Calculations for "Thermodynamic stability of a cosmological SU(2)-weak gauge field"

Daniel Friedan

New High Energy Theory Center and Department of Physics and Astronomy, Rutgers, The State University of New Jersey, Piscataway, New Jersey 08854-8019 U.S.A. and

Science Institute, The University of Iceland, Reykjavik, Iceland

dfriedan@gmail.com   *physics.rutgers.edu/~friedan*

March 22, 2022

Abstract

This note shows calculations for the paper *Thermodynamic stability of a cosmological SU(2)-weak gauge field*. The section numbers and headings match those of the paper. The numerical calculations and some algebra calculations are performed in two SageMath notebooks. The Supplemental Materials — this note and the SageMath notebooks along with printouts as html and pdf — are available at `arXiv.org` as ancillary files associated with the paper, at `cocalc.com/dfriedan/DM/SM`, and at `physics.rutgers.edu/~friedan`.

Contents

2 Spin(4)-symmetric CGF

S^3 is the unit 3-sphere in $\mathbb{R}^4$ with metric $\hat{g}_{ij}(\hat{x})$.

$$\hat{x} \in \mathbb{R}^4 \qquad \delta_{\mu\nu}\hat{x}^\mu\hat{x}^\nu = 1 \qquad (\hat{x}^\mu) = (\hat{x}^i, \hat{x}^4) \qquad \hat{g}_{ij}(\hat{x})d\hat{x}^i d\hat{x}^j = \delta_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu \tag{2.1}$$

S^3 is identified with SU(2) by

$$\hat{x} \in S^3 \quad \longleftrightarrow \quad g_{\hat{x}} = \hat{x}^4\mathbf{1} + \hat{x}^k i^{-1}\sigma_k \ \in \mathrm{SU}(2) \tag{2.2}$$

σ_k are the Pauli matrices,

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ij}{}^k\sigma_k \tag{2.3}$$

Spin(4) is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. It acts on S^3 as SO(4).

$$U = (g_L, g_R) \qquad \hat{x} \mapsto U\hat{x} \qquad g_{U\hat{x}} = g_L\, g_{\hat{x}}\, g_R^{-1} \tag{2.4}$$

The metric is

$$\hat{g}_{ij}(\hat{x}) = \frac{1}{2}\mathrm{tr}\left[\hat{\partial}_i g_{\hat{x}} \hat{\partial}_j (g_{\hat{x}}^{-1})\right] \qquad \hat{\partial}_i = \frac{\partial}{\partial\hat{x}^i} \tag{2.5}$$

The Dirac matrices at $\hat{x}$ are the components of an $\mathfrak{su}(2)$-valued 1-form

$$\hat{\gamma}_i(\hat{x}) = \frac{1}{2} g_{\hat{x}}\hat{\partial}_i(g_{\hat{x}}^{-1}) \tag{2.6}$$

At the north pole,

$$\hat{N} = (0,0,0,1) \qquad \hat{g}_{ij}(\hat{N}) = \delta_{ij} \qquad \hat{\gamma}_i(\hat{N}) = \frac{i}{2}\sigma_i \tag{2.7}$$

so everywhere on S^3

$$\hat{\gamma}_i(\hat{x})\hat{\gamma}_j(\hat{x}) = -\frac{1}{4}\hat{g}_{ij}(\hat{x}) - \frac{1}{2}\hat{\epsilon}_{ij}{}^k(\hat{x})\hat{\gamma}_k(\hat{x}) \tag{2.8}$$

where $\hat{\epsilon}_{ijk}(\hat{x})$ is the volume 3-form. Also at $\hat{N}$,

$$\begin{gathered} \hat{\partial}_{[i}\hat{\gamma}_{j]} = \frac{1}{2}\hat{\partial}_{[i} g_{\hat{x}} \hat{\partial}_{j]}(g_{\hat{x}}^{-1}) = \frac{1}{2} i^{-1}\sigma_{[i} i\sigma_{j]} = i\epsilon_{ij}{}^k\sigma_k = 2\epsilon_{ij}{}^k\hat{\gamma}_k = -2[\hat{\gamma}_i,\, \hat{\gamma}_j] \\ \hat{\partial}_i\hat{\gamma}_j + [\hat{\gamma}_i,\, \hat{\gamma}_j] - i \leftrightarrow j = 0 \end{gathered} \tag{2.9}$$

so the Spin(4)-symmetric covariant derivative acting on spinors is

$$\hat{\nabla}_i = \hat{\partial}_i + \hat{\gamma}_i \qquad \hat{\nabla}_{[i}\hat{\gamma}_{j]} = 0 \tag{2.10}$$

with curvature

$$[\hat{\nabla}_i,\, \hat{\nabla}_j] = \hat{\partial}_{[i}\hat{\gamma}_{j]} + [\hat{\gamma}_i,\, \hat{\gamma}_j] = \hat{\epsilon}_{ij}{}^k\hat{\gamma}_k \tag{2.11}$$

Extending $\hat{\nabla}_i$ to be the metric covariant derivative on tensors as well,

$$\hat{\nabla}_i\hat{\gamma}_j = 0 \tag{2.12}$$

because $\hat{\nabla}_{[i}\hat{\gamma}_{j]} = 0$ and $\hat{\nabla}_i\hat{\gamma}_j + \hat{\nabla}_j\hat{\gamma}_i$ would have to equal $A^k_{ij}\hat{\gamma}_k$ for A^k_{ij} a 3-tensor symmetric in i, j. There is no such SO(4)-symmetric 3-tensor.

6 Quadratic term in the action

6.1 Expand the action

The Spin(4)-symmetric classical gauge field has covariant derivative:

$$D_0^{\rm cl} = \partial_{\hat t} \quad D_k^{\rm cl} = \hat\nabla_k + \hat b(\hat t)\hat\gamma_k \tag{6.1}$$

The curvature is

$$\begin{aligned} F_{0j}^{\rm cl} &= [D_0^{\rm cl},\, D_j^{\rm cl}] = \partial_{\hat t}\hat b\,\hat\gamma_j \\ F_{ij}^{\rm cl} &= [D_i^{\rm cl},\, D_j^{\rm cl}] = [\hat\nabla_i,\,\hat\nabla_j] + \hat b^2[\hat\gamma_i,\,\hat\gamma_j] = (1-\hat b^2)\hat\epsilon_{ij}{}^k\hat\gamma_k \end{aligned} \tag{6.2}$$

The perturbed gauge field has covariant derivative

$$D_0 = D_0^{\rm cl} = \partial_{\hat t} \qquad D_i = D_i^{\rm cl} + B_i \tag{6.3}$$

and curvature

$$\begin{aligned} F_{0j} &= [D_0,\, D_j] = F_{0j}^{\rm cl} + \partial_{\hat t}B_j \\ F_{ij} &= [D_i,\, D_j] = F_{ij}^{\rm cl} + D_{[i}^{\rm cl}B_{j]} + [B_i,\, B_j] \\ D_i^{\rm cl}B_j &= \hat\nabla_i B_j + \hat b[\hat\gamma_i,\, B_j] \end{aligned} \tag{6.4}$$

The Yang-Mills action is

$$\frac{1}{\hbar}S_{\rm gauge} = \frac{1}{2g^2}\int {\rm tr}(-F_{\mu\nu}F^{\mu\nu})\,\sqrt{-\hat g}\,d^3\hat x\,d\hat t \tag{6.5}$$

$${\rm tr}(-F_{\mu\nu}F^{\mu\nu}) = {\rm tr}\left(2F_{0j}F_0{}^j - F_{ij}F^{ij}\right) \tag{6.6}$$

The zeroth order term in B is

$$\begin{aligned} {\rm tr}(-F_{\mu\nu}F^{\mu\nu})_0 &= {\rm tr}\left(2(\partial_{\hat t}\hat b)^2\hat\gamma_j\hat\gamma^j - (1-\hat b^2)^2\hat\epsilon_{ij}{}^k\hat\epsilon^{ijk'}\gamma_k\gamma_{k'}\right) \\ &= 3\left[-(\partial_{\hat t}\hat b)^2 + (1-\hat b^2)^2\right] \end{aligned} \tag{6.7}$$

