hold= True)
    print('\n')
```

formulas from Table 2

$$\alpha^2\langle b^2\rangle \;= \frac{E}{K} + m - 1$$

$$\alpha^2\mu^2 \;= -2\,m + 1$$

$$\alpha^4 E_{\rm EW} \;= -\frac{1}{2}\,(m-1)m$$

$$\alpha^2\hat{a}^2 \;= \frac{3}{2}\,\alpha^2\langle b^2\rangle + \frac{4\,\alpha^2\mu^2\lambda^2}{g^2}$$

$$\hat{\rho} \;= \frac{3\left(\frac{3\,\alpha^2\langle b^2\rangle^2}{\lambda^2} + \frac{32\,\alpha^4 E_{\rm EW}}{g^2}\right)}{32\,\alpha^2\hat{a}^{2\,2}}$$

$$\hat{p} \;= -\frac{\frac{9\,\alpha^2\langle b^2\rangle^2}{\lambda^2} + \frac{32\,\left(\alpha^2\langle b^2\rangle\alpha^2\mu^2 - \alpha^4 E_{\rm EW}\right)}{g^2}}{32\,\alpha^2\hat{a}^{2\,2}}$$

Next, substitutions are made in the formulas to define functions

$$m,\; E/K \mapsto \hat{\rho},\; \hat{p},\; \frac{d\hat{p}}{d\sigma}$$

For efficiency, the elliptic integrals $E,\ K$ will be calculated separately.

In [18]:

```
expr={}
for var_str in list(formula.keys()):
    formula_expanded = eval(preparse(str(formula[var_str])))
    locals()[var_str] = formula_expanded
    expr[var_str] = formula_expanded

mdot = -m
EoverKdot = 1/2 -EoverK +EoverK^2/(2*(1-m))
dot = lambda y: mdot*y.diff(m) + EoverKdot*y.diff(EoverK)
#
expr['dphatds'] = dphatds = dot(phat)

expr_n={}
for var_str,expre in expr.items():
    expr_n[var_str] = expre.subs(lam2==lam2_val,g2==g2_val)

fast={}
for var_str,expre in expr_n.items():
    fast[var_str] = fast_callable(expre,vars=[m,EoverK])
pretty_print(LE(r"\text{fast_callable versions of}"),list(fast.keys()))
```

fast_callable versions of $[\texttt{alpha2\_b2}, \texttt{alpha2\_mu2}, \texttt{alpha4\_ECGF}, \texttt{alpha2\_ahat2}, \texttt{rhohat}, \texttt{phat}, \texttt{dphatds}]$

define functions

$$m \mapsto (\hat{\rho}, \hat{p}, d\hat{p}/ds) \qquad m \mapsto (\hat{\rho}, \hat{p})$$

In [19]:

```
def rhohat_phat_dphatds(m_arg):
    m_val = mpf(m_arg)
    KoverE_val = mp.ellipe(m_val)/mp.ellipk(m_val)
    rhohat_val = fast['rhohat'](m_val,KoverE_val)
    phat_val   = fast['phat'](m_val,KoverE_val)
    dphatds_val = fast['dphatds'](m_val,KoverE_val)
    return (rhohat_val,phat_val,dphatds_val)
```

In [20]:

```
def rhohat_phat(m_arg):
    m_val = mpf(m_arg)
    KoverE_val = mp.ellipe(m_val)/mp.ellipk(m_val)
    rhohat_val = fast['rhohat'](m_val,KoverE_val)
    phat_val   = fast['phat'](m_val,KoverE_val)
    return [rhohat_val,phat_val]
```

define the function that provides the derivatives of the dependent variables to the ode solver.

$$\sigma, y \mapsto \left[\frac{d\hat{r}}{d\sigma}, \frac{d\hat{m}}{d\sigma}, \frac{d\hat{e}}{d\sigma}\right]$$

In [21]:

```
def F_shell(sigma,y_arg):
    rhat = y_arg[0]
    mhat = y_arg[1]
    m = (1-sigma)/2
    (rhohat,phat,dphatds) = rhohat_phat_dphatds(m)
    drhatds = -dphatds*(rhat-2*mhat)/((rhohat+phat)*(mhat/rhat+rhat^2*phat))
    drhatdsigma = drhatds/(1-sigma)
    dmhatdsigma = rhat^2*rhohat*drhatds/(1-sigma)
    dehatdrhat = rhohat*rhat^2/mp.sqrt(1-2*mhat/rhat)
    dehatdsigma = dehatdrhat*drhatdsigma
    return [drhatdsigma,dmhatdsigma,dehatdsigma]
```

How stars are represented

A **star** is generated from an initial condition $\hat{\rho}_{norm}(0)$.

For $\hat{\rho}_{norm}(0) > 1$, first the **core_ode** is constructed and solved to the endpoint $u = 1$ to get the core radius, mass, and energy. These are then used as initial conditions for the **shell_ode** which is solved to the endpoint $\sigma = 1$ to get the star radius, mass, and energy.

For $\hat{\rho}_{norm}(0) \leq 1$, only the **shell_ode** is constructed and run.

The results are stored in a **star** which is a dictionary with two keys, **'data'** and **'ode'**.

- **star['data']** is a dictionary with keys
    - **'central density'**
    - **'mass'**
    - **'radius'**
    - **'energy'**
    - **'core mass'**
    - **'core radius'**
    - **'core energy'**
    - **'core density'**
    - **'core pressure'**
    - **'core profile'**
    - **'shell profile'**
- **star['ode']** is a dictionary with keys
    - **'core ode'**
    - **'shell ode'**
    - **'initial value'**

Numerical values are mpf floating point numbers. **'core profile'** and **'shell profile'** are not used here. Their values would be the intermediate values of the TOV solution for the star. Here we only calculate at the ode endpoints: the data at the boundary of the core and at the boundary of the star.

How stars are generated and stored

The object is to generate a set of stars more or less evenly spaced on the star curve in the M, R plane.

The even space is more or less accomplished by an algorithm.

The data is collected in subsets to allow for twiddling the decimal precision and the spacing parameters.

A subset of stars is called a **star_set**. This is a dictionary of stars. The dictionary key is the star's initial value converted to a string. The value is a **star**.

- **star_set[skey] = star**

The **star_set**s are stored in two lists, one for the stars with cores, the other for the coreless stars.

code to make a store with core

star = **make_star(** $\hat{\rho}_{norm}(0)$**)**

In [22]:

```
skey = lambda x : "{:.8e}".format(RDF(x))
```

In [23]:

```
ode_tol = None
ode_degree = None
epsilon=mpf(1e-20)
#
F_core = lambda u,y: [dwdu_fn(u,y[0],y[1]),dxdu_fn(u,y[0],y[1]),dzdu_fn(u,y[0],y[1])]
#
def make_star(rhohat_norm_0_arg,return_ode=False):
    rhohat_norm_0 = mpf(rhohat_norm_0_arg)
    star_key = skey(rhohat_norm_0)
# make core
    rhohat_norm_0_actual = rhohat_norm_0 - epsilon
    u0 = mpf(1/rhohat_norm_0_actual)
    a_1_val = a_1_fn(u0)
    b_1_val = b_1_fn(u0)
    w0 = mpf((1-a_1_val*epsilon)/(3*u0))
    x0=  mpf(b_1_val*epsilon)
    z0 = mp.power(x0,3/2)/(3*u0)
    the_odefun = mp.odefun(F_core,u0,[w0,x0,z0],tol=ode_tol,degree=ode_degree)
    core_ode = lambda rhohat_norm: \
                rhat_mhat_ehat_rhohat_phat(1/rhohat_norm,the_odefun(1/rhohat_norm))
    core_data = core_ode(1)
# make shell
    y_init = core_data[0:3]
    shell_odefun =  mp.odefun(F_shell,0,y_init,tol=ode_tol,degree=ode_degree)
    shell_ode = lambda m: tuple(shell_odefun(mpf(1-2*m)) + rhohat_phat(m))
    shell_data = shell_ode(0)
# store star
    data={}
    data['central density'] = rhohat_norm_0*rhohat_EW_mpf
    data['radius'] = shell_data[0]
    data['mass']   = shell_data[1]
    data['energy'] = shell_data[2]
#
    data['core radius']   = core_data[0]
    data['core mass']     = core_data[1]
    data['core energy']   = core_data[2]
    data['core density']  = core_data[3]
    data['core pressure'] = core_data[4]
#   data['core_profile']=None
#   data['shell_profile']=None
    ode={}
    if return_ode:
        ode['core ode'] = core_ode
        ode['shell ode'] = shell_ode
    ode['initial value'] = rhohat_norm_0_actual
    star={}
    star['star key'] =star_key
    star['data']     =data
```

```
    star['ode']       =ode
    return star
```

code to adaptively sample parameter space

The last 3 points are used to calculate the radius of curvature A/B (radius of the circle passing through the points).

At the beginning of a run, 3 stars have to be generated to initialize the adaptive algorithm.

The spacing in the $\hat{M}$-$\hat{R}$ plane to the next point is

$$\frac{A}{A+cB}\epsilon'$$

If the radius of curvature is large, the spacing is ϵ'. If the radius of curvature is small, the spacing is $\frac{A}{cB}\epsilon'$.

Then linear extrapolation gives the parameter step that will give this spacing.

In [24]:

```
def next():
    q10 = p[1]-p[0]
    q21 = p[2]-p[1]
    n10 = norm(q10)
    n21 = norm(q21)
    n20 = norm(q21+q10)
    A = n10*n21*n20
    B = 2*abs(q10[0]*q21[1]-q10[1]*q21[0])
#  A/B is the radius of curvature
#    1 try R = A/(A+B)
#    2 try R = A/(A+0.5*B)
    R = A/(A+0.25*B)
    return R*(t[2]-t[1])/n21
```

In [25]:

```
def initialize_adaptive(rhohat_norms):
    global star_set,p,t
    if len(rhohat_norms) != 3:
        print("ERROR: NEED 3 stars")
        return
    this_star_set={}
    p=[]
    t=[]
    print("star_set[{0}]".format(len(star_set)))
    for rhohat_norm_0 in rhohat_norms:
        start = timer()
        star = make_star(rhohat_norm_0 )
        end = timer()
        star_key = star['star key']
        this_star_set[star_key]=star
        star_data = star['data']
        print_star(star_key,star_data,time=end-start)
        rhat = star_data['radius']
        mhat = star_data['mass']
        p.append(vector([Reals(mhat),Reals(rhat)]))
        t.append(Reals(rhohat_norm_0))
    star_set.append(this_star_set)
def make_stars_adaptive(rhohat_norm_max=10,eps=.1):
    global star_set,p,t
    this_star_set ={}
    print("star_set[{0}]".format(len(star_set)))
    rhohat_norm_0 = t[2]
    while rhohat_norm_0 <= rhohat_norm_max:
        rhohat_norm_0 = rhohat_norm_0+eps*next()
        start = timer()
        star = make_star(rhohat_norm_0)
        end = timer()
        star_key = star['star key']
        this_star_set[star_key]=star
        star_data = star['data']
        print_star(star_key,star_data,time=end-start)
        rhat = star_data['radius']
        mhat = star_data['mass']
        p.append(vector([Reals(mhat),Reals(rhat)]))
        p.pop(0)
        t.append(Reals(rhohat_norm_0))
        t.pop(0)
    star_set.append(this_star_set)
```

code for coreless stars

For $\hat{\rho}_{norm}(0) \leq 1$ the star has no core. Only the shell ode is run.

Again $\hat{r} = 0$ is a singular point of the ode. The initial condition is again shifted by ϵ.

$$\sigma = \sigma_0 + \epsilon$$

$$c_1 = \hat{r}\frac{d\hat{r}}{d\sigma}_{/\sigma_0} \qquad \hat{r}^2 = 2c_1\epsilon$$

$$\frac{d\hat{p}}{d\hat{r}} = -(\hat{\rho}_0 + \hat{p}_0)\left(\frac{1}{3}\hat{\rho}_0 + \hat{p}_0\right)\hat{r} \qquad \hat{p} = \hat{p}_0 - \hat{\rho}_0 + \hat{p}_0)\left(\frac{1}{3}\hat{\rho}_0 + \hat{p}_0\right)\frac{1}{2}\hat{r}^2$$

$$\hat{m} = \hat{e} = \frac{1}{3}\hat{\rho}_0\hat{r}^3$$

In [26]:

```
ode_tol = None
ode_degree = None
epsilon=mpf(1e-20)
#
#
def make_coreless_star(m_0_arg,return_ode=False):
    if m_0_arg > 1/2:
        print('ERROR: star_key must be <= 1/2')
        return
    m_0 = mpf(m_0_arg)
#     m_0 = mpf(Reals(m_0_arg))
    star_key = skey(m_0)
#
    (rhohat_norm_0_actual,phat_s) = rhohat_phat(m_0)
    rhat_s = var('rhat_s',latex_name=r"\hat r")
    (rhohat_s_mpf,phat_s,dphatds_s) = rhohat_phat_dphatds(m_0)
    rhohat_s=Reals(rhohat_s_mpf)
    phat_s=Reals(phat_s)
    dphatds_s=Reals(dphatds_s)
    mhat_s = rhohat_s*rhat_s^3/3
    drhatds_s = -dphatds_s*(rhat_s-2*mhat_s)/((rhohat_s+phat_s)*(mhat_s/rhat_s+rhat_s^2*phat_s))
    c1 = mpf((rhat_s*drhatds_s).simplify_rational().subs(rhat_s =0))
    c1 = mp.sqrt(2*c1)
#
    m_0_actual = m_0*(1-c1*epsilon*epsilon)
    rhat_init = c1*epsilon
    mhat_init = rhohat_s*mp.power(rhat_init,3)/3
    ehat_init = mhat_init
    y_init = [rhat_init,mhat_init,ehat_init]
    sigma_init = mpf(1 - 2*m_0_actual)
    shell_odefun =  mp.odefun(F_shell,sigma_init,y_init,tol=ode_tol,degree=ode_degree)
    shell_ode = lambda m: tuple(shell_odefun(mpf(1-2*m)) + rhohat_phat(m))
    shell_data = shell_ode(0)
# store star
    data={}
    data['central density'] = rhohat_s_mpf
    data['radius'] = shell_data[0]
    data['mass']   = shell_data[1]
    data['energy'] = shell_data[2]
#   data['shell_profile']=None
    ode={}
    if return_ode:
        ode['shell ode'] = shell_ode
    ode['initial value'] = m_0_actual
    star={}
    star['star key'] =star_key
    star['data']     =data
    star['ode']      =ode
    return star
```

In [27]:

```
def initialize_adaptive_coreless(ms):
    global coreless_star_set,p,t
    if len(ms) != 3:
        print("ERROR: NEED 3 stars")
        return
    this_star_set={}
    p=[]
    t=[]
    print("coreless_star_set[{0}]".format(len(coreless_star_set)))
    for m_0 in ms:
```

```
        start = timer()
        star = make_coreless_star(m_0)
        end = timer()
        star_key = star['star key']
        this_star_set[star_key]=star
        star_data = star['data']
        print_star(star_key,star_data,time=end-start)
        rhat = star_data['radius']
        mhat = star_data['mass']
        p.append(vector([Reals(mhat),Reals(rhat)]))
        t.append(Reals(m_0))
    coreless_star_set.append(this_star_set)
def make_coreless_stars_adaptive(m_min=0.4,eps=.1):
    global coreless_star_set,p,t
    this_star_set ={}
    print("coreless_star_set[{0}]".format(len(coreless_star_set)))
    m_0 = t[2]
    while m_0 >= m_min:
        m_0 = m_0+eps*next()
        start = timer()
        star = make_coreless_star(m_0)
        end = timer()
        star_key = star['star key']
        this_star_set[star_key]=star
        star_data = star['data']
        print_star(star_key,star_data,time=end-start)
        rhat = star_data['radius']
        mhat = star_data['mass']
        p.append(vector([Reals(mhat),Reals(rhat)]))
        p.pop(0)
        t.append(Reals(m_0))
        t.pop(0)
    coreless_star_set.append(this_star_set)
```

code to initialize the adaptive algorithm from the last 3 stars in a star_set

In [28]:

```
def set_pt(a_star_set):
    global p,t
    pf = []
    tf = []
    for star_key in list(a_star_set.keys())[-3:]:
        star_data = a_star_set[star_key]['data']
        mass = star_data['mass']
        radius = star_data['radius']
        param = a_star_set[star_key]['ode']['initial value']
        tf.append(param)
        pf.append(vector([Reals(mass),Reals(radius)]))
    p = pf
    t = tf
```

code for printing and plotting

In [29]:

```
def print_star(star_key,star_data,time=0):
    rhat = star_data['radius']
    mhat = star_data['mass']
    ehat = star_data['energy']
    BE = ehat -mhat
    BEratio = BE/mhat
    print(star_key,"r=%.6e" % rhat, "m=%.6e" % mhat, "e=%.6e" % ehat,"BE=%.6e" % BE, "BE/M=%.6e" % B
#def print_star(star_key,star_data,time=0):
#    rhat = star_data['radius']
#    mhat = star_data['mass']
#    ehat = star_data['energy']
#    BE = ehat -mhat
#    BEratio = BE/mhat
#    print("'" + star_key + "'"," rhat= %.6e" % rhat, " mhat= %.6e" % mhat, " ehat= %.6e" % ehat," B
#    pretty_print("'" + star_key + "'", \
#        LE(r"\quad\hat r = "),"%.6e" % rhat, LE(r"\quad\hat m = "),\
#            "%.6e" % mhat, LE(r"\quad\hat e = "), "%.6e" % ehat,\
#        LE(r"\quad BE = "),"%.6e" % BE, LE(r"\quad BE/\hat m = "),\
#            "%.6e" % BEratio,LE(r"\quad"),'time = ',round(time,1))
```

In [30]:

```
def plots(star_dict,xstr,ystr,joined=True):
    if joined:
        the_plot = list_plot( [[star_data[xstr],star_data[ystr]]\
                        for star_key,star_data in star_dict.items()],\
```

```
                         marker=None,plotjoined=True,thickness=0.5)
    else:
        the_plot = list_plot( [[star_data[xstr],star_data[ystr]]\
                         for star_key,star_data in star_dict.items()],\
                         marker='.',size=10)
    return the_plot
```

In [31]:

```
def plot_stars(xstr,ystr):
    return list_plot( [[star_data[xstr],star_data[ystr]]\
                        for star_key,star_data in stars.items()],\
                         marker='.',size=10)
def plot_stars_joined(xstr,ystr):
    return list_plot( [[star_data[xstr],star_data[ystr]]\
                        for star_key,star_data in stars.items()],\
                         marker=None,plotjoined=True,thickness=0.5)
```

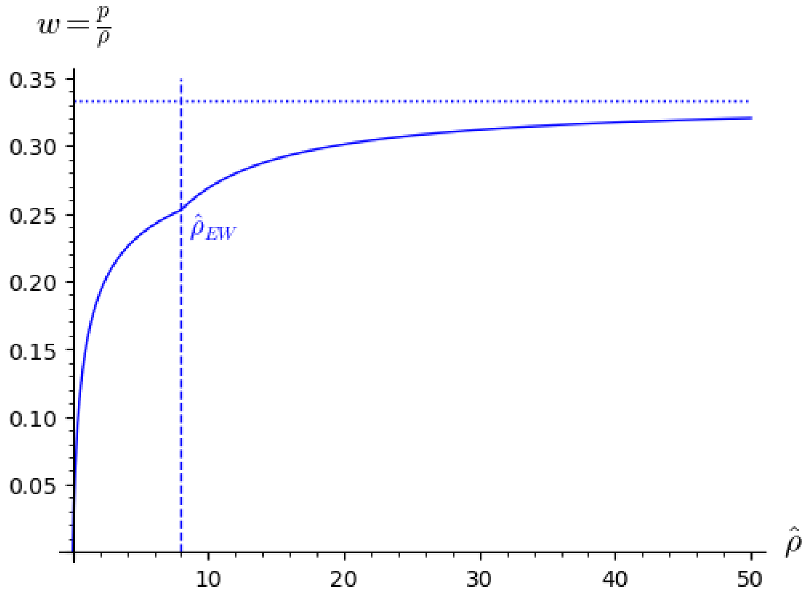
In [32]:

```
def plot_coreless_stars(xstr,ystr):
    return list_plot( [[star_data[xstr],star_data[ystr]]\
                        for star_key,star_data in coreless_stars.items()],\
                         marker='.',size=10)
def plot_coreless_stars_joined(xstr,ystr):
    return list_plot( [[star_data[xstr],star_data[ystr]]\
                        for star_key,star_data in coreless_stars.items()],\
                         marker=None,plotjoined=True,thickness=0.5)
```

Plot equation of state

In [33]:

```
rho_low = lambda m: rhohat_phat(m)[0]
w_low = lambda m: rhohat_phat(m)[1]/ rhohat_phat(m)[0]
# p_low =lambda m: rhohat_phat(m)[1]
#
w_low_density_plot = parametric_plot((rho_low,w_low), (m,1e-6,0.5))
rhohat_max =50
w_high_density_plot = plot(lambda xx: (1-c_b_val*rhohat_EW_val/xx)/3,(rhohat_EW_val, rhohat_max))
#
plt = w_low_density_plot+w_high_density_plot
#plt += text(r"$\hat\rho_{EW}$",(rhohat_EW_val-1,-.02),fontsize="large")
plt += text(r"$\hat\rho_{EW}$",(rhohat_EW_val+2.5,.24),fontsize="large")
plt += line([(rhohat_EW_val,0),(rhohat_EW_val,0.35)],linestyle="--")
plt +=line([(0,1/3),(rhohat_max,1/3)],linestyle=":")
plt.set_aspect_ratio(2*rhohat_max )
plt.set_axes_range(ymax=.35)
plt.axes_labels([r"$\hat \rho$",r"$w=\frac{p}{\rho}$"])
show(plt,fontsize=10,axes_labels_size=1.5)
plt.save('plots/rhohat_w.pdf',dpi=600,fontsize=10,axes_labels_size=1.5)
```



generate stars with cores

In [34]:

```
star_set=[]
initialize_adaptive([1.0001,1.0011,1.0021])
```

```
star_set[0]
1.00010000e+00 r=1.296425e+00 m=2.324147e-01 e=2.991668e-01 BE=6.675207e-02 BE/M=2.872110e-01 t=9
1.00110000e+00 r=1.296270e+00 m=2.323664e-01 e=2.991125e-01 BE=6.674607e-02 BE/M=2.872449e-01 t=8
1.00210000e+00 r=1.296115e+00 m=2.323182e-01 e=2.990582e-01 BE=6.674008e-02 BE/M=2.872788e-01 t=8
```

In [35]:

```
make_stars_adaptive(rhohat_norm_max=1e9,eps=.02)
```

```
star_set[1]
1.09387377e+00 r=1.282999e+00 m=2.281176e-01 e=2.943168e-01 BE=6.619929e-02 BE/M=2.901981e-01 t=7
1.19720666e+00 r=1.270457e+00 m=2.238962e-01 e=2.895208e-01 BE=6.562456e-02 BE/M=2.931026e-01 t=7
1.31989022e+00 r=1.258032e+00 m=2.194890e-01 e=2.844832e-01 BE=6.499422e-02 BE/M=2.961161e-01 t=6
1.46238263e+00 r=1.246289e+00 m=2.150529e-01 e=2.793804e-01 BE=6.432755e-02 BE/M=2.991243e-01 t=6
1.62513990e+00 r=1.235655e+00 m=2.107082e-01 e=2.743487e-01 BE=6.364054e-02 BE/M=3.020317e-01 t=6
1.80734628e+00 r=1.226476e+00 m=2.065668e-01 e=2.695175e-01 BE=6.295069e-02 BE/M=3.047474e-01 t=6
2.00469227e+00 r=1.219027e+00 m=2.027569e-01 e=2.650390e-01 BE=6.228216e-02 BE/M=3.071765e-01 t=6
2.20742739e+00 r=1.213444e+00 m=1.994188e-01 e=2.610852e-01 BE=6.166636e-02 BE/M=3.092304e-01 t=6
2.40080697e+00 r=1.209636e+00 m=1.966694e-01 e=2.578049e-01 BE=6.113555e-02 BE/M=3.108544e-01 t=6
2.57030402e+00 r=1.207268e+00 m=1.945460e-01 e=2.552554e-01 BE=6.070939e-02 BE/M=3.120567e-01 t=6
2.70870885e+00 r=1.205889e+00 m=1.929808e-01 e=2.533661e-01 BE=6.038533e-02 BE/M=3.129085e-01 t=6
2.81807217e+00 r=1.205100e+00 m=1.918383e-01 e=2.519813e-01 BE=6.014306e-02 BE/M=3.135092e-01 t=6
2.90547009e+00 r=1.204640e+00 m=1.909795e-01 e=2.509370e-01 BE=5.995756e-02 BE/M=3.139477e-01 t=6
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4.24746491e+04 r=1.402521e+00 m=1.951125e-01 e=2.534753e-01 BE=5.836279e-02 BE/M=2.991237e-01 t=7
6.17390691e+04 r=1.404196e+00 m=1.954417e-01 e=2.538435e-01 BE=5.840178e-02 BE/M=2.988195e-01 t=7
8.78613638e+04 r=1.405212e+00 m=1.957388e-01 e=2.541894e-01 BE=5.845063e-02 BE/M=2.986154e-01 t=7
1.16332669e+05 r=1.405649e+00 m=1.959482e-01 e=2.544412e-01 BE=5.849302e-02 BE/M=2.985127e-01 t=7
1.38232288e+05 r=1.405769e+00 m=1.960608e-01 e=2.545799e-01 BE=5.851914e-02 BE/M=2.984745e-01 t=7
1.49604811e+05 r=1.405791e+00 m=1.961078e-01 e=2.546387e-01 BE=5.853092e-02 BE/M=2.984630e-01 t=7
1.54479354e+05 r=1.405794e+00 m=1.961260e-01 e=2.546616e-01 BE=5.853564e-02 BE/M=2.984594e-01 t=7
1.57260978e+05 r=1.405794e+00 m=1.961359e-01 e=2.546742e-01 BE=5.853826e-02 BE/M=2.984576e-01 t=7
```