The first order term is

$$\begin{aligned} {\rm tr}(-F_{\mu\nu}F^{\mu\nu})_1 &= {\rm tr}\left(4F_{0j}^{\rm cl}\partial_{\hat t}B^j - 4F_{ij}^{\rm cl}D^{{\rm cl}\,i}B^j\right) \\ &= {\rm tr}\left(4\partial_{\hat t}\hat b\,\hat\gamma_j\partial_{\hat t}B^j - 4(1-\hat b^2)\hat\epsilon^{ijk}\hat\gamma_k(\hat\nabla_iB_j + \hat b[\hat\gamma_i,\,B_j])\right) \\ &= \partial_{\hat t}\left(4\partial_{\hat t}\hat b\,{\rm tr}(\hat\gamma_jB^j)\right) - 4\left(\partial_{\hat t}^2\hat b + 2(\hat b^2-1)\hat b\right){\rm tr}(\hat\gamma_jB^j) \end{aligned} \tag{6.8}$$

The second order term is

$${\rm tr}(-F_{\mu\nu}F^{\mu\nu})_2 = {\rm tr}(-2\partial_{\hat t}B_j\partial_{\hat t}B^j - D_{[i}^{\rm cl}B_{j]}D^{{\rm cl}\,[i}B^{j]} - 2F_{ij}^{\rm cl}[B^i,\,B^j]) \tag{6.9}$$

The zeroth order term in the action is

$$\frac{1}{\hbar}S_0 = {\rm Vol}(S^3)\frac{3}{g^2}\int\left[-\frac12\left(\frac{d\hat b}{d\hat t}\right)^2 + \frac12(\hat b^2-1)^2\right]d\hat t \qquad {\rm Vol}(S^3) = 2\pi^2 \tag{6.10}$$

The first order term is

$$\begin{aligned} \frac{1}{\hbar}S_1 &= \frac{2}{g^2}\int \left[\partial_{\hat t}\left(\partial_{\hat t}\hat b\, B_0\right) - \left(\partial_{\hat t}^2\hat b + 2(\hat b^2-1)\hat b\right) B_0\right] d\hat t \\ B_0(\hat t) &= \int \mathrm{tr}(\hat\gamma_j B^j)\,\sqrt{-\hat g}\, d^3\hat x \end{aligned} \tag{6.11}$$

The second order term in the action is

$$\begin{aligned} \frac{1}{\hbar}S_2 &= \frac{1}{2g^2}\int \mathrm{tr}(2\partial_{\hat t}B_j\partial_{\hat t}B^j - D^{\mathrm{cl}}_{[i}B_{j]}D^{\mathrm{cl}\,[i}B^{j]} - 2F^{\mathrm{cl}}_{ij}[B^i,\,B^j])\,\sqrt{-\hat g}\,d^3\hat x\,d\hat t \\ &= \frac{1}{g^2}\int \mathrm{tr}(\partial_{\hat t}B_j\partial_{\hat t}B^j - B_i\Box B^i)\,\sqrt{-\hat g}\,d^3\hat x\,d\hat t \end{aligned} \tag{6.12}$$

$$\int \mathrm{tr}(B_i\Box B^i)\,\sqrt{-\hat g}\,d^3\hat x = \int \mathrm{tr}(D^{\mathrm{cl}}_{[i}B_{j]}D^{\mathrm{cl}\,i}B^j + F^{\mathrm{cl}}_{ij}[B^i,\,B^j]))\,\sqrt{-\hat g}\,d^3\hat x \tag{6.13}$$

Changing variables

$$\hat t = \epsilon z \qquad \hat b(\hat t) = \frac{1}{\epsilon}b(z) \tag{6.14}$$

the second order term becomes

$$\frac{1}{\hbar}S_2 = \frac{1}{\epsilon g^2}\int \mathrm{tr}(\partial_z B_i\partial_z B^i - B_i\mathbf{K}B^i)\,\sqrt{-\hat g}\,d^3\hat x\,dz \tag{6.15}$$

$$\mathbf{K} = \epsilon^2\Box \tag{6.16}$$

$$\int \mathrm{tr}(B_i\mathbf{K}B^i)\,\sqrt{-\hat g}\,d^3\hat x = \int \mathrm{tr}(\epsilon^2 D^{\mathrm{cl}}_{[i}B_{j]}D^{\mathrm{cl}\,i}B^j + \epsilon^2 F^{\mathrm{cl}}_{ij}[B^i,\,B^j]))\,\sqrt{-\hat g}\,d^3\hat x \tag{6.17}$$

6.2 Operators Γ and $*\hat\nabla$

Define linear operators on the perturbations B_i.

$$\Gamma B_i = \epsilon_i{}^{jk}[\gamma_j,\,B_k] \qquad *\hat\nabla B_i = \epsilon_i{}^{jk}\hat\nabla_j B_k \tag{6.18}$$

so

$$\begin{aligned} *D^{\mathrm{cl}}B_i &= \epsilon_i{}^{jk}D^{\mathrm{cl}}_j B_k = *\hat\nabla B_i + \hat b\,\Gamma B_i \\ D^{\mathrm{cl}}_{[i}B_{j]}D^{\mathrm{cl}\,i}B^j &= (*D^{\mathrm{cl}}B_k)(*D^{\mathrm{cl}}B^k) \\ \mathrm{tr}(F^{\mathrm{cl}}_{ij}[B^i,\,B^j]) &= (1-\hat b^2)\mathrm{tr}(2\epsilon_{ij}{}^k\hat\gamma_k B^i\,B^j) = (1-\hat b^2)\mathrm{tr}(B_k\Gamma B^k) \end{aligned} \tag{6.19}$$

Γ and $*\hat\nabla$ are symmetric operators.

$$\begin{aligned} \int \mathrm{tr}\left(B'^i *\hat\nabla B_i\right) &= \int \mathrm{tr}\left(B'^i\epsilon_i{}^{jk}\hat\nabla_j B_k\right) = \int \mathrm{tr}\left(\epsilon_k{}^{ji}\hat\nabla_j B'{}_i B^k\right) \\ &= \int \mathrm{tr}\left(*\hat\nabla B'_k B^k\right) \end{aligned} \tag{6.20}$$

$$\begin{aligned} \int \mathrm{tr}\left(B'^i\Gamma B_i\right) &= \int \mathrm{tr}\left(B'^i\epsilon_i{}^{jk}(\gamma_j B_k - B_k\gamma_j)\right) = \int \mathrm{tr}\left(\epsilon_i{}^{jk}(B'^i\gamma_j B_k - \gamma_j B'^i B_k)\right) \\ &= \int \mathrm{tr}\left(\epsilon_k{}^{ji}[\gamma_j,\,B'{}_i]B^k\right) = \int \mathrm{tr}\left(\Gamma B'_k B^k\right) \end{aligned} \tag{6.21}$$

So

$$\begin{aligned}\mathbf{K} &= \epsilon^2(*\hat{\nabla} + \hat{b}\Gamma)^2 + \epsilon^2(1-\hat{b}^2)\Gamma \\ &= (\Gamma^2 - \Gamma)b^2 + (*\hat{\nabla}\Gamma + \Gamma*\hat{\nabla})\epsilon b + \left((*\hat{\nabla})^2 + \Gamma\right)\epsilon^2\end{aligned} \tag{6.22}$$

7 Spin(4) decomposition

The perturbations of the gauge field form the Spin(4) representation

$$\begin{aligned}\left\{B_i^j(\hat{x})\right\} &= \bigoplus_{j_3} (j_1 \otimes j_2 \otimes j_3, j_3) \qquad j_1 = j_2 = 1 \\ &= \bigoplus_{j_3,J} \mathbb{C}^{N(J,j_3)} \otimes (J, j_3) \qquad (J, j_3) = (j_L, j_R)\end{aligned} \tag{7.1}$$

The infinitesimal gauge transformations form the representation

$$\begin{gathered}\left\{v^i(\hat{x})\right\} = \bigoplus_{j_3} (1 \otimes j_3, j_3) = \bigoplus_{j_3,J} \mathbb{C}^{N_{\text{gauge}}(J,j_3)} \otimes (J, j_3) \\ \mathbb{C}^{N_{\text{gauge}}(J,j_3)} \subseteq \mathbb{C}^{N(J,j_3)}\end{gathered} \tag{7.2}$$

The triple tensor product $j_1 \otimes j_2 \otimes j_3$ can be decomposed in two ways,

$$\begin{aligned}(j_1 \otimes j_2) \otimes j_3 &= \left(\bigoplus_{J_{12}} J_{12}\right) \otimes j_3 = \bigoplus_{J} \left(\bigoplus_{J_{12}} J\right) = \bigoplus_{J} \left(\mathbb{C}^{N(J,j_3)} \otimes J\right) \\ j_1 \otimes (j_2 \otimes j_3) &= j_1 \otimes \left(\bigoplus_{J_{23}} J_{23}\right) = \bigoplus_{J} \left(\bigoplus_{J_{23}} J\right) = \bigoplus_{J} \left(\mathbb{C}^{N(J,j_3)} \otimes J\right)\end{aligned} \tag{7.3}$$

with

$$J_{12} \in \{0, 1, 2\} \qquad J_{23} - j_3 \in \{-1, 0, 1\} \qquad J - j_3 \in \{0, \pm1, \pm2\} \tag{7.4}$$

The tensor product for the gauge symmetries decomposes

$$1 \otimes j_3 = \bigoplus_{J} \left(\mathbb{C}^{N_{\text{gauge}}(J,j_3)} \otimes \mathbf{J}\right) \qquad J - j_3 \in \{-1, 0, 1\} \tag{7.5}$$

The representations that occur in the decompositions are listed in Table 1. The physical degrees of freedom are the reprentations with $N_{\text{phys}} = N - N_{\text{gauge}} > 0$. The physical representations are listed in Table 2. The infinite series are parametrized by j rather than j_R so that the symmetry $j_L \leftrightarrow j_R$ becomes $j \leftrightarrow -j$.

8 Five ODEs

8.1 $\mathfrak{su}(2)$ notation

The $\mathfrak{su}(2)$ generators in the $j = 1/2$ representation are

$$L_a^{(1/2)} = \mu_a = \frac{i\sigma_a}{2} \qquad [\mu_a,\, \mu_b] = -\epsilon_{ab}{}^c\mu_c \tag{8.1}$$

The generators L_a of a general representation satisfy the same commutation relations,

$$[L_a,\, L_b] = -\epsilon_{ab}{}^c L_c \tag{8.2}$$

$J-j_3$	j_3	J_{12}	$J_{23}-j_3$	N	N_{gauge}	N_{phys}
0	0	0	1	1	0	1
0	$\frac{1}{2}$	$0,1$	$0,1$	2	1	1
0	≥ 1	$0,1,2$	$-1,0,1$	3	1	2
1	0	1	1	1	1	0
1	$\geq \frac{1}{2}$	$1,2$	$0,1$	2	1	1
-1	1	1	0	1	1	0
-1	$\geq \frac{3}{2}$	$1,2$	$-1,0$	2	1	1
2	≥ 0	2	1	1	0	1
-2	≥ 2	2	-1	1	0	1

Table 1: Decomposition of $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{j_R}$.

	j_L	j_R		J_{12}	J_{23}	N	N_{gauge}	N_{phys}
$\mathbf{1}$	0	0		0	1	1	0	1
$\mathbf{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$0,1$	$\frac{1}{2},\frac{3}{2}$	2	1	1
$\mathbf{3}_j$	$j-\frac{1}{2}$	$j-\frac{1}{2}$	$\frac{3}{2}\leq j$	$0,1,2$	$j-\frac{3}{2},j-\frac{1}{2},j+\frac{1}{2}$	3	1	2
$\mathbf{2}_j$	j	$j-1$	$\frac{3}{2}\leq j$	$1,2$	$j-1,j$	2	1	1
	$-j-1$	$-j$	$j\leq -\frac{3}{2}$	$1,2$	$-j-1,-j$	2	1	1
$\mathbf{1}_j$	$j+\frac{1}{2}$	$j-\frac{3}{2}$	$\frac{3}{2}\leq j$	2	$j-\frac{1}{2}$	1	0	1
	$-j-\frac{3}{2}$	$-j+\frac{1}{2}$	$j\leq -\frac{3}{2}$	2	$-j-\frac{1}{2}$	1	0	1

Table 2: The physical representations $N_{\text{phys}} = N - N_{\text{gauge}} > 0$.

The Casimir operator is

$$C = -\frac{1}{2}L^a L_a \tag{8.3}$$

using δ_{ab} to raise/lower indices. The Casimir for the irreducible representation j is

$$C_j = \frac{1}{2}j(j+1) \tag{8.4}$$

The adjoint representation is $j=1$,

$$L_a^{(1)}v_b = \epsilon_{ab}{}^c v_c \qquad C_1 = 1 \tag{8.5}$$

8.2 The representation $\{B_i\}$

The Dirac matrices on S^3 form an $\mathfrak{su}(2)$-valued 1-form which can be written

$$\hat\gamma_i(\hat x) = \hat\gamma_i^a(\hat x)\mu_a \qquad \hat\gamma_i^a\hat\gamma_a^j = \delta_i^j \qquad \hat\gamma_i^a\hat\gamma_b^i = \delta_b^a \tag{8.6}$$

where indices are raised/lowered with $\hat{g}_{ij}$ and δ_{ab}. The gauge field perturbation B_i can be written

$$B_i(\hat{x}) = B_i^a(\hat{x})\mu_a = B_{ab}(\hat{x})\hat{\gamma}_i^b\mu^a \tag{8.7}$$

making explicit the representation

$$\{B_{ab}(\hat{x})\} = \bigoplus_{j_3}(j_1 \otimes j_2 \otimes j_3, j_3) \qquad j_1 = j_2 = 1 \tag{8.8}$$

Let L_a^1, L_a^2, L_a^3 be the generators of the representations j_1, j_2, j_3.

$$L_a^1 B_{bc} = \epsilon_{ab}{}^d B_{dc} \qquad L_a^2 B_{bc} = \epsilon_{ac}{}^d B_{bd} \qquad L_a^3 B_{bc} = \frac{1}{2}\hat{\gamma}_a^i(\hat{x})\hat{\partial}_i B_{bc} \tag{8.9}$$

The last formula is verified by noting that L_a^3 is the generator of left multiplication acting on functions so should act as a right-invariant vector field and by checking at the north pole.

$$\hat{\gamma}_a^i(\hat{N}) = \delta_a^i \qquad g_{\hat{N}+\epsilon^a\hat{\gamma}_a^i/2} = 1 - \epsilon^a\mu_a \tag{8.10}$$

Tthe Casimirs of the individual factors are

$$C_{j_1} = C_{j_2} = 1 \qquad C_{j_3} = \frac{1}{2}j_3(j_3+1) \tag{8.11}$$

Write

$$\begin{aligned} L_a^{12} &= L_a^1 + L_a^2 \qquad C_{12} = -\frac{1}{2}\delta^{ab}L_a^{12}L_b^{12} = -\frac{1}{2}\delta^{ab}(L_a^1 + L_a^2)(L_b^1 + L_b^2) \\ L_a^{23} &= L_a^2 + L_a^3 \qquad C_{23} = -\frac{1}{2}\delta^{ab}L_a^{23}L_b^{23} = -\frac{1}{2}\delta^{ab}(L_a^2 + L_a^3)(L_b^2 + L_b^3) \end{aligned} \tag{8.12}$$