```
1.59615424e+05 r=1.405793e+00 m=1.961441e-01 e=2.546845e-01 BE=5.854042e-02 BE/M=2.984563e-01 t=7
1.61907453e+05 r=1.405792e+00 m=1.961518e-01 e=2.546943e-01 BE=5.854250e-02 BE/M=2.984551e-01 t=7
1.64212971e+05 r=1.405790e+00 m=1.961593e-01 e=2.547039e-01 BE=5.854454e-02 BE/M=2.984540e-01 t=7
1.66565743e+05 r=1.405788e+00 m=1.961668e-01 e=2.547134e-01 BE=5.854659e-02 BE/M=2.984531e-01 t=8
1.68996056e+05 r=1.405785e+00 m=1.961744e-01 e=2.547230e-01 BE=5.854867e-02 BE/M=2.984522e-01 t=7
1.71535991e+05 r=1.405781e+00 m=1.961820e-01 e=2.547328e-01 BE=5.855081e-02 BE/M=2.984514e-01 t=7
1.74221205e+05 r=1.405776e+00 m=1.961899e-01 e=2.547429e-01 BE=5.855302e-02 BE/M=2.984508e-01 t=7
1.77092755e+05 r=1.405771e+00 m=1.961980e-01 e=2.547533e-01 BE=5.855533e-02 BE/M=2.984502e-01 t=7
1.80199337e+05 r=1.405764e+00 m=1.962065e-01 e=2.547643e-01 BE=5.855778e-02 BE/M=2.984497e-01 t=7
1.83600187e+05 r=1.405756e+00 m=1.962155e-01 e=2.547759e-01 BE=5.856039e-02 BE/M=2.984494e-01 t=7
1.87368966e+05 r=1.405746e+00 m=1.962250e-01 e=2.547882e-01 BE=5.856321e-02 BE/M=2.984493e-01 t=7
1.91599138e+05 r=1.405734e+00 m=1.962353e-01 e=2.548016e-01 BE=5.856629e-02 BE/M=2.984493e-01 t=7
1.96411582e+05 r=1.405719e+00 m=1.962464e-01 e=2.548161e-01 BE=5.856968e-02 BE/M=2.984497e-01 t=7
2.01965631e+05 r=1.405700e+00 m=1.962585e-01 e=2.548320e-01 BE=5.857344e-02 BE/M=2.984505e-01 t=7
2.08475423e+05 r=1.405677e+00 m=1.962718e-01 e=2.548495e-01 BE=5.857767e-02 BE/M=2.984518e-01 t=7
2.16234679e+05 r=1.405646e+00 m=1.962865e-01 e=2.548690e-01 BE=5.858247e-02 BE/M=2.984539e-01 t=7
2.25655186e+05 r=1.405607e+00 m=1.963029e-01 e=2.548908e-01 BE=5.858797e-02 BE/M=2.984570e-01 t=7
2.37328057e+05 r=1.405556e+00 m=1.963210e-01 e=2.549153e-01 BE=5.859432e-02 BE/M=2.984618e-01 t=7
2.52123935e+05 r=1.405488e+00 m=1.963411e-01 e=2.549428e-01 BE=5.860170e-02 BE/M=2.984689e-01 t=7
2.71361575e+05 r=1.405395e+00 m=1.963631e-01 e=2.549734e-01 BE=5.861034e-02 BE/M=2.984794e-01 t=8
2.97099682e+05 r=1.405269e+00 m=1.963866e-01 e=2.550071e-01 BE=5.862043e-02 BE/M=2.984950e-01 t=7
3.32656335e+05 r=1.405094e+00 m=1.964106e-01 e=2.550427e-01 BE=5.863213e-02 BE/M=2.985182e-01 t=8
3.83555947e+05 r=1.404852e+00 m=1.964326e-01 e=2.550780e-01 BE=5.864542e-02 BE/M=2.985524e-01 t=8
4.59282128e+05 r=1.404520e+00 m=1.964482e-01 e=2.551080e-01 BE=5.865984e-02 BE/M=2.986021e-01 t=8
5.76511989e+05 r=1.404079e+00 m=1.964504e-01 e=2.551246e-01 BE=5.867420e-02 BE/M=2.986719e-01 t=8
7.64840300e+05 r=1.403527e+00 m=1.964301e-01 e=2.551164e-01 BE=5.868627e-02 BE/M=2.987642e-01 t=8
1.07562699e+06 r=1.402902e+00 m=1.963801e-01 e=2.550732e-01 BE=5.869307e-02 BE/M=2.988748e-01 t=8
1.59021594e+06 r=1.402295e+00 m=1.963018e-01 e=2.549940e-01 BE=5.869218e-02 BE/M=2.989895e-01 t=8
2.40577644e+06 r=1.401821e+00 m=1.962100e-01 e=2.548939e-01 BE=5.868390e-02 BE/M=2.990872e-01 t=8
3.54163811e+06 r=1.401548e+00 m=1.961284e-01 e=2.548005e-01 BE=5.867211e-02 BE/M=2.991516e-01 t=9
4.76209689e+06 r=1.401443e+00 m=1.960739e-01 e=2.547358e-01 BE=5.866197e-02 BE/M=2.991830e-01 t=9
5.64199217e+06 r=1.401419e+00 m=1.960468e-01 e=2.547030e-01 BE=5.865613e-02 BE/M=2.991945e-01 t=9
6.04996460e+06 r=1.401417e+00 m=1.960367e-01 e=2.546905e-01 BE=5.865376e-02 BE/M=2.991978e-01 t=9
6.21052475e+06 r=1.401417e+00 m=1.960331e-01 e=2.546859e-01 BE=5.865288e-02 BE/M=2.991989e-01 t=9
6.30897924e+06 r=1.401417e+00 m=1.960309e-01 e=2.546833e-01 BE=5.865235e-02 BE/M=2.991995e-01 t=9
6.39966777e+06 r=1.401417e+00 m=1.960290e-01 e=2.546809e-01 BE=5.865188e-02 BE/M=2.992000e-01 t=9
6.49116536e+06 r=1.401418e+00 m=1.960271e-01 e=2.546785e-01 BE=5.865141e-02 BE/M=2.992005e-01 t=9
6.58514464e+06 r=1.401418e+00 m=1.960252e-01 e=2.546761e-01 BE=5.865093e-02 BE/M=2.992010e-01 t=9
6.68281731e+06 r=1.401419e+00 m=1.960233e-01 e=2.546737e-01 BE=5.865045e-02 BE/M=2.992014e-01 t=9
6.78550817e+06 r=1.401420e+00 m=1.960213e-01 e=2.546713e-01 BE=5.864995e-02 BE/M=2.992019e-01 t=9
6.89472241e+06 r=1.401422e+00 m=1.960193e-01 e=2.546687e-01 BE=5.864942e-02 BE/M=2.992023e-01 t=8
7.01222132e+06 r=1.401423e+00 m=1.960172e-01 e=2.546661e-01 BE=5.864888e-02 BE/M=2.992027e-01 t=8
7.14012303e+06 r=1.401425e+00 m=1.960150e-01 e=2.546633e-01 BE=5.864829e-02 BE/M=2.992030e-01 t=8
7.28103549e+06 r=1.401427e+00 m=1.960127e-01 e=2.546604e-01 BE=5.864766e-02 BE/M=2.992034e-01 t=8
7.43823804e+06 r=1.401430e+00 m=1.960102e-01 e=2.546572e-01 BE=5.864698e-02 BE/M=2.992037e-01 t=8
7.61593594e+06 r=1.401434e+00 m=1.960075e-01 e=2.546537e-01 BE=5.864624e-02 BE/M=2.992040e-01 t=8
7.81962591e+06 r=1.401438e+00 m=1.960046e-01 e=2.546500e-01 BE=5.864541e-02 BE/M=2.992043e-01 t=8
8.05663307e+06 r=1.401443e+00 m=1.960014e-01 e=2.546459e-01 BE=5.864449e-02 BE/M=2.992045e-01 t=8
8.33691772e+06 r=1.401450e+00 m=1.959978e-01 e=2.546413e-01 BE=5.864345e-02 BE/M=2.992045e-01 t=8
8.67431730e+06 r=1.401459e+00 m=1.959939e-01 e=2.546362e-01 BE=5.864226e-02 BE/M=2.992044e-01 t=8
9.08850739e+06 r=1.401470e+00 m=1.959896e-01 e=2.546305e-01 BE=5.864089e-02 BE/M=2.992041e-01 t=8
9.60818449e+06 r=1.401485e+00 m=1.959848e-01 e=2.546241e-01 BE=5.863930e-02 BE/M=2.992034e-01 t=8
1.02763844e+07 r=1.401505e+00 m=1.959794e-01 e=2.546169e-01 BE=5.863744e-02 BE/M=2.992020e-01 t=8
1.11596481e+07 r=1.401533e+00 m=1.959736e-01 e=2.546089e-01 BE=5.863527e-02 BE/M=2.991998e-01 t=8
1.23643236e+07 r=1.401571e+00 m=1.959676e-01 e=2.546003e-01 BE=5.863274e-02 BE/M=2.991961e-01 t=8
1.40664539e+07 r=1.401624e+00 m=1.959617e-01 e=2.545915e-01 BE=5.862984e-02 BE/M=2.991903e-01 t=8
1.65679832e+07 r=1.401698e+00 m=1.959568e-01 e=2.545834e-01 BE=5.862662e-02 BE/M=2.991813e-01 t=8
2.04038101e+07 r=1.401798e+00 m=1.959545e-01 e=2.545778e-01 BE=5.862329e-02 BE/M=2.991678e-01 t=8
2.65423532e+07 r=1.401929e+00 m=1.959571e-01 e=2.545774e-01 BE=5.862031e-02 BE/M=2.991488e-01 t=8
3.67301766e+07 r=1.402085e+00 m=1.959668e-01 e=2.545852e-01 BE=5.861840e-02 BE/M=2.991242e-01 t=8
5.39286695e+07 r=1.402248e+00 m=1.959846e-01 e=2.546030e-01 BE=5.861833e-02 BE/M=2.990966e-01 t=9
8.22311000e+07 r=1.402386e+00 m=1.960079e-01 e=2.546282e-01 BE=5.862031e-02 BE/M=2.990711e-01 t=9
1.24208335e+08 r=1.402474e+00 m=1.960306e-01 e=2.546542e-01 BE=5.862357e-02 BE/M=2.990532e-01 t=9
1.73872855e+08 r=1.402511e+00 m=1.960470e-01 e=2.546736e-01 BE=5.862666e-02 BE/M=2.990439e-01 t=9
2.14355339e+08 r=1.402521e+00 m=1.960557e-01 e=2.546843e-01 BE=5.862861e-02 BE/M=2.990406e-01 t=9
2.35086916e+08 r=1.402522e+00 m=1.960591e-01 e=2.546886e-01 BE=5.862946e-02 BE/M=2.990397e-01 t=9
2.42908309e+08 r=1.402522e+00 m=1.960603e-01 e=2.546901e-01 BE=5.862975e-02 BE/M=2.990394e-01 t=9
2.46970250e+08 r=1.402522e+00 m=1.960609e-01 e=2.546908e-01 BE=5.862990e-02 BE/M=2.990393e-01 t=9
2.50525736e+08 r=1.402522e+00 m=1.960614e-01 e=2.546914e-01 BE=5.863003e-02 BE/M=2.990392e-01 t=9
2.54109072e+08 r=1.402522e+00 m=1.960618e-01 e=2.546920e-01 BE=5.863016e-02 BE/M=2.990391e-01 t=9
2.57797539e+08 r=1.402522e+00 m=1.960623e-01 e=2.546926e-01 BE=5.863029e-02 BE/M=2.990390e-01 t=9
2.61638244e+08 r=1.402521e+00 m=1.960628e-01 e=2.546932e-01 BE=5.863042e-02 BE/M=2.990390e-01 t=9
2.65683716e+08 r=1.402521e+00 m=1.960633e-01 e=2.546939e-01 BE=5.863055e-02 BE/M=2.990389e-01 t=9
2.69994113e+08 r=1.402521e+00 m=1.960638e-01 e=2.546945e-01 BE=5.863069e-02 BE/M=2.990388e-01 t=9
2.74640151e+08 r=1.402520e+00 m=1.960644e-01 e=2.546952e-01 BE=5.863084e-02 BE/M=2.990388e-01 t=9
2.79707214e+08 r=1.402520e+00 m=1.960649e-01 e=2.546959e-01 BE=5.863100e-02 BE/M=2.990387e-01 t=9
2.85300806e+08 r=1.402519e+00 m=1.960655e-01 e=2.546967e-01 BE=5.863117e-02 BE/M=2.990387e-01 t=9
2.91554029e+08 r=1.402519e+00 m=1.960661e-01 e=2.546975e-01 BE=5.863136e-02 BE/M=2.990386e-01 t=9
2.98638112e+08 r=1.402518e+00 m=1.960668e-01 e=2.546984e-01 BE=5.863156e-02 BE/M=2.990386e-01 t=9
3.06777585e+08 r=1.402517e+00 m=1.960676e-01 e=2.546994e-01 BE=5.863178e-02 BE/M=2.990386e-01 t=9
```

```
3.16272659e+08 r=1.402515e+00 m=1.960684e-01 e=2.547004e-01 BE=5.863204e-02 BE/M=2.990387e-01 t=9
3.27533004e+08 r=1.402513e+00 m=1.960693e-01 e=2.547016e-01 BE=5.863232e-02 BE/M=2.990387e-01 t=9
3.41129972e+08 r=1.402511e+00 m=1.960703e-01 e=2.547030e-01 BE=5.863265e-02 BE/M=2.990389e-01 t=9
3.57879481e+08 r=1.402508e+00 m=1.960715e-01 e=2.547045e-01 BE=5.863302e-02 BE/M=2.990391e-01 t=9
3.78977297e+08 r=1.402504e+00 m=1.960727e-01 e=2.547062e-01 BE=5.863346e-02 BE/M=2.990394e-01 t=9
4.06226515e+08 r=1.402499e+00 m=1.960741e-01 e=2.547080e-01 BE=5.863397e-02 BE/M=2.990399e-01 t=9
4.42432347e+08 r=1.402491e+00 m=1.960755e-01 e=2.547101e-01 BE=5.863457e-02 BE/M=2.990407e-01 t=9
4.92109729e+08 r=1.402481e+00 m=1.960771e-01 e=2.547124e-01 BE=5.863527e-02 BE/M=2.990419e-01 t=9
5.62791377e+08 r=1.402467e+00 m=1.960786e-01 e=2.547146e-01 BE=5.863606e-02 BE/M=2.990437e-01 t=9
6.67508046e+08 r=1.402447e+00 m=1.960797e-01 e=2.547167e-01 BE=5.863695e-02 BE/M=2.990465e-01 t=9
8.29543818e+08 r=1.402421e+00 m=1.960801e-01 e=2.547180e-01 BE=5.863785e-02 BE/M=2.990504e-01 t=9
1.09135005e+09 r=1.402386e+00 m=1.960792e-01 e=2.547179e-01 BE=5.863865e-02 BE/M=2.990559e-01 t=9
```

generate coreless stars

In [36]:

```
coreless_star_set=[]
initialize_adaptive_coreless([0.5,0.499,0.498])
```

```
coreless_star_set[0]
5.00000000e-01 r=1.296441e+00 m=2.324196e-01 e=2.991722e-01 BE=6.675267e-02 BE/M=2.872076e-01 t=11
4.99000000e-01 r=1.299815e+00 m=2.334598e-01 e=3.003398e-01 BE=6.688004e-02 BE/M=2.864735e-01 t=11
4.98000000e-01 r=1.303226e+00 m=2.344930e-01 e=3.014964e-01 BE=6.700343e-02 BE/M=2.857375e-01 t=11
```

In [37]:

```
make_coreless_stars_adaptive(m_min=0.01,eps=0.025)
```

```
coreless_star_set[1]
4.92774117e-01 r=1.321575e+00 m=2.397770e-01 e=3.073618e-01 BE=6.758479e-02 BE/M=2.818652e-01 t=11
4.87524670e-01 r=1.340742e+00 m=2.448846e-01 e=3.129496e-01 BE=6.806499e-02 BE/M=2.779472e-01 t=11
4.82258661e-01 r=1.360541e+00 m=2.498029e-01 e=3.182499e-01 BE=6.844708e-02 BE/M=2.740044e-01 t=11
4.76928416e-01 r=1.381030e+00 m=2.545698e-01 e=3.233070e-01 BE=6.873718e-02 BE/M=2.700131e-01 t=11
4.71558240e-01 r=1.402020e+00 m=2.591576e-01 e=3.280937e-01 BE=6.893606e-02 BE/M=2.660005e-01 t=11
4.66160607e-01 r=1.423382e+00 m=2.635528e-01 e=3.325991e-01 BE=6.904635e-02 BE/M=2.619830e-01 t=11
4.60741224e-01 r=1.445027e+00 m=2.677497e-01 e=3.368212e-01 BE=6.907144e-02 BE/M=2.579701e-01 t=11
4.55302500e-01 r=1.466891e+00 m=2.717467e-01 e=3.407616e-01 BE=6.901486e-02 BE/M=2.539676e-01 t=11
4.49845040e-01 r=1.488927e+00 m=2.755440e-01 e=3.444241e-01 BE=6.888014e-02 BE/M=2.499788e-01 t=11
4.44368505e-01 r=1.511102e+00 m=2.791426e-01 e=3.478132e-01 BE=6.867064e-02 BE/M=2.460056e-01 t=11
4.38872115e-01 r=1.533388e+00 m=2.825440e-01 e=3.509335e-01 BE=6.838955e-02 BE/M=2.420492e-01 t=11
4.33354925e-01 r=1.555767e+00 m=2.857498e-01 e=3.537897e-01 BE=6.803987e-02 BE/M=2.381100e-01 t=11
4.27815979e-01 r=1.578221e+00 m=2.887613e-01 e=3.563858e-01 BE=6.762447e-02 BE/M=2.341881e-01 t=12
4.22254394e-01 r=1.600738e+00 m=2.915797e-01 e=3.587258e-01 BE=6.714605e-02 BE/M=2.302837e-01 t=12
4.16669397e-01 r=1.623307e+00 m=2.942060e-01 e=3.608132e-01 BE=6.660720e-02 BE/M=2.263964e-01 t=12
4.11060344e-01 r=1.645921e+00 m=2.966410e-01 e=3.626513e-01 BE=6.601039e-02 BE/M=2.225262e-01 t=12
4.05426724e-01 r=1.668571e+00 m=2.988850e-01 e=3.642430e-01 BE=6.535803e-02 BE/M=2.186728e-01 t=12
3.99768157e-01 r=1.691252e+00 m=3.009385e-01 e=3.655909e-01 BE=6.465242e-02 BE/M=2.148360e-01 t=12
3.94084385e-01 r=1.713959e+00 m=3.028016e-01 e=3.666974e-01 BE=6.389584e-02 BE/M=2.110156e-01 t=12
3.88375263e-01 r=1.736686e+00 m=3.044742e-01 e=3.675647e-01 BE=6.309049e-02 BE/M=2.072113e-01 t=12
3.82640754e-01 r=1.759429e+00 m=3.059562e-01 e=3.681948e-01 BE=6.223855e-02 BE/M=2.034230e-01 t=12
3.76880914e-01 r=1.782185e+00 m=3.072474e-01 e=3.685895e-01 BE=6.134214e-02 BE/M=1.996506e-01 t=12
3.71095889e-01 r=1.804951e+00 m=3.083473e-01 e=3.687507e-01 BE=6.040338e-02 BE/M=1.958940e-01 t=13
3.65285904e-01 r=1.827723e+00 m=3.092555e-01 e=3.686798e-01 BE=5.942435e-02 BE/M=1.921529e-01 t=13
3.59451256e-01 r=1.850499e+00 m=3.099714e-01 e=3.683785e-01 BE=5.840713e-02 BE/M=1.884275e-01 t=13
3.53592307e-01 r=1.873276e+00 m=3.104944e-01 e=3.678482e-01 BE=5.735375e-02 BE/M=1.847175e-01 t=13
3.47709480e-01 r=1.896051e+00 m=3.108239e-01 e=3.670901e-01 BE=5.626628e-02 BE/M=1.810230e-01 t=13
3.41803251e-01 r=1.918823e+00 m=3.109591e-01 e=3.661058e-01 BE=5.514674e-02 BE/M=1.773441e-01 t=13
3.35874144e-01 r=1.941588e+00 m=3.108993e-01 e=3.648964e-01 BE=5.399716e-02 BE/M=1.736805e-01 t=13
3.29922729e-01 r=1.964346e+00 m=3.106438e-01 e=3.634633e-01 BE=5.281956e-02 BE/M=1.700326e-01 t=13
3.23949617e-01 r=1.987094e+00 m=3.101917e-01 e=3.618077e-01 BE=5.161595e-02 BE/M=1.664001e-01 t=13
3.17955453e-01 r=2.009829e+00 m=3.095424e-01 e=3.599308e-01 BE=5.038834e-02 BE/M=1.627833e-01 t=13
3.11940916e-01 r=2.032551e+00 m=3.086951e-01 e=3.578338e-01 BE=4.913875e-02 BE/M=1.591822e-01 t=13
3.05906718e-01 r=2.055258e+00 m=3.076489e-01 e=3.555181e-01 BE=4.786918e-02 BE/M=1.555968e-01 t=13
2.99853594e-01 r=2.077947e+00 m=3.064031e-01 e=3.529847e-01 BE=4.658163e-02 BE/M=1.520273e-01 t=13
2.93782307e-01 r=2.100616e+00 m=3.049569e-01 e=3.502350e-01 BE=4.527809e-02 BE/M=1.484737e-01 t=13
2.87693639e-01 r=2.123265e+00 m=3.033097e-01 e=3.472703e-01 BE=4.396057e-02 BE/M=1.449362e-01 t=13
2.81588393e-01 r=2.145892e+00 m=3.014608e-01 e=3.440918e-01 BE=4.263104e-02 BE/M=1.414149e-01 t=12
2.75467392e-01 r=2.168494e+00 m=2.994093e-01 e=3.407008e-01 BE=4.129150e-02 BE/M=1.379099e-01 t=12
2.69331471e-01 r=2.191070e+00 m=2.971548e-01 e=3.370987e-01 BE=3.994392e-02 BE/M=1.344213e-01 t=12
2.63181481e-01 r=2.213619e+00 m=2.946965e-01 e=3.332868e-01 BE=3.859028e-02 BE/M=1.309492e-01 t=12
2.57018284e-01 r=2.236139e+00 m=2.920340e-01 e=3.292665e-01 BE=3.723254e-02 BE/M=1.274939e-01 t=12
2.50842754e-01 r=2.258629e+00 m=2.891666e-01 e=3.250393e-01 BE=3.587265e-02 BE/M=1.240553e-01 t=12
2.44655772e-01 r=2.281086e+00 m=2.860940e-01 e=3.206065e-01 BE=3.451257e-02 BE/M=1.206337e-01 t=12
2.38458228e-01 r=2.303510e+00 m=2.828156e-01 e=3.159698e-01 BE=3.315424e-02 BE/M=1.172292e-01 t=13
2.32251016e-01 r=2.325898e+00 m=2.793310e-01 e=3.111305e-01 BE=3.179957e-02 BE/M=1.138419e-01 t=13
2.26035035e-01 r=2.348250e+00 m=2.756399e-01 e=3.060904e-01 BE=3.045049e-02 BE/M=1.104720e-01 t=13
2.19811190e-01 r=2.370564e+00 m=2.717420e-01 e=3.008509e-01 BE=2.910889e-02 BE/M=1.071196e-01 t=13
2.13580385e-01 r=2.392838e+00 m=2.676370e-01 e=2.954137e-01 BE=2.777667e-02 BE/M=1.037848e-01 t=13
2.07343525e-01 r=2.415071e+00 m=2.633248e-01 e=2.897805e-01 BE=2.645568e-02 BE/M=1.004679e-01 t=13
2.01101517e-01 r=2.437261e+00 m=2.588052e-01 e=2.839530e-01 BE=2.514779e-02 BE/M=9.716879e-02 t=12
1.94855264e-01 r=2.459408e+00 m=2.540782e-01 e=2.779330e-01 BE=2.385483e-02 BE/M=9.388777e-02 t=12
```

```
1.88605668e-01 r=2.481509e+00 m=2.491436e-01 e=2.717222e-01 BE=2.257862e-02 BE/M=9.062493e-02 t=12
1.82353629e-01 r=2.503564e+00 m=2.440016e-01 e=2.653225e-01 BE=2.132095e-02 BE/M=8.738038e-02 t=12
1.76100040e-01 r=2.525570e+00 m=2.386521e-01 e=2.587357e-01 BE=2.008360e-02 BE/M=8.415426e-02 t=12
1.69845791e-01 r=2.547527e+00 m=2.330955e-01 e=2.519638e-01 BE=1.886831e-02 BE/M=8.094669e-02 t=12
1.63591763e-01 r=2.569433e+00 m=2.273318e-01 e=2.450086e-01 BE=1.767682e-02 BE/M=7.775777e-02 t=12
1.57338835e-01 r=2.591286e+00 m=2.213614e-01 e=2.378722e-01 BE=1.651082e-02 BE/M=7.458762e-02 t=12
1.51087873e-01 r=2.613087e+00 m=2.151845e-01 e=2.305565e-01 BE=1.537200e-02 BE/M=7.143635e-02 t=12
1.44839738e-01 r=2.634832e+00 m=2.088016e-01 e=2.230636e-01 BE=1.426200e-02 BE/M=6.830406e-02 t=12
1.38595280e-01 r=2.656522e+00 m=2.022132e-01 e=2.153956e-01 BE=1.318245e-02 BE/M=6.519085e-02 t=12
1.32355341e-01 r=2.678154e+00 m=1.954196e-01 e=2.075545e-01 BE=1.213493e-02 BE/M=6.209682e-02 t=12
1.26120751e-01 r=2.699727e+00 m=1.884215e-01 e=1.995425e-01 BE=1.112102e-02 BE/M=5.902205e-02 t=12
1.19892328e-01 r=2.721241e+00 m=1.812195e-01 e=1.913617e-01 BE=1.014224e-02 BE/M=5.596662e-02 t=12
1.13670881e-01 r=2.742694e+00 m=1.738143e-01 e=1.830144e-01 BE=9.200098e-03 BE/M=5.293062e-02 t=12
1.07457203e-01 r=2.764085e+00 m=1.662066e-01 e=1.745027e-01 BE=8.296056e-03 BE/M=4.991411e-02 t=12
1.01252078e-01 r=2.785412e+00 m=1.583973e-01 e=1.658288e-01 BE=7.431552e-03 BE/M=4.691717e-02 t=12
9.50562739e-02 r=2.806676e+00 m=1.503871e-01 e=1.569951e-01 BE=6.607988e-03 BE/M=4.393985e-02 t=12
8.88705458e-02 r=2.827873e+00 m=1.421771e-01 e=1.480038e-01 BE=5.826732e-03 BE/M=4.098222e-02 t=12
8.26956342e-02 r=2.849004e+00 m=1.337681e-01 e=1.388572e-01 BE=5.089116e-03 BE/M=3.804433e-02 t=12
7.65322651e-02 r=2.870068e+00 m=1.251611e-01 e=1.295576e-01 BE=4.396436e-03 BE/M=3.512621e-02 t=12
7.03811494e-02 r=2.891063e+00 m=1.163574e-01 e=1.201073e-01 BE=3.749955e-03 BE/M=3.222791e-02 t=12
6.42429824e-02 r=2.911988e+00 m=1.073578e-01 e=1.105087e-01 BE=3.150896e-03 BE/M=2.934947e-02 t=12
5.81184438e-02 r=2.932843e+00 m=9.816377e-02 e=1.007642e-01 BE=2.600448e-03 BE/M=2.649091e-02 t=12
5.20081971e-02 r=2.953626e+00 m=8.877635e-02 e=9.087612e-02 BE=2.099761e-03 BE/M=2.365226e-02 t=12
4.59128895e-02 r=2.974337e+00 m=7.919687e-02 e=8.084682e-02 BE=1.649950e-03 BE/M=2.083352e-02 t=12
3.98331516e-02 r=2.994975e+00 m=6.942663e-02 e=7.067872e-02 BE=1.252090e-03 BE/M=1.803472e-02 t=12
3.37695971e-02 r=3.015539e+00 m=5.946698e-02 e=6.037420e-02 BE=9.072200e-04 BE/M=1.525586e-02 t=12
2.77228228e-02 r=3.036027e+00 m=4.931933e-02 e=4.993567e-02 BE=6.163407e-04 BE/M=1.249694e-02 t=12
2.16934077e-02 r=3.056440e+00 m=3.898512e-02 e=3.936553e-02 BE=3.804149e-04 BE/M=9.757953e-03 t=12
1.56819130e-02 r=3.076776e+00 m=2.846582e-02 e=2.866619e-02 BE=2.003678e-04 BE/M=7.038890e-03 t=12
9.68887987e-03 r=3.097035e+00 m=1.776298e-02 e=1.784007e-02 BE=7.708666e-05 BE/M=4.339737e-03 t=12
```

In [38]:

```
set_precision(30)
Rhat_a = Reals(pi*sqrt(c_a_val))
c_M = Reals(pi*c_a_val^(3/2))
c_E = Reals((3*pi/4)*c_a_val^(5/2))
pretty_print(LE(r"\hat R_{asymp} = "),"{:.8f}".format(Rhat_a),\
             LE(r"\qquad\hat M_{asymp} = "),"{:.8f}".format(c_M),LE(r"\:\hat{\rho}(0)"),\
             LE(r"\qquad\hat{B}E_{asymp} = "),"{:.8f}".format(c_E),LE(r"\:\hat{\rho}(0)^2"))
this_star_set ={}
print("coreless_star_set[{0}]".format(len(coreless_star_set)))
for n in range(15):
    m_0 = 5e-3/2^n
    start = timer()
    star = make_coreless_star(m_0)
    end = timer()
    star_key = star['star key']
    this_star_set[star_key]=star
    star_data = star['data']
    print_star(star_key,star_data,time=end-start)
    central_density = Reals(star_data['central density'])
    mass = Reals(star_data['mass'])
    radius = Reals(star_data['radius'])
    energy = Reals(star_data['energy'])
    BE = energy-mass
    pretty_print(LE(r"\frac{R}{R_{asymp}}="), RDF(radius/Rhat_a),\
            LE(r"\qquad\frac{M}{M_{asymp}}="), RDF(mass/(c_M*central_density)),\
            LE(r"\qquad\frac{BE}{BE_{asymp}}="), RDF(BE/(c_E*central_density^2)))
coreless_star_set.append(this_star_set)
set_precision()
```

mp.dps =30 mp decimal precision =31 sage decimal precision =34

$$\hat{R}_{asymp} = 3.12975969 \qquad \hat{M}_{asymp} = 3.10622730\,\hat{\rho}(0) \qquad \hat{B}E_{asymp} = 2.31215389\,\hat{\rho}(0)^2$$

```
coreless_star_set[2]
5.00000000e-03 r=3.112877e+00 m=9.237803e-03 e=9.258458e-03 BE=2.065435e-05 BE/M=2.235851e-03 t=69
```

$$\frac{R}{R_{asymp}} = 0.9946056049803801 \qquad \frac{M}{M_{asymp}} = 0.9796428920439193 \qquad \frac{BE}{BE_{asymp}} = 0.9693029209427709$$