In the decompositions (7.3)

$$C_{12} = \frac{1}{2}J_{12}(J_{12}+1) \qquad C_{23} = \frac{1}{2}J_{23}(J_{23}+1) \tag{8.13}$$

8.3 Γ and $*\hat{\nabla}$ in terms of Casimirs

The operator Γ defined in (6.18) is

$$\Gamma B_i = \epsilon_i{}^{jk}[\hat{\gamma}_j,\, B_k] = \epsilon_i{}^{jk}[\hat{\gamma}_j^c\mu_c,\, B_{a'b'}\hat{\gamma}_k^{b'}\mu^{a'}] = -\epsilon_i{}^{jk}\hat{\gamma}_j^c\hat{\gamma}_k^{b'}B_{a'b'}\epsilon_c{}^{a'}{}_a\mu^a \tag{8.14}$$

Now use

$$\epsilon_{abc} = \epsilon_{ijk}\hat{\gamma}_a^i\hat{\gamma}_b^j\hat{\gamma}_c^k \tag{8.15}$$

to get

$$\begin{aligned} \Gamma(B_{ab}\mu^a\hat{\gamma}_i^b) = -\epsilon_{i'}{}^{jk}\hat{\gamma}_b^{i'}\gamma_j^c\hat{\gamma}_k^{b'}\epsilon_c{}^{a'}{}_a B_{a'b'}\mu^a\hat{\gamma}_i^b &= -\epsilon^{cb'}\epsilon_c{}^{a'}{}_a B_{a'b'}\mu^a\hat{\gamma}_i^b \\ \Gamma B_{ab} = -\delta^{cd}\epsilon_{ca}{}^{a'}\epsilon_{db}{}^{b'}B_{a'b'} & \end{aligned} \tag{8.16}$$

$$\Gamma = -\delta^{cd}L_c^1L_d^2 = C_{12} - C_{j_1} - C_{j_2} = C_{12} - 2 \tag{8.17}$$

Next, define

$$*d = *\hat{\nabla} - \Gamma \tag{8.18}$$

$$
\begin{aligned}
*d(B_{ab}\hat{\gamma}^b_i\mu^a) &= \epsilon_i{}^{jk}\hat{\partial}_j(B_{ab}\hat{\gamma}^b_k\mu^a) = \epsilon_i{}^{jk}(\hat{\partial}_j B_{ab})\hat{\gamma}^b_k\mu^a + \epsilon_i{}^{jk}B_{ab}\hat{\partial}_j\hat{\gamma}^b_k\mu^a \\
&= -\epsilon^c{}_b{}^{b'}(\hat{\gamma}^j_c\hat{\partial}_j B_{ab'})\hat{\gamma}^b_i\mu^a + \epsilon_{i'}{}^{jk}\hat{\gamma}^{i'}_b\hat{\partial}_j\hat{\gamma}^{b'}_k B_{ab'}\hat{\gamma}^b_i\mu^a \\
*dB_{ab} &= -2\delta^{cd}L^2_c L^3_d + A^{b'}_b B_{ab'} \qquad A^{b'}_b = \epsilon_{i'}{}^{jk}\hat{\gamma}^{i'}_b\hat{\partial}_j\hat{\gamma}^{b'}_k \\
A^{b'}_b\mu_{b'} &= \epsilon_{i'}{}^{jk}\hat{\gamma}^{i'}_b\hat{\partial}_j\hat{\gamma}_k = \epsilon_{i'}{}^{jk}\hat{\gamma}^{i'}_b\frac{1}{2}\hat{\partial}_{[j}\hat{\gamma}_{k]} = -\epsilon_{i'}{}^{jk}\hat{\gamma}^{i'}_b[\hat{\gamma}_j,\,\hat{\gamma}_k] \\
&= \epsilon_{i'}{}^{jk}\hat{\gamma}^{i'}_b\epsilon_{jk}{}^i\hat{\gamma}_i = 2\gamma^i_b\hat{\gamma}_i = 2\gamma^i_b\hat{\gamma}^c_i\mu_c = 2\mu_b \\
A^{b'}_b &= 2\delta^{b'}_b \\
*d &= 2(C_{23} - C_{j_2} - C_{j_3}) + 2 = 2(C_{23} - C_{j_3})
\end{aligned} \tag{8.19}
$$

8.4 Wigner 6-j symbol and Racah W-coefficient

The two decompositions (7.3) give two canonical bases for $\mathbb{C}^{N(J,j_3)}$,

$$\mathbb{C}^{N(J,j_3)} = \bigoplus_{J_{12}} \hat{e}_{12}(J_{12}) \otimes \mathbb{C} = \bigoplus_{J_{23}} \hat{e}_{23}(J_{23}) \otimes \mathbb{C} \tag{8.20}$$

C_{12} is diagonal in the first basis, C_{23} in the second.

$$C_{12}\,\hat{e}_{12}(J_{12}) = \frac{1}{2}J_{12}(J_{12}+1)\,\hat{e}_{12}(J_{12}) \qquad C_{23}\,\hat{e}_{23}(J_{23}) = \frac{1}{2}J_{23}(J_{23}+1)\,\hat{e}_{23}(J_{23}) \tag{8.21}$$

But they cannot be diagonalized simultaneously unless $N = 1$. The two bases are related by a matrix,

$$\hat{e}_{12}(J_{12}) = \sum_{J_{23}} U(J_{12}, J_{23})\,\hat{e}_{23}(J_{23}) \tag{8.22}$$

The matrix elements $U(J_{12}, J_{23})$ are called the *recoupling coefficients*. Traditionally the recoupling coefficients are written

$$U(J_{12}, J_{23}) = \langle\,(j_1,(j_2j_3)J_{23})J\,|\,((j_1j_2)J_{12}j_3)J\,\rangle \tag{8.23}$$

They are related to the Wigner 6-j symbol and the Racah W-coefficient,

$$\frac{U(J_{12}, J_{23}))}{\sqrt{(2J_{12}+1)(2J_{23}+1)}} = (-1)^{j_1+j_2+j_3+J}\left\{\begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23}\end{array}\right\} = W(j_1j_2Jj_3;\,J_{12}J_{23}) \tag{8.24}$$

Given U, the two Casimirs can be written as matrices in the same basis,

$$C_{12} = \frac{1}{2}J_{12}(J_{12}+1) \qquad C_{23} = U\,\frac{1}{2}J_{23}(J_{23}+1)\,U^t \tag{8.25}$$

Racah's algorithm for the W-coefficient [1] is:

$$
\begin{gathered}
W(j_1j_2Jj_3;\,J_{12}J_{23}) = W(abcd;\,ef) \\
a = j_1 \quad b = j_2 \quad c = J \quad d = j_3 \quad e = J_{12} \quad f = J_{23}
\end{gathered} \tag{8.26}
$$

$$
\begin{gathered}
W(abcd;ef) = \Delta(a,b,e)\,\Delta(c,d,e)\,\Delta(a,c,f)\,\Delta(b,d,f)\;w(abcd;ef) \\
\Delta(a,b,c) = \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{1/2}
\end{gathered} \tag{8.27}
$$

$$w(abcd;ef) = \sum_z \frac{(-1)^{z+\beta_1}(z+1)!}{\prod_{i=1}^{4}(z-\alpha_i)!\ \prod_{i=1}^{3}(\beta_i - z)!} \tag{8.28}$$

The sum is over integers z in the range

$$\max(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq z \leq \min(\beta_1, \beta_2, \beta_3)$$
$$\alpha_1 = a+b+e \qquad \alpha_2 = c+d+e \qquad \alpha_3 = a+c+f \qquad \alpha_4 = b+d+f \tag{8.29}$$
$$\beta_1 = a+b+c+d \qquad \beta_2 = a+d+e+f \qquad \beta_3 = b+c+e+f$$

8.5 The five $\mathbf{K}(z)$ matrices

The quadratic action is (6.15)

$$\frac{1}{\hbar}S_2 = \frac{1}{\epsilon g^2}\int \mathrm{tr}(\partial_z B_i \partial_z B^i - B_i \mathbf{K} B^i)\,\sqrt{-\hat{g}}\,d^3\hat{x}\,dz \tag{8.30}$$

with equation of motion

$$\left(\frac{d^2}{dz^2} + \mathbf{K}(z)\right) B_i = 0 \tag{8.31}$$

Equation (6.22) gives $\mathbf{K}(z)$ as

$$\mathbf{K} = \left(\Gamma^2 - \Gamma\right) b^2 + \left(*\hat{\nabla}\Gamma + \Gamma *\hat{\nabla}\right)\epsilon b + \left[(*\hat{\nabla})^2 + \Gamma\right]\epsilon^2 \tag{8.32}$$

now expressed in terms of Casimirs,

$$\Gamma = C_{12} - 2 \qquad *\hat{\nabla} - \Gamma = 2C_{23} - j_3(j_3+1) \tag{8.33}$$

For each irreducible (j_L, j_R) occuring in the space $\{B_i\}$ of gauge field perturbations, $\mathbf{K}(z)$ is an $N\times N$ matrix acting on $\mathbb{C}^N$, $N = N(j_L, j_R)$.

The $\mathbf{K}(z)$ matrix for **Case 1** $(j_L, j_R) = (0, 0)$ is calculated by hand below. The $\mathbf{K}(z)$ matrices for the other four sets of representations listed in Table 2 are calculated in Sagemath in the notebook *K(z) matrices*. For the series of representations indexed by j, the parameter p is defined as

$$p^2 = 4\epsilon^2 C_{(j_L,j_R)} = 2\epsilon^2\left[j_L(j_L+1) + j_R(j_R+1)\right] \tag{8.34}$$

The representations are indexed by $(j_L, j_R) = (J, j_3)$.

Case 1

	j_L	j_R	J_{12}	$J_{23} - j_3$	N	N_{gauge}	N_{phys}
1	0	0	0	1	1	0	1

$$C_{12} = 0 \quad C_{23} = 1 \qquad \Gamma = -2 \quad *\hat{\nabla} = 0$$
$$\mathbf{K} = 6b^2 - 2\epsilon^2 \tag{8.35}$$

Case 2

	j_L	j_R	J_{12}	$J_{23}-j_3$	N	N_{gauge}	N_{phys}
2	$\frac{1}{2}$	$\frac{1}{2}$	$0,1$	$0,1$	2	1	1

$$\mathbf{K} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} b^2 + \begin{pmatrix} 0 & -3\sqrt{2} \\ -3\sqrt{2} & 0 \end{pmatrix} \epsilon b + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \epsilon^2 \tag{8.36}$$

Case $\mathbf{3_j}$

	j_L	j_R		J_{12}	$J_{23}-j_3$	N	N_{gauge}	N_{phys}
$\mathbf{3}_j$	$j-\frac{1}{2}$	$j-\frac{1}{2}$	$\frac{3}{2}\leq j$	$0,1,2$	$-1,0,1$	3	1	2

$$\begin{gathered} \mathbf{K}_2 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} b^2 + \begin{pmatrix} 0 & -6 & 0 \\ -6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sigma b + \begin{pmatrix} \alpha^2 & 0 & \alpha \\ 0 & \alpha^2+1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} 2\sigma^2 \\ p^2 = 4\epsilon^2 \left(j^2 - \frac{1}{4} \right) \qquad \sigma = \sqrt{\frac{2}{3}} \sqrt{j^2 - \frac{1}{4}}\, \epsilon = \frac{p}{\sqrt{6}} \qquad \alpha = \sqrt{2} \sqrt{\frac{j^2-1}{j^2-\frac{1}{4}}} \end{gathered} \tag{8.37}$$

Case $\mathbf{2_j}$

	j_L	j_R		J_{12}	$J_{23}-j_3$	N	N_{gauge}	N_{phys}
$\mathbf{2}_j$	j	$j-1$	$\frac{3}{2}\leq j$	$1,2$	$0,1$	2	1	1
	$-j-1$	$-j$	$j\leq -\frac{3}{2}$	$1,2$	$-1,0$	2	1	1

$$\begin{gathered} \mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} b^2 + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} 2\sigma b + \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix} 2\sigma^2 \\ p^2 = 4\epsilon^2 j^2 \qquad \sigma = \epsilon j = \frac{1}{2} p \qquad \alpha = \sqrt{1 - \frac{1}{j^2}} \end{gathered} \tag{8.38}$$

Case $\mathbf{1_j}$

	j_L	j_R		J_{12}	$J_{23}-j_3$	N	N_{gauge}	N_{phys}
$\mathbf{1}_j$	$j+\frac{1}{2}$	$j-\frac{3}{2}$	$\frac{3}{2}\leq j$	2	1	1	0	1
	$-j-\frac{3}{2}$	$-j+\frac{1}{2}$	$j\leq -\frac{3}{2}$	2	-1	1	0	1

$$\begin{gathered} \mathbf{K} = 4\sigma b + 4\alpha^2\sigma^2 \\ p^2 = 4\epsilon^2 \left(j^2 + \frac{3}{4} \right) \qquad \sigma = \epsilon j = \frac{1}{2}\sqrt{p^2 - 3\epsilon^2} \qquad \alpha = \sqrt{1 + \frac{1}{4j^2}} \end{gathered} \tag{8.39}$$

8.6 Gauge symmetries

A gauge variation of the classical solution is a solution of the equation of motion that is first order in $b(z)$.

$$w_{\text{gauge}}(z) = w_1 b(z) + w_0 \tag{8.40}$$

Use

$$b'' = 2\epsilon^2 b - 2b^3 \tag{8.41}$$

and write

$$\mathbf{K} = \mathbf{K}_2 b^2 + \mathbf{K}_1 b + \mathbf{K}_0 \tag{8.42}$$

to calculate

$$\begin{aligned} 0 &= \frac{d^2 w}{dz^2} + \mathbf{K}(z) w \\ &= w_1(2\epsilon^2 b - 2b^3) + \left(\mathbf{K}_2 b^2 + \mathbf{K}_1 b + \mathbf{K}_0\right)(w_1 b + w_0) \\ &= (\mathbf{K}_2 - 2) w_1 b^3 + (\mathbf{K}_2 w_0 + \mathbf{K}_1 w_1) b^2 \\ &\qquad + (2\epsilon^2 w_1 + \mathbf{K}_0 w_1 + \mathbf{K}_1 w_0) b + \mathbf{K}_0 w_0 \end{aligned} \tag{8.43}$$

So the coefficients w_0, w_1 are the solutions of

$$\begin{aligned} 0 &= \mathbf{K}_0 w_0 \\ 0 &= (\mathbf{K}_2 - 2) w_1 \\ 0 &= \mathbf{K}_2 w_0 + \mathbf{K}_1 w_1 \\ 0 &= \mathbf{K}_1 w_0 + (\mathbf{K}_0 + 2\epsilon^2) w_1 \end{aligned} \tag{8.44}$$

Case 2

$$\mathbf{K}_2 = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mathbf{K}_1 = \begin{pmatrix} 0 & -3\sqrt{2} \\ -3\sqrt{2} & 0 \end{pmatrix} \epsilon \qquad \mathbf{K}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \epsilon^2 \tag{8.45}$$

$$\begin{gathered} w_1 = B \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad w_0 = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 0 = A \begin{pmatrix} 6 \\ 0 \end{pmatrix} + B \begin{pmatrix} -3\sqrt{2} \\ 0 \end{pmatrix} \epsilon \\ A = \frac{\epsilon}{\sqrt{2}} B \qquad 0 = A \begin{pmatrix} 0 \\ -3\sqrt{2} \end{pmatrix} \epsilon + B \begin{pmatrix} 0 \\ 3 \end{pmatrix} \epsilon^2 = 0 \end{gathered} \tag{8.46}$$

$$w_{\text{gauge}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} b + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\epsilon}{\sqrt{2}} \tag{8.47}$$