```
2.50000000e-03 r=3.121319e+00 m=4.637931e-03 e=4.643111e-03 BE=5.180341e-06 BE/M=1.116951e-03 t=70
```

$$\frac{R}{R_{asymp}} = 0.9973031546211996 \qquad \frac{M}{M_{asymp}} = 0.9897892048650972 \qquad \frac{BE}{BE_{asymp}} = 0.9845660984846787$$

```
1.25000000e-03 r=3.125540e+00 m=2.323730e-03 e=2.325027e-03 BE=1.297181e-06 BE/M=5.582323e-04 t=75
```

$$\frac{R}{R_{asymp}} = 0.9986516640065255 \quad \frac{M}{M_{asymp}} = 0.9948865283759023 \quad \frac{BE}{BE_{asymp}} = 0.9922616348599246$$

```
6.25000000e-04 r=3.127650e+00 m=1.163057e-03 e=1.163381e-03 BE=3.245573e-07 BE/M=2.790555e-04 t=69
```

$$\frac{R}{R_{asymp}} = 0.9993258535106289 \quad \frac{M}{M_{asymp}} = 0.9974412439490421 \quad \frac{BE}{BE_{asymp}} = 0.9961254545828799$$

```
3.12500000e-04 r=3.128705e+00 m=5.818264e-04 e=5.819076e-04 BE=8.117210e-08 BE/M=1.395126e-04 t=67
```

$$\frac{R}{R_{asymp}} = 0.9996629321113595 \quad \frac{M}{M_{asymp}} = 0.9987201166989614 \quad \frac{BE}{BE_{asymp}} = 0.9980613854222472$$

```
1.56250000e-04 r=3.129232e+00 m=2.909877e-04 e=2.910080e-04 BE=2.029712e-08 BE/M=6.975250e-05 t=67
```

$$\frac{R}{R_{asymp}} = 0.9998314673920934 \quad \frac{M}{M_{asymp}} = 0.9993599320035969 \quad \frac{BE}{BE_{asymp}} = 0.999030357099086$$

```
7.81250000e-05 r=3.129496e+00 m=1.455125e-04 e=1.455176e-04 BE=5.074792e-09 BE/M=3.487530e-05 t=67
```

$$\frac{R}{R_{asymp}} = 0.9999157340298255 \quad \frac{M}{M_{asymp}} = 0.9996799344119525 \quad \frac{BE}{BE_{asymp}} = 0.9995150946284376$$

```
3.90625000e-05 r=3.129628e+00 m=7.276091e-05 e=7.276218e-05 BE=1.268762e-09 BE/M=1.743741e-05 t=66
```

$$\frac{R}{R_{asymp}} = 0.9999578670983169 \quad \frac{M}{M_{asymp}} = 0.9998399593080929 \quad \frac{BE}{BE_{asymp}} = 0.9997575263316804$$

```
1.95312500e-05 r=3.129694e+00 m=3.638162e-05 e=3.638194e-05 BE=3.171985e-10 BE/M=8.718648e-06 t=66
```

$$\frac{R}{R_{asymp}} = 0.9999789335700044 \quad \frac{M}{M_{asymp}} = 0.999919977679523 \quad \frac{BE}{BE_{asymp}} = 0.999878757919923$$

```
9.76562500e-06 r=3.129727e+00 m=1.819110e-05 e=1.819118e-05 BE=7.930063e-11 BE/M=4.359309e-06 t=66
```

$$\frac{R}{R_{asymp}} = 0.9999894667902132 \quad \frac{M}{M_{asymp}} = 0.9999599883461242 \quad \frac{BE}{BE_{asymp}} = 0.999939377648447$$

```
4.88281250e-06 r=3.129743e+00 m=9.095623e-06 e=9.095643e-06 BE=1.982528e-11 BE/M=2.179651e-06 t=66
```

$$\frac{R}{R_{asymp}} = 0.9999947333964092 \quad \frac{M}{M_{asymp}} = 0.9999799940496521 \quad \frac{BE}{BE_{asymp}} = 0.9999696884963407$$

```
2.44140625e-06 r=3.129751e+00 m=4.547830e-06 e=4.547835e-06 BE=4.956336e-12 BE/M=1.089825e-06 t=65
```

$$\frac{R}{R_{asymp}} = 0.9999973666985303 \quad \frac{M}{M_{asymp}} = 0.9999899969939735 \quad \frac{BE}{BE_{asymp}} = 0.9999848441661993$$

```
1.22070312e-06 r=3.129756e+00 m=2.273919e-06 e=2.273921e-06 BE=1.239086e-12 BE/M=5.449120e-07 t=65
```

$$\frac{R}{R_{asymp}} = 0.9999986833493466 \quad \frac{M}{M_{asymp}} = 0.9999949984892738 \quad \frac{BE}{BE_{asymp}} = 0.9999924220626071$$

```
6.10351562e-07 r=3.129758e+00 m=1.136961e-06 e=1.136961e-06 BE=3.097718e-13 BE/M=2.724560e-07 t=65
```

$$\frac{R}{R_{asymp}} = 0.9999993416746935 \quad \frac{M}{M_{asymp}} = 0.9999974992427088 \quad \frac{BE}{BE_{asymp}} = 0.9999962110261806$$

```
3.05175781e-07 r=3.129759e+00 m=5.684807e-07 e=5.684808e-07 BE=7.744297e-14 BE/M=1.362280e-07 t=64
```

$$\frac{R}{R_{asymp}} = 0.9999996708373515 \quad \frac{M}{M_{asymp}} = 0.9999987496208727 \quad \frac{BE}{BE_{asymp}} = 0.99999810551181$$

`mp.dps` =20 `mp decimal precision` =21 `sage decimal precision` =24

The star at $\rho(0) = \rho_{EW}$

In [39]:

```
star_data_EW =coreless_star_set[0][skey(0.5)]['data']
mass_EW = myR(star_data_EW['mass'])
radius_EW = myR(star_data_EW['radius'])
EW = (mass_EW,radius_EW)
EW
```

Out[39]:

$$(0.23241956266102788692000, 1.29644054300631910450000)$$

The star at $\hat{\rho}(0) = 10^8$

In [40]:

```
rhohat_0_norm_BSM = mpf(1e8)/rhohat_EW_mpf
star_BSM = make_star(rhohat_0_norm_BSM)
mass_BSM = myR(star_BSM['data']['mass'])
radius_BSM = myR(star_BSM['data']['radius'])
BSM = (mass_BSM,radius_BSM)
BSM
```

Out[40]:

$$(0.19596684210709988734000, 1.40157671446392885230000)$$

Plots

collect star_data in two dictionaries, **cored_stars** and **coreless_stars**

stars is the union of the two, a dictionary of all the stars

The star's key in these dictionaries is **central_density** (number, not string), for sorting.

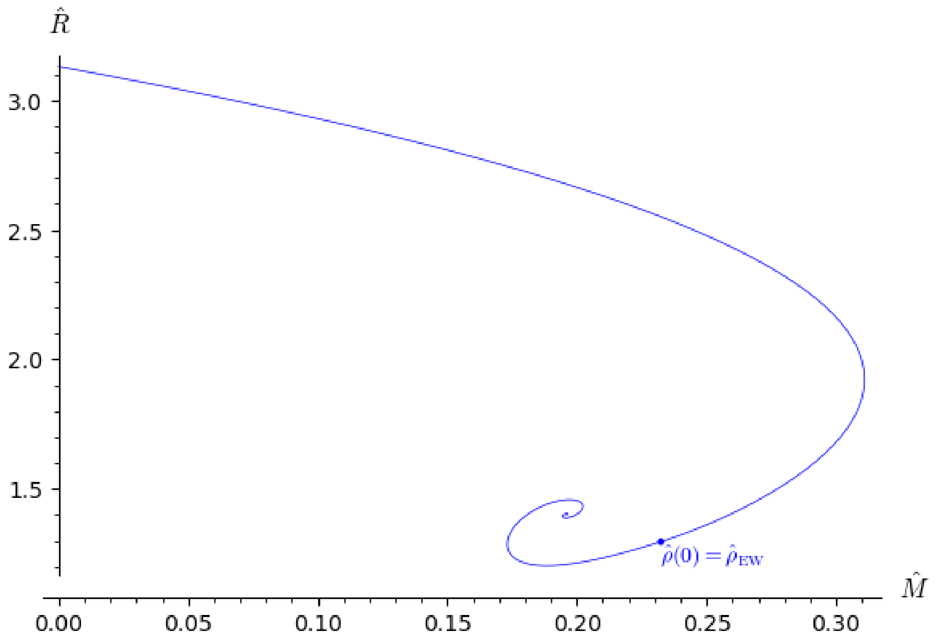
In [41]:

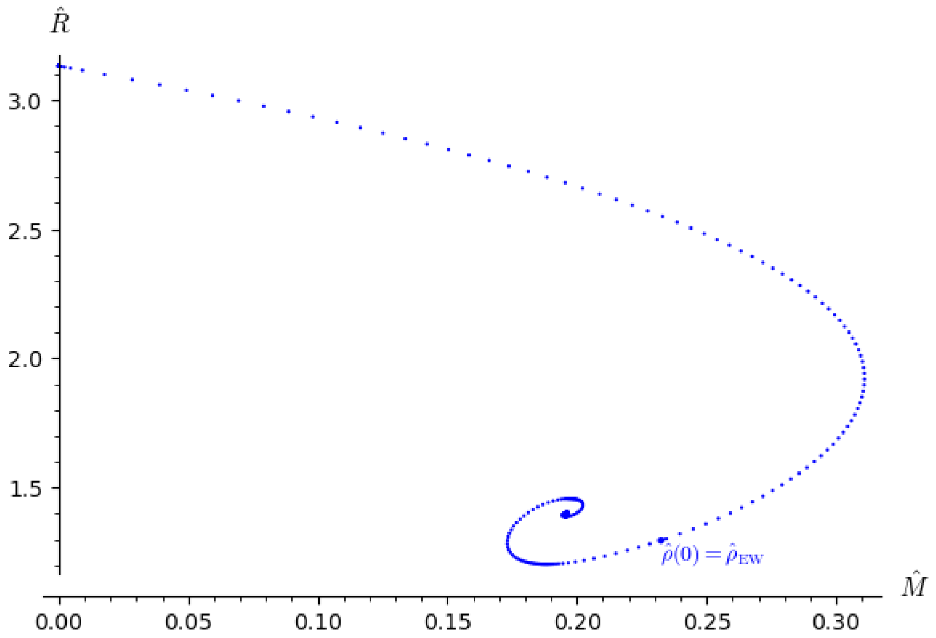
```
cored_stars = {}
for starset in star_set:
    for star_key, star in starset.items():
        if star_key in cored_stars:
            print(star_key, ' DUPLICATE')
        star_data = star['data']
        central_density = star_data['central density']
        cored_stars[central_density] = star['data']
cored_stars = dict(sorted(cored_stars.items()))
#
coreless_stars = {}
for starset in coreless_star_set:
    for star_key, star in starset.items():
        if star_key in coreless_stars:
            print(star_key, ' DUPLICATE')
        star_data = star['data']
        central_density = star_data['central density']
        coreless_stars[central_density] = star['data']
coreless_stars = dict(sorted(coreless_stars.items()))
#
stars = coreless_stars | cored_stars
stars = dict(sorted(stars.items()))
```

In [42]:

```
star_plot_joined = plots(stars,'mass','radius',joined=True)
#show(star_plot_joined,ymin=0,xmin=0)
plt=star_plot_joined
#plt.set_aspect_ratio(.05)
plt.set_axes_range(xmin=0,ymin=0)
plt.axes_labels([r"$\hat M$",r"$\hat R$"])
plt_bo1=plt
plt_bo2=plt
plt_bo0=plt
plt += point(EW)
plt += text(r"$\hat\rho(0) =\hat \rho_{\mathrm{EW}}$", (mass_EW+.02,radius_EW-.05),\
            fontsize='medium')
plt.axes_labels([r"$\hat M$",r"$\hat R$"])
plt.set_aspect_ratio(.1)
show(plt,axes_labels_size=1.2)
plt.save('plots/M_R.pdf',dpi=600,axes_labels_size=1.2)
#
star_plot_unjoined = plots(stars,'mass','radius',joined=False)
#show(star_plot_joined,ymin=0,xmin=0)
plt=star_plot_unjoined
#plt.set_aspect_ratio(.05)
plt.set_axes_range(xmin=0,ymin=0)
plt.axes_labels([r"$\hat M$",r"$\hat R$"])
plt_bo1_unj=plt
plt_bo2_unj=plt
plt_bo0_unj=plt
plt += point(EW)
plt += text(r"$\hat\rho(0) =\hat \rho_{\mathrm{EW}}$", (mass_EW+.02,radius_EW-.05),\
```

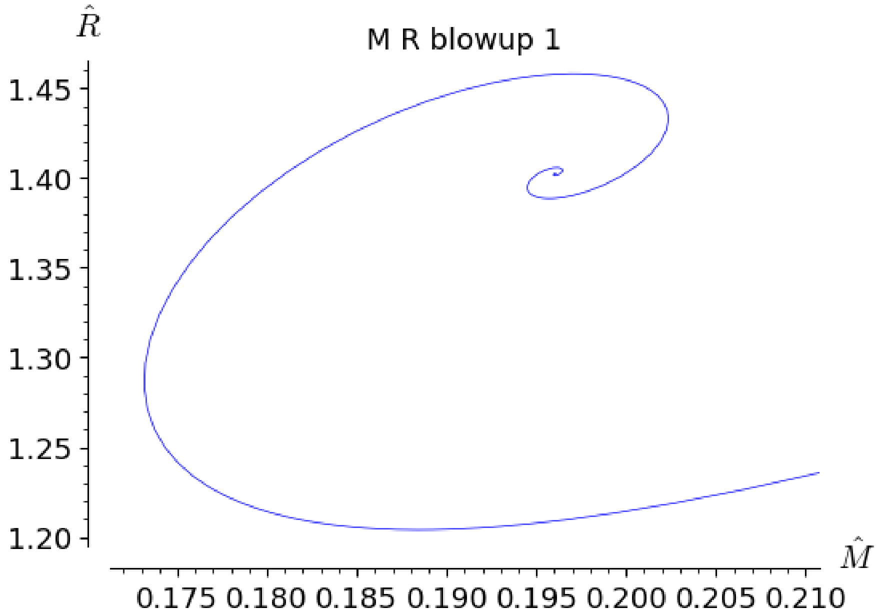
```
            fontsize='medium')
plt.axes_labels([r"$\hat M$",r"$\hat R$"])
plt.set_aspect_ratio(.1)
show(plt,axes_labels_size=1.2)
```

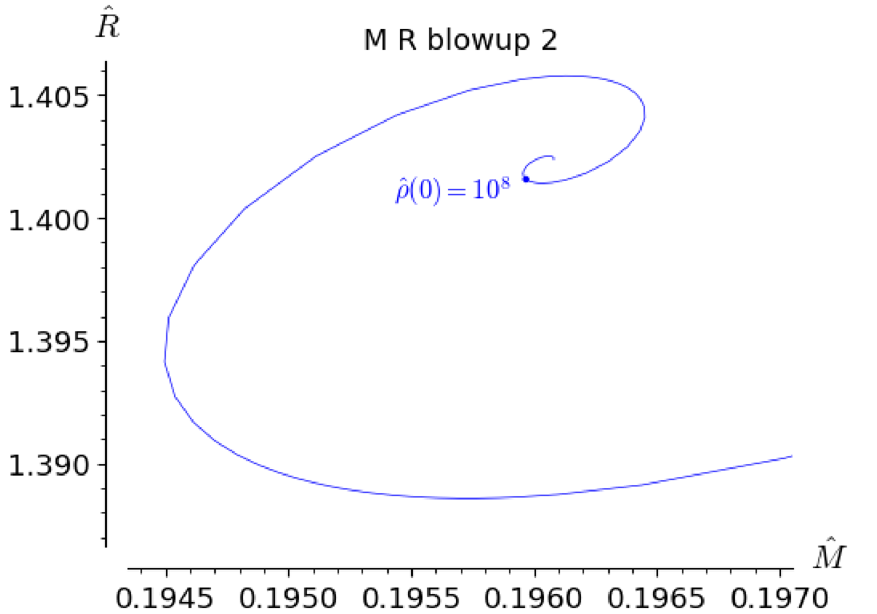


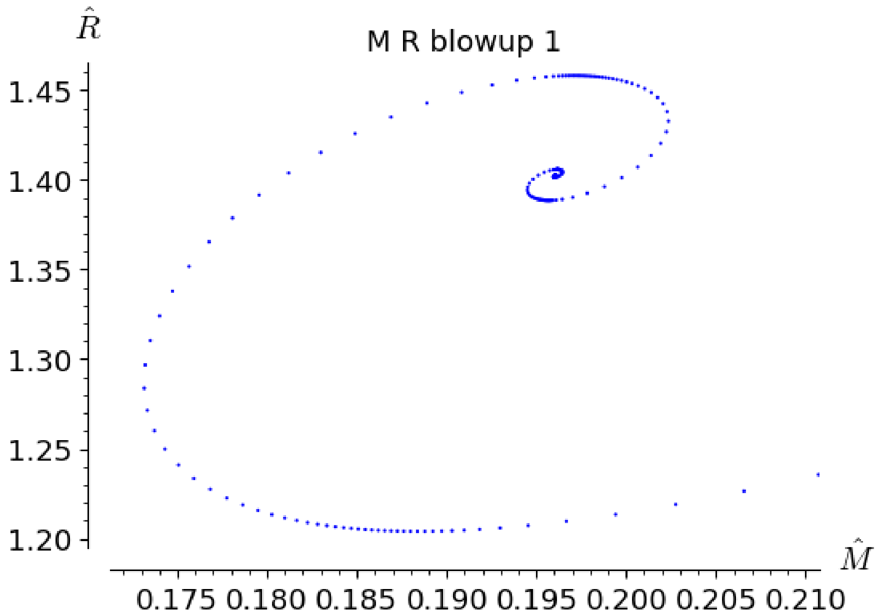


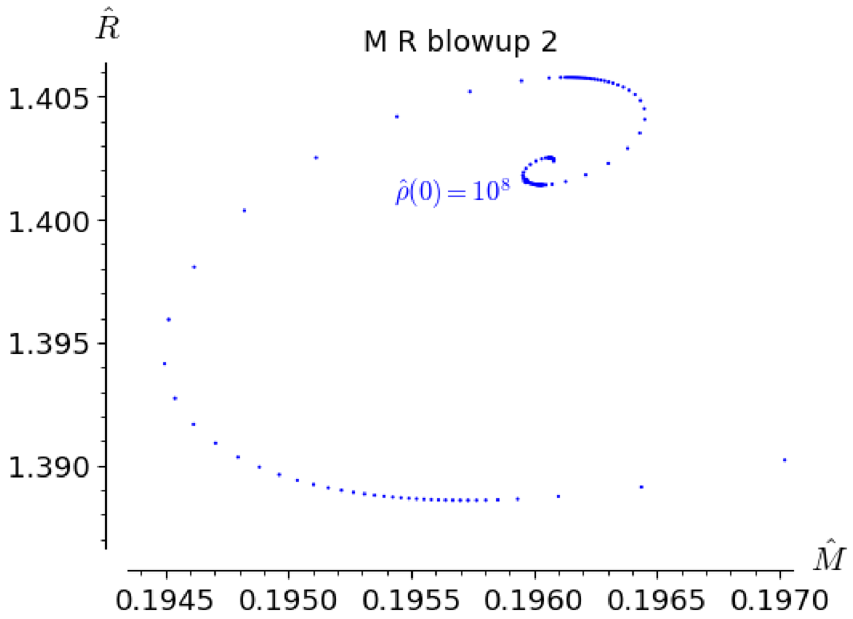
In [43]:
```
plt_bo1.set_aspect_ratio(.1)
plt_bo1.set_axes_range(xmin=0.172,xmax=.21,ymin=1.2,ymax=1.46)
show(plt_bo1,title="M R blowup 1",fontsize=13,axes_labels_size=1.2)
plt_bo1.save('plots/M_Rblowup1.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
#
plt_bo2 = plt_bo0
plt_bo2 += point(BSM)
plt_bo2 += text(r"$\hat\rho(0) =10^8$", (mass_BSM-0.0003,radius_BSM-.0005),fontsize='large')
plt_bo2.set_aspect_ratio(.1)
plt_bo2.set_axes_range(xmin=0.1944,xmax=.197,ymin=1.387,ymax=1.406)
plt_bo2.axes_labels([r"$\hat M$",r"$\hat R$"])
show(plt_bo2,title="M R blowup 2",fontsize=13,axes_labels_size=1.2)
plt_bo2.save('plots/M_Rblowup2.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
#
plt_bo1_unj.set_aspect_ratio(.1)
plt_bo1_unj.set_axes_range(xmin=0.172,xmax=.21,ymin=1.2,ymax=1.46)
show(plt_bo1_unj,title="M R blowup 1",fontsize=13,axes_labels_size=1.2)
#
plt_bo2_unj = plt_bo0_unj
plt_bo2_unj += point(BSM)
plt_bo2_unj += text(r"$\hat\rho(0) =10^8$", (mass_BSM-0.0003,radius_BSM-.0005),fontsize='large')
plt_bo2_unj.set_aspect_ratio(.1)
```

```
plt_bo2_unj.set_axes_range(xmin=0.1944,xmax=.197,ymin=1.387,ymax=1.406)
plt_bo2_unj.axes_labels([r"$\hat M$",r"$\hat R$"])
show(plt_bo2_unj,title="M R blowup 2",fontsize=13,axes_labels_size=1.2)
```



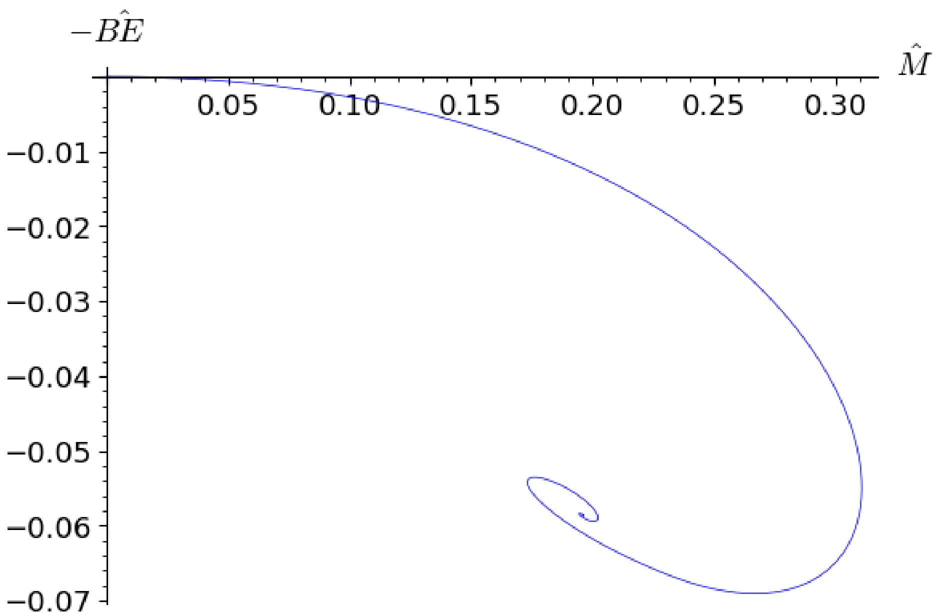


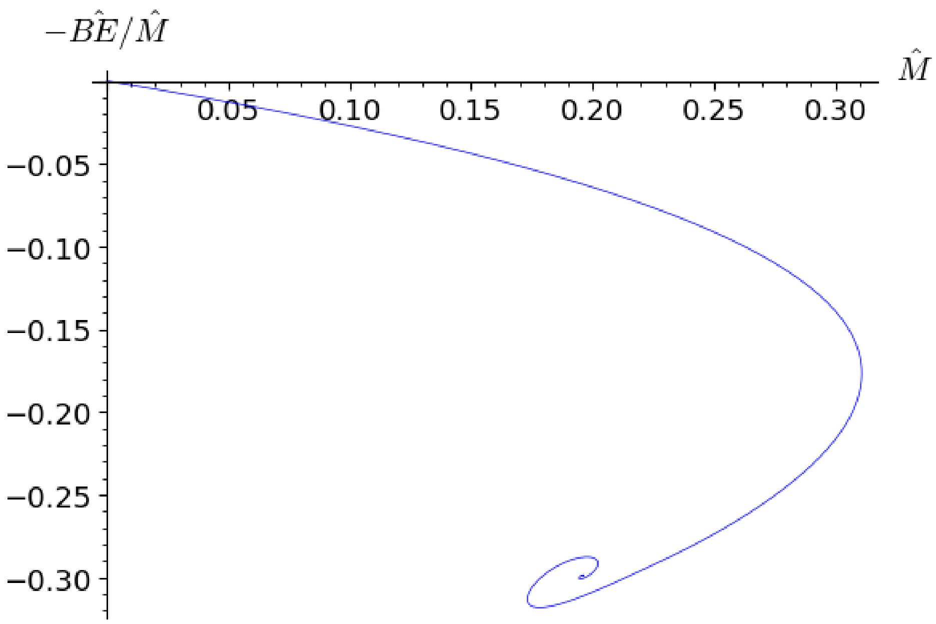




In [44]:

```
BElist = []
for central_density,star_data in stars.items():
    mass = star_data['mass']
    energy = star_data['energy']
    BE = energy-mass
    BE_ratio = (energy-mass)/mass
    BElist.append([mass,-BE])
plt = list_plot(BElist,marker=None,plotjoined=True,thickness=0.5)
#plt.set_axes_range(xmin=0,ymin=0)
plt.axes_labels([r"$\hat M$",r"$-\hat{BE}$"])
show(plt,fontsize=13,axes_labels_size=1.2)
plt.save('plots/M_BE.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
#
BEratiolist = []
for central_density,star_data in stars.items():
    mass = star_data['mass']
    energy = star_data['energy']
    BE_ratio = (energy-mass)/mass
    BEratiolist.append([mass,-BE_ratio])
plt = list_plot(BEratiolist,marker=None,plotjoined=True,thickness=0.5)
#plt.set_axes_range(xmin=0,ymin=0)
plt.axes_labels([r"$\hat M$",r"$-\hat{BE}/\hat{M}$"])
show(plt,fontsize=13,axes_labels_size=1.2)
plt.save('plots/M_BEratio.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
```





Special stars

The star of minumum radius.

3.20219250e+00 r=1.204027e+00 m=1.883777e-01 e=2.477540e-01 BE=5.937627e-02 BE/M=3.151979e-01 t=6

In [45]:
```
star_data_min_R =star_set[1]['3.20219250e+00']['data']
mass_min_R = myR(star_data_min_R['mass'])
radius_min_R = myR(star_data_min_R['radius'])
min_R = (mass_min_R,radius_min_R)
min_R
```

Out[45]:

$$(0.18837772352835232517000, 1.2040273735813345141000)$$

The star of largest mass

3.47709480e-01 r=1.896051e+00 m=3.108239e-01 e=3.670901e-01 BE=5.626628e-02 BE/M=1.810230e-01 t=12
3.41803251e-01 r=1.918823e+00 m=3.109591e-01 e=3.661058e-01 BE=5.514674e-02 BE/M=1.773441e-01 t=12
3.35874144e-01 r=1.941588e+00 m=3.108993e-01 e=3.648964e-01 BE=5.399716e-02 BE/M=1.736805e-01 t=12

In [46]:
```
for star_key in ['3.47709480e-01','3.41803251e-01','3.35874144e-01']:
    print(coreless_star_set[1][star_key]['ode']['initial value'])
```

```
0.34770947997778476825
0.34180325054452595958
0.335874143802153429
```

In [47]:
```
star_set_search_largest_m={}
for x in srange(0.336,0.347,.001,universe=RDF):
    x=Reals(round(x,4))
    star = make_coreless_star(x)
    star_data = star['data']
    star_key = star['star key']
    print_star(star_key,star_data)
    star_set_search_largest_m[star_key]=star
```