Case $3_\mathbf{j}$

$$\begin{gathered} \mathbf{K}_2 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K}_1 = \begin{pmatrix} 0 & -6 & 0 \\ -6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sigma_3 \qquad \mathbf{K}_0 = \begin{pmatrix} \alpha_3^2 & 0 & \alpha_3 \\ 0 & \alpha_3^2 + 1 & 0 \\ \alpha_3 & 0 & 1 \end{pmatrix} 2\sigma_3^2 \\ \sigma_3 = \sqrt{\frac{2}{3}} \sqrt{j^2 - \frac{1}{4}}\, \epsilon \qquad \alpha_3 = \sqrt{2} \sqrt{\frac{j^2 - 1}{j^2 - \frac{1}{4}}} \end{gathered} \tag{8.48}$$

$$w_1 = B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad w_0 = A \begin{pmatrix} 1 \\ 0 \\ -\alpha_3 \end{pmatrix}$$

$$0 = A \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + B \begin{pmatrix} -6\sigma_3 \\ 0 \\ 0 \end{pmatrix} \qquad A = \sigma_3 B \tag{8.49}$$

$$0 = A \begin{pmatrix} 0 \\ -6\sigma_3 \\ 0 \end{pmatrix} + B \begin{pmatrix} 0 \\ 2\epsilon^2 + (\alpha_3^2 + 1) 2\sigma_3^2 \\ 0 \end{pmatrix} \tag{8.50}$$

$$\begin{aligned} 0 &= B\left[-6\sigma_3^2 + 2\epsilon^2 + 2\sigma_3^2(\alpha_3^2 + 1)\right] = 2B\left[\epsilon^2 + \sigma_3(\alpha_3^2 - 2)\right] \\ &= 2B\epsilon^2 \left[1 + \frac{2}{3}\left(j^2 - \frac{1}{4}\right) 2 \left(\frac{-\frac{3}{4}}{j^2 - \frac{1}{4}}\right)\right] \\ &= 0 \end{aligned} \tag{8.51}$$

$$w_{\text{gauge}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} b + \begin{pmatrix} 1 \\ 0 \\ -\alpha_3 \end{pmatrix} \sigma_3 \tag{8.52}$$

Case $\mathbf{2_j}$

$$\mathbf{K}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{K}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} 2\sigma_2 \qquad \mathbf{K}_0 = \begin{pmatrix} \alpha_2^2 & \alpha_2 \\ \alpha_2 & 1 \end{pmatrix} 2\sigma_2^2$$

$$\sigma_2 = \epsilon j \qquad \alpha_2 = \sqrt{1 - \frac{1}{j^2}} \tag{8.53}$$

$$w_1 = B \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad w_0 = A \begin{pmatrix} 1 \\ -\alpha_2 \end{pmatrix}$$

$$0 = A \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\sigma_2 \begin{pmatrix} -B \\ 0 \end{pmatrix} \qquad A = \sigma_2 B \tag{8.54}$$

$$0 = B \begin{pmatrix} 2\epsilon^2 + 2\alpha_2^2\sigma_2^2 \\ 2\alpha_2\sigma_2^2 \end{pmatrix} + \sigma_2 B \begin{pmatrix} -2\sigma_2 \\ -2\alpha_2\sigma_2 \end{pmatrix} = 2B \begin{pmatrix} \epsilon^2 + \alpha_2^2\sigma_2^2 - \sigma_2^2 \\ 0 \end{pmatrix} = 0$$

$$w_{\text{gauge}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} b + \begin{pmatrix} 1 \\ -\alpha_2 \end{pmatrix} \sigma_2 \tag{8.55}$$

9 Time-translation zero-mode

The zero-mode is written as a free particle in the reparametrized time $\tilde{z}$ given by

$$\frac{d\tilde{z}}{dz} = \frac{1}{\mathrm{cn}'(z)^2} \tag{9.1}$$

The following calculates the periodicities of $\tilde{z}(z)$. The results are not used in the paper, but will be relevant for further investigation of the zero-mode integral.

The identities in section 9.1 below are used to integrate.

$$\begin{aligned}\tilde{z} &= \int \frac{dz}{\operatorname{sn}(z)^2\operatorname{dn}(z)^2} = \int \frac{k^2\operatorname{sn}(z)^2+\operatorname{dn}(z)^2}{\operatorname{sn}(z)^2\operatorname{dn}(z)^2}\,dz\\ &= k^2\int \operatorname{nd}^2\,dz + \int \operatorname{ns}^2\,dz\\ &= \frac{k^2}{k'^2}(\mathcal{E}-k^2\operatorname{sn}\operatorname{cd}) + z - \operatorname{dn}\operatorname{cs} - \mathcal{E} \qquad (9.2)\\ &= \left(\frac{k^2}{k'^2}-1\right)\mathcal{E} + z - \frac{k^4}{k'^2}\frac{\operatorname{sn}\operatorname{cn}}{\operatorname{dn}} - \frac{\operatorname{dn}\operatorname{cn}}{\operatorname{sn}}\\ &= \left(\frac{k^2}{k'^2}-1\right)\mathcal{E} + z - \frac{\operatorname{cn}}{k'^2\operatorname{sn}\operatorname{dn}}\left(k^4\operatorname{sn}^2+k'^2\operatorname{dn}^2\right)\end{aligned}$$

$$k^4\operatorname{sn}^2+k'^2\operatorname{dn}^2 = k^4\operatorname{sn}^2+k'^2(1-k^2\operatorname{sn}^2) = k'^2+k^2(k^2-k'^2)\operatorname{sn}^2 \qquad (9.3)$$

$$\begin{aligned}\tilde{z} &= \left(\frac{k^2}{k'^2}-1\right)\mathcal{E} + z - \frac{\operatorname{cn}}{\operatorname{sn}\operatorname{dn}} - \frac{\operatorname{cn}}{k'^2\operatorname{sn}\operatorname{dn}}k^2(k^2-k'^2)\operatorname{sn}^2\\ &= -\frac{\operatorname{cn}}{\operatorname{sn}\operatorname{dn}} + z + \left(\frac{k^2}{k'^2}-1\right)\left(\mathcal{E}-k^2\frac{\operatorname{cn}\operatorname{sn}}{\operatorname{dn}}\right)\end{aligned} \qquad (9.4)$$

So $\tilde{z}$ has periodicities

$$\begin{aligned}\tilde{z}(z+2K)-\tilde{z}(z) &= 2K + \left(\frac{k^2}{k'^2}-1\right)(\mathcal{E}(z+2K)-\mathcal{E}(z))\\ &= 2K + \left(\frac{k^2}{k'^2}-1\right)2E\end{aligned} \qquad (9.5)$$

$$\begin{aligned}\tilde{z}(z+2K'i)-\tilde{z}(z) &= 2K'i + \left(\frac{k^2}{k'^2}-1\right)(\mathcal{E}(z+2K'i)-\mathcal{E}(z))\\ &= 2K'i + \left(\frac{k^2}{k'^2}-1\right)(2K'-2E')i\end{aligned} \qquad (9.6)$$

9.1 More on Jacobi elliptic functions

These identities are from the DLMF [2, Chapters 22 and 19].

$\operatorname{sn}(z,k)$ **and** $\operatorname{dn}(z,k)$

$$\begin{aligned}\operatorname{sn}^2+\operatorname{cn}^2 = 1 &\qquad k^2\operatorname{sn}^2+\operatorname{dn}^2 = 1\\ \operatorname{sn}(z)\sim z+O(z^3) &\qquad \operatorname{dn}(z)\sim 1+O(z^2)\end{aligned} \qquad (9.7)$$

$$\begin{aligned}\operatorname{cn}' = -\operatorname{sn}\operatorname{dn} &\qquad \operatorname{dn}' = -k^2\operatorname{sn}\operatorname{cn}\\ \operatorname{dn}'' = (1+k'^2)\operatorname{dn}-2\operatorname{dn}^3 &\qquad (\operatorname{dn}')^2 = (1-\operatorname{dn}^2)(\operatorname{dn}^2-k'^2)\end{aligned} \qquad (9.8)$$

half-periods

$$
\begin{aligned}
&\operatorname{cn}(z+2K) = -\operatorname{cn}(z) \qquad \operatorname{sn}(z+2K) = -\operatorname{sn}(z) \qquad \operatorname{dn}(z+2K) = \operatorname{dn}(z)\\
&\operatorname{cn}(z+2K'i) = -\operatorname{cn}(z) \qquad \operatorname{sn}(z+2K'i) = \operatorname{sn}(z) \qquad \operatorname{dn}(z+2K'i) = -\operatorname{dn}(z)
\end{aligned} \tag{9.9}
$$

the other Jacobi elliptic functions

$$
\mathrm{nc} = \frac{1}{\mathrm{cn}} \qquad \mathrm{ns} = \frac{1}{\mathrm{sn}} \qquad \mathrm{nd} = \frac{1}{\mathrm{dn}} \qquad \mathrm{cd} = \frac{\mathrm{cn}}{\mathrm{dn}} \qquad \mathrm{cs} = \frac{\mathrm{cn}}{\mathrm{sn}} \tag{9.10}
$$

Jacobi's amplitude function $\phi = \operatorname{am}(z,k)$

$$
\begin{gathered}
\phi = \operatorname{am}(z) = \int_0^z \mathrm{dn} \qquad \operatorname{sn}(z) = \sin(\phi) \qquad \operatorname{cn}(z) = \cos(\phi)\\
\operatorname{am}(z+2K) = \operatorname{am}(z) + \pi \qquad \operatorname{am}(2K) = \pi
\end{gathered} \tag{9.11}
$$

Jacobi epsilon function $\mathcal{E}(z,k)$ **and zeta function** $\mathcal{Z}(z,k)$

$$
\mathcal{E}(z) = \int_0^z \mathrm{dn}^2 = k^2 \,\mathrm{sn}\,\mathrm{cd} + k'^2 \int \mathrm{nd}^2 = z - \mathrm{dn}\,\mathrm{cs} - \int \mathrm{ns}^2 \tag{9.12}
$$

$$
\mathcal{Z}(z) = \mathcal{E}(z) - \frac{E(k)}{K(k)} z = \frac{\theta_0'}{\theta_0} \tag{9.13}
$$

$E(k)$ is the complete elliptic integral of the second kind,

$$
E(k) = \mathcal{E}(K) = E(\pi/2, k) \qquad E(\phi,k) = \mathcal{E}(z,k) \tag{9.14}
$$

satisfying

$$
EK' + E'K - KK' = \frac{\pi}{2} \tag{9.15}
$$

$$
\mathcal{E}(-z) = -\mathcal{E}(z) \qquad \mathcal{E}(z) = z + O(z^3) \tag{9.16}
$$

$$
\begin{aligned}
\mathcal{E}(u+v) &= \mathcal{E}(u) + \mathcal{E}(v) - k^2 \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{sn}(u+v)\\
\mathcal{Z}(u+v) &= \mathcal{Z}(u) + \mathcal{Z}(v) - k^2 \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{sn}(u+v)
\end{aligned} \tag{9.17}
$$

$$
\mathcal{Z}(z+2K) = \mathcal{Z}(z) \qquad \mathcal{Z}(z+2K'i) = \mathcal{Z}(z) - \frac{\pi}{K} i \tag{9.18}
$$

$$
\mathcal{E}(z+2K) = \mathcal{E}(z) + 2E \qquad \mathcal{E}(z+2K'i) = \mathcal{E}(z) + 2\,(K' - E')\, i \tag{9.19}
$$

$$
\begin{aligned}
\mathcal{E}(z+2K'i) &= \mathcal{E}(z) + \mathcal{Z}(z+2K'i) - \mathcal{Z}(z) + \frac{E}{K} 2K'i\\
&= \mathcal{E}(z) - \frac{\pi}{K} i + \frac{E}{K} 2K'i\\
&= \mathcal{E}(z) + \frac{2i}{K}\left(-\frac{\pi}{2} + EK'\right)\\
&= \mathcal{E}(z) + 2\,(K' - E')\, i
\end{aligned} \tag{9.20}
$$

10 Classical mechanics analytic in complex time z

$$\mathcal{A}(z) = \begin{pmatrix} 0 & -1 \\ \mathbf{K}(z) & 0 \end{pmatrix} \qquad \Omega = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.1}$$

$$\begin{aligned} \mathcal{A}(z)^t\Omega + \Omega\mathcal{A}(z) &= \begin{pmatrix} 0 & \mathbf{K}(z)^t \\ -1 & 0 \end{pmatrix} i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \mathbf{K}(z) & 0 \end{pmatrix} \\ &= i \begin{pmatrix} \mathbf{K}(z) - \mathbf{K}(z)^t & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{10.2}$$

$$\mathbf{K}(z) = \mathbf{K}(z)^t \quad \Longleftrightarrow \quad \mathcal{A}(z)^t\Omega + \Omega\mathcal{A}(z) = 0 \tag{10.3}$$

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Canonical commutation relations

$$\begin{gathered} \mathcal{Q} = \begin{pmatrix} q \\ p \end{pmatrix} \qquad \mathcal{Q}^t = \begin{pmatrix} q^t & p^t \end{pmatrix} \qquad \mathcal{Q}\mathcal{Q}^t = \begin{pmatrix} qq^t & qp^t \\ pq^t & pp^t \end{pmatrix} \\ \left(\mathcal{Q}\mathcal{Q}^t\right)^t - \mathcal{Q}\mathcal{Q}^t = \begin{pmatrix} (qq^t)^t - qq^t & (pq^t)^t - qp^t \\ (qp^t)^t - pq^t & (pp^t)^t - pp^t \end{pmatrix} \end{gathered} \tag{11.1}$$

$$\left(\mathcal{Q}\mathcal{Q}^t\right)^t - \mathcal{Q}\mathcal{Q}^t = \Omega \quad \Longleftrightarrow \quad \begin{aligned} (qq^t)^t - qq^t &= 0 \\ (pp^t)^t - pp^t &= 0 \\ (pq^t)^t - qp^t &= i \end{aligned} \tag{11.2}$$

$$\begin{gathered} \left[(qq^t)^t - qq^t\right]^{ab} = q^b q^a - q^a q^b \qquad \left[(pp^t)^t - pp^t\right]_{ab} = p_b p_a - p_a p_b \\ [(pq^t)^t - qp^t]^a_b = p_b q^a - q^a p_b \end{gathered} \tag{11.3}$$

$$\left(\mathcal{Q}\mathcal{Q}^t\right)^t - \mathcal{Q}\mathcal{Q}^t = \Omega \quad \Longleftrightarrow \quad \begin{aligned} &[q^a,\, q^b] = 0 \\ &[p_a,\, p_b] = 0 \\ &[p_b,\, q^a] = i\delta^a_b \end{aligned} \tag{11.4}$$

Phase-space action

$$\begin{aligned} \frac{1}{2i}\mathcal{Q}^t\Omega\left(\frac{d}{dz} + \mathcal{A}\right)\mathcal{Q} &= \frac{1}{2}\begin{pmatrix} q^t & p^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{d}{dz} + \begin{pmatrix} 0 & -1 \\ \mathbf{K}(z) & 0 \end{pmatrix}\right) \begin{pmatrix} q \\ p \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} q^t & p^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{dq}{dz} - p \\ \frac{dp}{dz} + Kq \end{pmatrix} \\ &= -\frac{1}{2}p^t\frac{dq}{dz} + \frac{1}{2}q^t\frac{dp}{dz} + \frac{1}{2}p^t p + \frac{1}{2}q^t K q \end{aligned} \tag{11.5}$$

18 Numerical evidence for Property P

18.1 Imaginary period monodromy $\mathcal{M}_{\mathrm{i}}$ at $t = K$

Identities for Jacobi elliptic functions. For $\tau \in \mathbb{R}$,

$$\begin{gathered} \operatorname{cn}(K+\tau i, k) = -k' \operatorname{sd}(\tau i, k) = -k' \operatorname{sd}(\tau, k') i \\ \operatorname{sd}(\tau, k') = \overline{\operatorname{sd}(\tau, k')} \\ \operatorname{sd}(-\tau, k') = -\operatorname{sd}(\tau, k') \qquad \operatorname{sd}(\tau + 2K, k') = -\operatorname{sd}(\tau, k') \end{gathered} \tag{18.1}$$

Let $t = K$ and let C_K be the vertical path passing through K.

$$z = K + \tau i \qquad b(z) = k \operatorname{cn}(z, k) = -kk' \operatorname{sd}(\tau, k') i = F(\tau) i \tag{18.2}$$

$$F(\tau) = -kk' \operatorname{sd}(\tau, k') \qquad \mathbf{K}(z) = -\mathbf{K}_2 F(\tau)^2 + \mathbf{K}_0 + \mathbf{K}_1 F(\tau) i \tag{18.3}$$

$$F(\tau) = \overline{F(\tau)} \qquad F(-\tau) = -F(\tau) \qquad F(\tau + 2K') = -F(\tau) \tag{18.4}$$

Write the propagator along C_K

$$\mathcal{P}(\tau_2, \tau_1) = \mathcal{P}_{C_K}(K + \tau_2 i, K + \tau_2 i) \tag{18.5}$$

The symmetries of $F(\tau)$ give

$$\mathcal{P}(4K' - \tau_2, 4K' - \tau_1) = \overline{\mathcal{P}(\tau_2, \tau_1)} \qquad \mathcal{P}(2K' - \tau_2, 2K' - \tau_1) = \mathcal{R}\mathcal{P}(\tau_2, \tau_1)\mathcal{R} \tag{18.6}$$

The imaginary period monodromy matrix is

$$\mathcal{M}_{\mathrm{i}} = \mathcal{P}(4K', 0) \tag{18.7}$$

Define

$$\mathcal{M}_{\mathrm{i}/4} = \mathcal{P}(K', 0) \qquad \mathcal{M}_{\mathrm{i}/2} = \mathcal{P}(2K', 0) \tag{18.8}$$

Then

$$\begin{aligned} \mathcal{M}_{\mathrm{i}/2} &= \mathcal{P}(2K', K')\mathcal{P}(K', 0) = \mathcal{R}\mathcal{P}(0, K')\mathcal{R}\mathcal{P}(K', 0) \\ &= \mathcal{R}\mathcal{M}_{\mathrm{i}/4}^{-1}\mathcal{R}\mathcal{M}_{\mathrm{i}/4} = \mathcal{R}\Omega\mathcal{M}_{\mathrm{i}/4}^{t}\Omega\mathcal{R}\mathcal{M}_{\mathrm{i}/4} \\ \mathcal{M}_{\mathrm{i}} &= \mathcal{P}(4K', 2K')\mathcal{P}(2K', 0) = \overline{\mathcal{P}(0, 2K')}\mathcal{P}(2K', 0) \\ &= \overline{\mathcal{M}_{\mathrm{i}/2}}^{-1}\mathcal{M}_{\mathrm{i}/2} = \Omega\mathcal{M}_{\mathrm{i}/2}^{\dagger}\Omega\mathcal{M}_{\mathrm{i}/2} \end{aligned} \tag{18.9}$$

so to calculate $\mathcal{M}_{\mathrm{i}}$ it is enough to calculate $\mathcal{M}_{\mathrm{i}/4}$. It is enough to integrate the ode from K to $K + K'i$.

18.2 $\mathcal{V}_{\text{phys}}(t)$

For an ode with $N_{\text{gauge}} > 0$, the gauge solution is $\mathcal{W}_{\text{gauge}}(z)$. The physical phase-space $\mathcal{V}_{\text{phys}}(t)$ is the quotient $\mathcal{W}_{\text{gauge}}^{\perp}/\mathbb{C}\mathcal{W}_{\text{gauge}}$ where $\mathcal{W}_{\text{gauge}}^{\perp}\mathbb{C}^N$ is the Ω-complement of $\mathcal{W}_{\text{gauge}}(z)$. For the numerical computations it is useful to represent $\mathcal{V}_{\text{phys}}(t)$ as a subspace of $\mathcal{W}_{\text{gauge}}^{\perp}$. Write

$$\mathcal{V} = \mathcal{V} \qquad \mathcal{V}_{\text{phys}} = \mathcal{V}_{\text{phys}}(t) \qquad \mathcal{W}_{\text{gauge}} = \mathcal{W}_{\text{gauge}}(t) \tag{18.10}$$

Define the vector

$$\tilde{\mathcal{W}}_{\text{gauge}} = i\Omega\mathcal{W}_{\text{gauge}} \tag{18.11}$$

satisfying

$$\begin{gathered}\mathcal{W}_{\text{gauge}}^t\tilde{\mathcal{W}}_{\text{gauge}} = 0 \qquad \mathcal{W}_{\text{gauge}}^t\mathcal{W}_{\text{gauge}} = \tilde{\mathcal{W}}_{\text{gauge}}^t\tilde{\mathcal{W}}_{\text{gauge}} \\ \mathcal{W}_{\text{gauge}}^t\Omega\tilde{\mathcal{W}}_{\text{gauge}} = i\mathcal{W}_{\text{gauge}}^t\mathcal{W}_{\text{gauge}}\end{gathered} \tag{18.12}$$

$\mathcal{W}_{\text{gauge}}$ is real so $\tilde{\mathcal{W}}_{\text{gauge}}$ is also real. The orthogonal complement $\tilde{\mathcal{W}}_{\text{gauge}}^{\perp}$ is the Ω-orthogonal complement of $\mathcal{W}_{\text{gauge}}$.

$$\tilde{\mathcal{W}}_{\text{gauge}}^t\mathcal{W} = 0 \quad \Leftrightarrow \quad \mathcal{W}_{\text{gauge}}^t\Omega\mathcal{W} = 0 \tag{18.13}$$

$\tilde{\mathcal{W}}$ is not a natural vector since Ω is a bilinear form on $\mathcal{V}$, not a linear operator. $\tilde{\mathcal{W}}$ depends on a choice of bilinear form on $\mathcal{V}$. Choosing $\tilde{\mathcal{W}}_{\text{gauge}}$ give the decomposition

$$\mathcal{V} = \mathbb{C}\tilde{\mathcal{W}}_{\text{gauge}} \oplus \mathbb{C}\mathcal{W}_{\text{gauge}} \oplus \mathcal{V}_{\text{phys}} \tag{18.14}$$

representing $\mathcal{V}_{\text{phys}}$ as a codimension two subspace of $\mathcal{V}$. Define

$$P = \mathcal{W}_{\text{gauge}}(\mathcal{W}_{\text{gauge}}^t\mathcal{W}_{\text{gauge}})^{-1}\mathcal{W}_{\text{gauge}}^t \qquad \tilde{P} = \tilde{\mathcal{W}}_{\text{gauge}}(\tilde{\mathcal{W}}_{\text{gauge}}^t\tilde{\mathcal{W}}_{\text{gauge}})^{-1}\tilde{\mathcal{W}}_{\text{gauge}}^t \tag{18.15}$$

which are commuting projections

$$P^2 = P \qquad \tilde{P}^2 = \tilde{P} \qquad P\tilde{P} = \tilde{P}P = 0 \qquad P^t = P \qquad \tilde{P}^t = \tilde{P} \tag{18.16}$$

Then define

$$P_{\text{gauge}} = 1 - \tilde{P} \qquad P_{\text{phys}} = P_{\text{gauge}} - P = 1 - \tilde{P} - P \tag{18.17}$$

so

$$\