```
3.36000000e-01 r=1.941106e+00 m=3.109026e-01 e=3.649244e-01 BE=5.402181e-02 BE/M=1.737580e-01 t=0
3.37000000e-01 r=1.937273e+00 m=3.109256e-01 e=3.651430e-01 BE=5.421733e-02 BE/M=1.743739e-01 t=0
3.38000000e-01 r=1.933436e+00 m=3.109431e-01 e=3.653553e-01 BE=5.441216e-02 BE/M=1.749907e-01 t=0
3.39000000e-01 r=1.929598e+00 m=3.109551e-01 e=3.655614e-01 BE=5.460630e-02 BE/M=1.756083e-01 t=0
3.40000000e-01 r=1.925756e+00 m=3.109615e-01 e=3.657613e-01 BE=5.479974e-02 BE/M=1.762267e-01 t=0
3.41000000e-01 r=1.921912e+00 m=3.109624e-01 e=3.659549e-01 BE=5.499246e-02 BE/M=1.768460e-01 t=0
3.42000000e-01 r=1.918065e+00 m=3.109577e-01 e=3.661422e-01 BE=5.518446e-02 BE/M=1.774661e-01 t=0
3.43000000e-01 r=1.914216e+00 m=3.109474e-01 e=3.663231e-01 BE=5.537572e-02 BE/M=1.780871e-01 t=0
3.44000000e-01 r=1.910364e+00 m=3.109316e-01 e=3.664978e-01 BE=5.556624e-02 BE/M=1.787089e-01 t=0
3.45000000e-01 r=1.906509e+00 m=3.109101e-01 e=3.666661e-01 BE=5.575601e-02 BE/M=1.793316e-01 t=0
3.46000000e-01 r=1.902652e+00 m=3.108831e-01 e=3.668281e-01 BE=5.594500e-02 BE/M=1.799551e-01 t=0
```

In [48]:
```
star = star_set_search_largest_m['3.41000000e-01']
star_data = star['data']
mass = Reals(star_data['mass'])
radius = Reals(star_data['radius'])
energy = Reals(star_data['energy'])
BE = energy-mass
pretty_print(LE(r"\hat M = "),mass,LE(r"\qquad M = "),mass*m_b_kg,LE(r" = "),mass*m_b_solar,LE(r"M_\
pretty_print(LE(r"\hat R = "),radius,LE(r"\qquad R = "),radius*r_b_m)
pretty_print(LE(r"\frac{BE}{M} = "),BE/mass)
```

$$\hat{M} = 0.31096240159359108330000 \qquad M = 1.8183901269497294 \times 10^{25}\,kg = 9.14499158594714 \times 10^{-06} M_\odot$$

$$\hat{R} = 1.92191225036405961940000 \qquad R = 0.08345974544494067\,m$$

$$\frac{BE}{M} = 0.176846008156575028345820$$

The mass where transitions become possible

9.23643135e+00 r=1.271494e+00 m=1.733260e-01 e=2.278757e-01 BE=5.454964e-02 BE/M=3.147227e-01 t=6
1.02705121e+01 r=1.283822e+00 m=1.731541e-01 e=2.274269e-01 BE=5.427287e-02 BE/M=3.134369e-01 t=6
1.14397938e+01 r=1.296846e+00 m=1.732126e-01 e=2.272499e-01 BE=5.403726e-02 BE/M=3.119707e-01 t=6

In [49]:
```
star_set_search_transition={}
for x in srange(9.24,11.43,.2,universe=RDF):
    x=Reals(round(x,4))
    star = make_star(x)
    star_data = star['data']
    star_key = star['star key']
    print_star(star_key,star_data)
    star_set_search_transition[star_key]=star
```

```
9.24000000e+00 r=1.271538e+00 m=1.733250e-01 e=2.278735e-01 BE=5.454855e-02 BE/M=3.147183e-01 t=0
9.44000000e+00 r=1.273979e+00 m=1.732717e-01 e=2.277609e-01 BE=5.448919e-02 BE/M=3.144725e-01 t=0
9.64000000e+00 r=1.276393e+00 m=1.732288e-01 e=2.276616e-01 BE=5.443281e-02 BE/M=3.142249e-01 t=0
9.84000000e+00 r=1.278779e+00 m=1.731956e-01 e=2.275748e-01 BE=5.437924e-02 BE/M=3.139759e-01 t=0
1.00400000e+01 r=1.281138e+00 m=1.731714e-01 e=2.274998e-01 BE=5.432836e-02 BE/M=3.137259e-01 t=0
1.02400000e+01 r=1.283469e+00 m=1.731558e-01 e=2.274358e-01 BE=5.428003e-02 BE/M=3.134752e-01 t=0
1.04400000e+01 r=1.285771e+00 m=1.731480e-01 e=2.273822e-01 BE=5.423413e-02 BE/M=3.132240e-01 t=0
1.06400000e+01 r=1.288044e+00 m=1.731478e-01 e=2.273384e-01 BE=5.419054e-02 BE/M=3.129727e-01 t=0
1.08400000e+01 r=1.290289e+00 m=1.731546e-01 e=2.273037e-01 BE=5.414916e-02 BE/M=3.127215e-01 t=0
1.10400000e+01 r=1.292504e+00 m=1.731679e-01 e=2.272778e-01 BE=5.410987e-02 BE/M=3.124706e-01 t=0
1.12400000e+01 r=1.294691e+00 m=1.731874e-01 e=2.272600e-01 BE=5.407259e-02 BE/M=3.122201e-01 t=0
```

In [50]:
```
star = star_set_search_transition['1.06400000e+01']
star_data = star['data']
mass = Reals(star_data['mass'])
radius = Reals(star_data['radius'])
energy = Reals(star_data['energy'])
BE = energy-mass
pretty_print(LE(r"\hat M = "),mass,LE(r"\qquad M = "),mass*m_b_kg,LE(r" = "),mass*m_b_solar,LE(r"M_\
pretty_print(LE(r"\hat R = "),radius,LE(r"\qquad R = "),radius*r_b_m)
pretty_print(LE(r"\frac{BE}{M} = "),BE/mass)
```

$$\hat{M} = 0.17314781111503715215000 \qquad M = 1.0125026968566777 \times 10^{25}\,kg = 5.092047358965388 \times 10^{-06} M_\odot$$

$$\hat{R} = 1.28804427498964687920000 \qquad R = 0.05593379577662084\,m$$

$$\frac{BE}{M} = 0.312972719242518580542140$$

The star of largest BE/M

In [51]:
```
star_set_search={}
for x in srange(5.21,5.30,.01,universe=RDF):
    x=Reals(round(x,4))
    star = make_star(x)
```

```
    star_data = star['data']
    star_key = star['star key']
    print_star(star_key,star_data)
    star_set_search[star_key]=star
```

```
5.21000000e+00 r=1.219663e+00 m=1.784181e-01 e=2.351704e-01 BE=5.675226e-02 BE/M=3.180858e-01 t=0
5.22000000e+00 r=1.219782e+00 m=1.783891e-01 e=2.351322e-01 BE=5.674309e-02 BE/M=3.180861e-01 t=0
5.23000000e+00 r=1.219902e+00 m=1.783602e-01 e=2.350941e-01 BE=5.673395e-02 BE/M=3.180864e-01 t=0
5.24000000e+00 r=1.220022e+00 m=1.783314e-01 e=2.350563e-01 BE=5.672484e-02 BE/M=3.180866e-01 t=0
5.25000000e+00 r=1.220142e+00 m=1.783028e-01 e=2.350186e-01 BE=5.671576e-02 BE/M=3.180867e-01 t=0
5.26000000e+00 r=1.220262e+00 m=1.782743e-01 e=2.349810e-01 BE=5.670670e-02 BE/M=3.180868e-01 t=0
5.27000000e+00 r=1.220383e+00 m=1.782460e-01 e=2.349436e-01 BE=5.669767e-02 BE/M=3.180867e-01 t=0
5.28000000e+00 r=1.220503e+00 m=1.782177e-01 e=2.349064e-01 BE=5.668867e-02 BE/M=3.180866e-01 t=0
5.29000000e+00 r=1.220624e+00 m=1.781896e-01 e=2.348693e-01 BE=5.667970e-02 BE/M=3.180864e-01 t=0
```

In [52]:

```
star = star_set_search['5.26000000e+00']
star_data = star['data']
mass = Reals(star_data['mass'])
radius = Reals(star_data['radius'])
energy = Reals(star_data['energy'])
BE = energy-mass
pretty_print(LE(r"\hat M = "),mass,LE(r"\qquad M = "),mass*m_b_kg,LE(r" = "),mass*m_b_solar,LE(r"M_\(
pretty_print(LE(r"\hat R = "),radius,LE(r"\qquad R = "),radius*r_b_m)
pretty_print(LE(r"\frac{BE}{M} = "),BE/mass)
```

$$\hat{M} = 0.17827431985146075410000 \qquad M = 1.04248057746459594 \times 10^{25}\, kg = 5.24281117806256 \times 10^{-06} M_{\odot}$$

$$\hat{R} = 1.2202620922647479658000 \qquad R = 0.0529903295934738 7\, m$$

$$\frac{BE}{M} = 0.31808676121034497896813$$

Not used in the paper

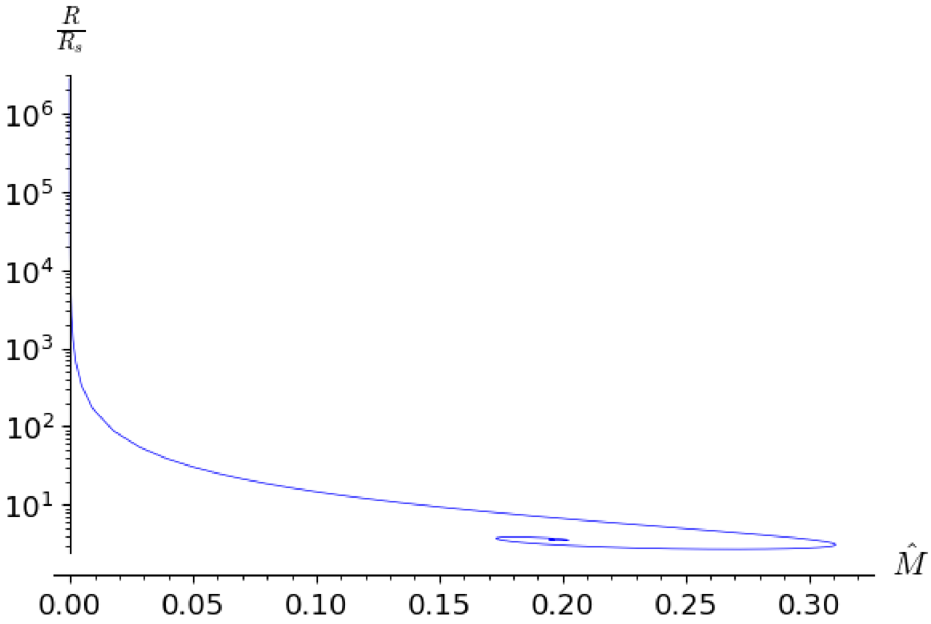
radius / Schwarzschild radius

In [53]:

```
Rratiolist = []
for central_density,star_data in stars.items():
    mass = star_data['mass']
    radius = star_data['radius']
    Rratiolist.append([mass,radius/(2*mass)])
plt = list_plot(Rratiolist,marker=None,plotjoined=True,thickness=0.5)
```

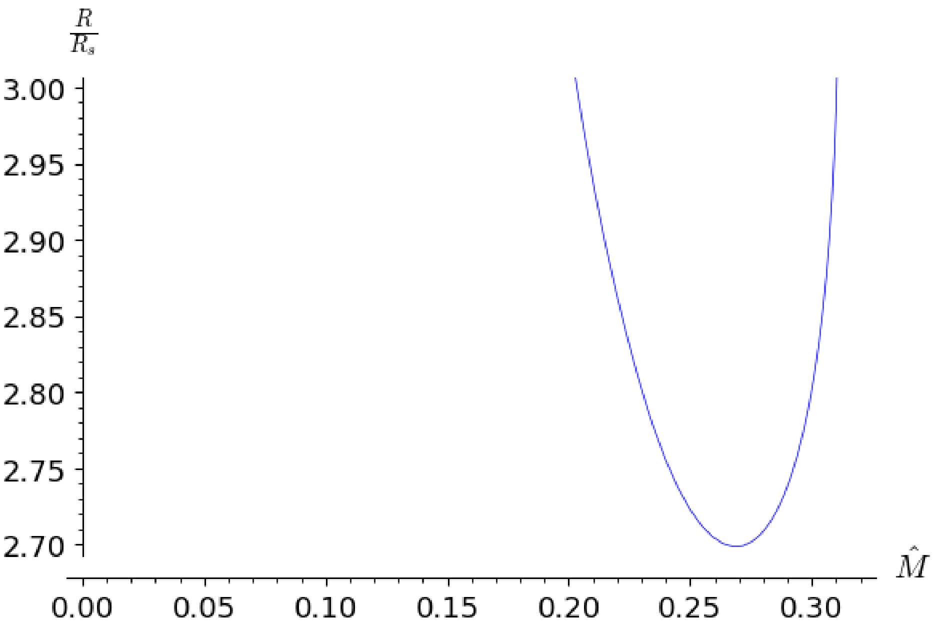
In [54]:

```
plt.set_axes_range(xmin=0,xmax=0.32)
plt.axes_labels([r"$\hat M$",r"$\frac{R}{R_s}$"])
show(plt,fontsize=13,axes_labels_size=1.2,scale="semilogy")
#plt.save('plots/M_BEratio.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
```



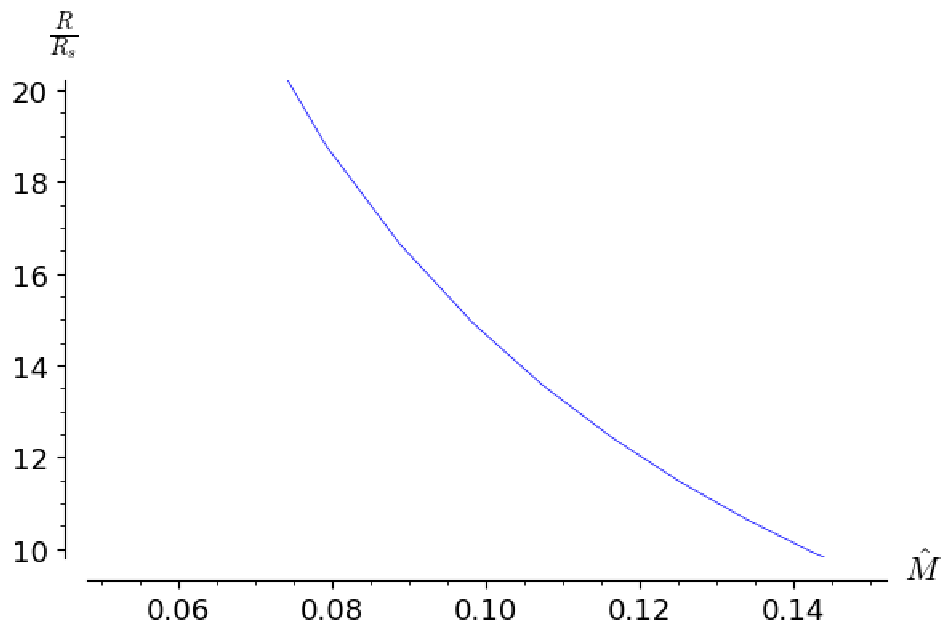
In [55]:

```
plt.set_axes_range(xmin=0,xmax=0.32,ymax=3)
plt.axes_labels([r"$\hat M$",r"$\frac{R}{R_s}$"])
show(plt,fontsize=13,axes_labels_size=1.2)
```



In [56]:

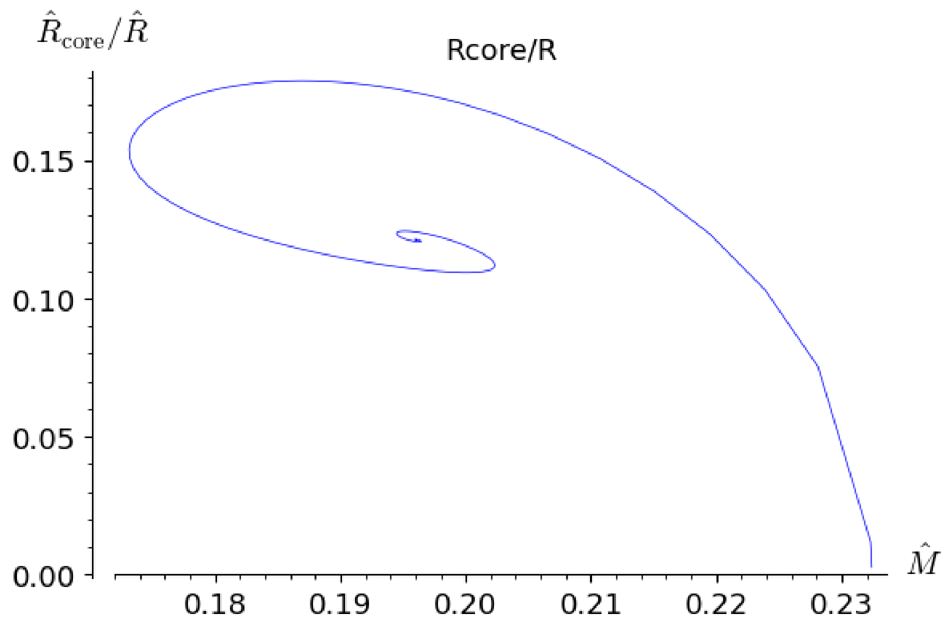
```
plt.set_axes_range(xmin=0.05,xmax=0.15,ymin=10,ymax=20)
plt.axes_labels([r"$\hat M$",r"$\frac{R}{R_s}$"])
show(plt,fontsize=13,axes_labels_size=1.2)
```

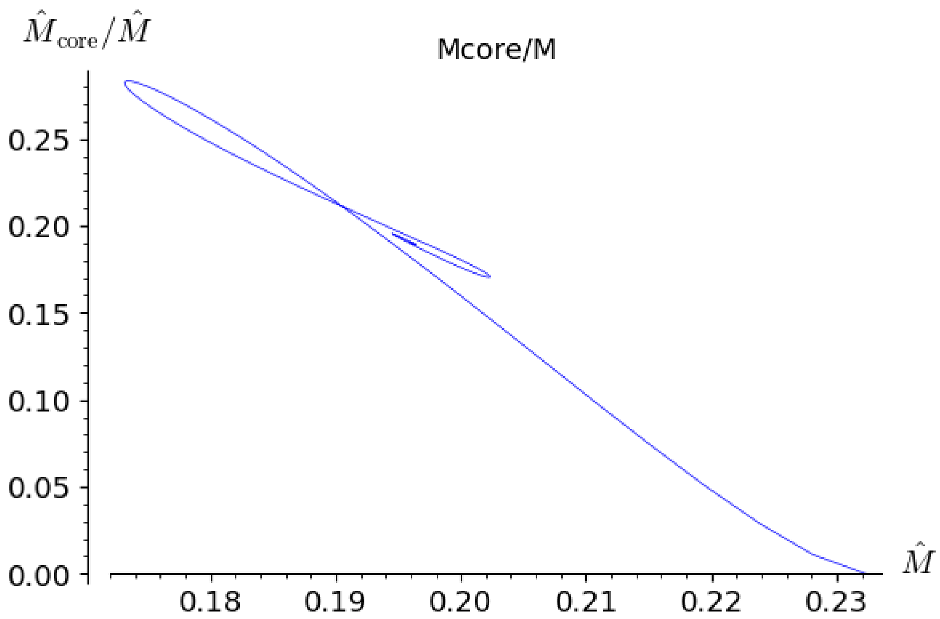


Core radius / radius and core mass / mass

In [57]:

```
core_radius_ratios = []
for central_density,star_data in cored_stars.items():
    mass = star_data['mass']
    core_radius_ratio = star_data['core radius']/star_data['radius']
    core_radius_ratios.append([mass,core_radius_ratio])
#list_plot(core_radius_ratios,marker='.',size=10)
plt = list_plot(core_radius_ratios,marker=None,plotjoined=True,thickness=0.5)
#plt.set_axes_range(xmin=0,ymin=0)
plt.axes_labels([r"$\hat M$",r"${\hat R_{\mathrm{core}}}/{\hat R}$"])
show(plt,title="Rcore/R",fontsize=13,axes_labels_size=1.2)
plt.save('plots/M_Rcoreratio.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
#
core_mass_ratios = []
for central_density,star_data in cored_stars.items():
    mass = star_data['mass']
    core_mass_ratio = star_data['core mass']/star_data['mass']
    core_mass_ratios.append([mass,core_mass_ratio])
plt = list_plot(core_mass_ratios,marker=None,plotjoined=True,thickness=0.5)
plt.axes_labels([r"$\hat M$",r"${\hat M_{\mathrm{core}}}/{\hat M}$"])
show(plt,title="Mcore/M",fontsize=13,axes_labels_size=1.2)
plt.save('plots/M_Mcoreratio.pdf',dpi=600,fontsize=13,axes_labels_size=1.2)
#list_plot(core_mass_ratios,marker='.',size=10)
```



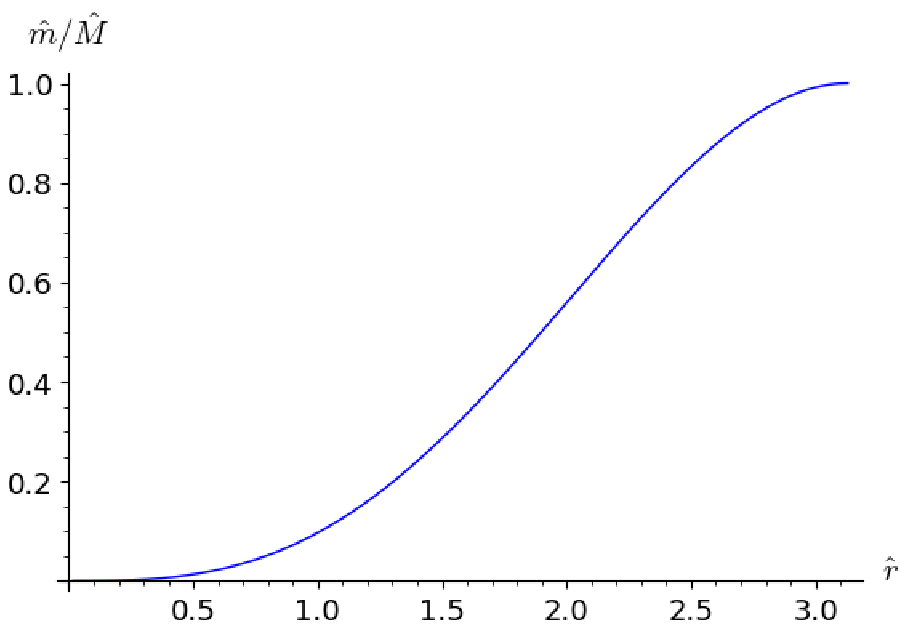


mass profile of a cloud

In [58]:
```
set_precision(30)
star = make_coreless_star(3.0517578125e-7,return_ode=True)
ode = star['ode']
star_data = star['data']
shell_ode = ode['shell ode']
initial_value = ode['initial value']
mass = star_data['mass']
#
r_fun = lambda m: Reals(shell_ode(m)[0])
m_fun = lambda m: Reals(shell_ode(m)[1]/mass)
plt = parametric_plot((r_fun,m_fun),(m,0,initial_value))
#
set_precision()
#
plt.set_aspect_ratio(2)
plt.axes_labels([r"$\hat r$",r"$\hat m/\hat M$"])
show(plt,fontsize=13,axes_labels_size=1.2)
```

mp.dps =30   mp decimal precision =31   sage decimal precision =34

mp.dps =20   mp decimal precision =21   sage decimal precision =24



mass profiles of massive stars

3 stars of same mass, different radius and binding energy

1.19892328e-01 r=2.721241e+00 m=1.812195e-01 e=1.913617e-01 BE=1.014224e-02 BE/M=5.596662e-02 t=12

2.86784664e+01 r=1.403815e+00 m=1.812014e-01 e=2.350428e-01 BE=5.384138e-02 BE/M=2.971356e-01 t=6

4.47166984e+00 r=1.211533e+00 m=1.809907e-01 e=2.385063e-01 BE=5.751560e-02 BE/M=3.177822e-01 t=6

In [59]:

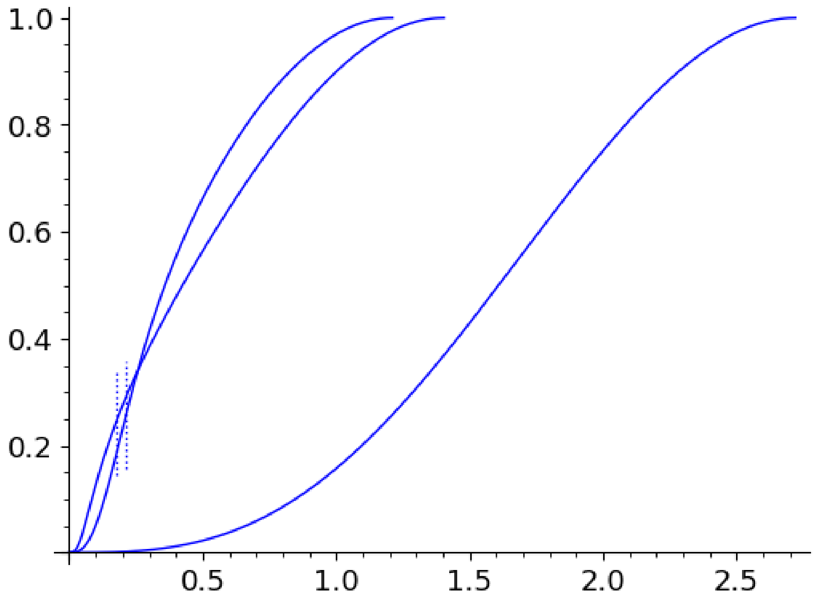
```
star_key1 = '1.19892328e-01'
star1 = coreless_star_set[1][star_key1]
star_key2 = '2.86784664e+01'
star2 = star_set[1][star_key2]
star_key3 = '4.47166984e+00'
star3 = star_set[1][star_key3]
print(star1['ode']['initial value'])
print(star2['ode']['initial value'])
print(star3['ode']['initial value'])
```

```
0.11989232813047308982
28.678466396330248299
4.4716698404048722934
```

In [60]:

```
star1 = make_coreless_star(0.11989232813047308982,return_ode=True)
star2 = make_star(28.678466396330248299,return_ode=True)
star3 = make_star(4.4716698404048722934,return_ode=True)
star_key1 = '1.19892328e-01'
star_key2 = '2.86784664e+01'
star_key3 = '4.47166984e+00'

star_key = star_key1
star = star1
ode = star['ode']
star_data = star['data']
shell_ode = ode['shell ode']
initial_value = ode['initial value']
mass = star_data['mass']
#
r_fun = lambda m: Reals(shell_ode(m)[0])
m_fun = lambda m: Reals(shell_ode(m)[1]/mass)
plt1 = parametric_plot((r_fun,m_fun),(m,0,initial_value))
#
plt1.set_aspect_ratio(2)
plt1.axes_labels([r"$\hat r$",r"$\hat m/\hat M$"])
#show(plt,fontsize=13,axes_labels_size=1.2)

star_key = star_key2
star = star2
ode = star['ode']
star_data = star['data']
shell_ode = ode['shell ode']
core_ode = ode['core ode']
initial_value = ode['initial value']
mass = star_data['mass']
#
rhoh = var('rhoh')
core_r = lambda rhoh: Reals(core_ode(rhoh)[0])
core_m = lambda rhoh: Reals(core_ode(rhoh)[1]/mass)
bdry = core_ode(1)
r_bdry =  Reals(bdry[0])
m_bdry = Reals(bdry[1])/mass
plt_core = parametric_plot((core_r,core_m),(rhoh,1,initial_value))
plt_core += line([(r_bdry,m_bdry-.1),(r_bdry,m_bdry+.1)],linestyle=":")
#show(plt_core)
#
r_fun = lambda m: Reals(shell_ode(m)[0])
m_fun = lambda m: Reals(shell_ode(m)[1]/mass)
plt = parametric_plot((r_fun,m_fun),(m,0,0.5))
plt2 = plt+plt_core
#
plt2.set_aspect_ratio(2)
plt2.axes_labels([r"$\hat r$",r"$\hat m/\hat M$"])
#show(plt2,fontsize=13,axes_labels_size=1.2)

star_key = star_key3
star = star3
ode = star['ode']
star_data = star['data']
shell_ode = ode['shell ode']
core_ode = ode['core ode']
initial_value = ode['initial value']
```

```
mass = star_data['mass']
#
rhoh = var('rhoh')
core_r = lambda rhoh: Reals(core_ode(rhoh)[0])
core_m = lambda rhoh: Reals(core_ode(rhoh)[1]/mass)
bdry = core_ode(1)
r_bdry =  Reals(bdry[0])
m_bdry = Reals(bdry[1])/mass
plt_core = parametric_plot((core_r,core_m),(rhoh,1,initial_value))
plt_core += line([(r_bdry,m_bdry-.1),(r_bdry,m_bdry+.1)],linestyle=":")
#show(plt_core)
#
r_fun = lambda m: Reals(shell_ode(m)[0])
m_fun = lambda m: Reals(shell_ode(m)[1]/mass)
plt = parametric_plot((r_fun,m_fun),(m,0,0.5))
plt3 = plt+plt_core
#
plt3.set_aspect_ratio(2)
plt3.axes_labels([r"$\hat r$",r"$\hat m/\hat M$"])
#show(plt3,fontsize=13,axes_labels_size=1.2)

show(plt1+plt2+plt3,fontsize=13,axes_labels_size=1.2)
```



In [ ]: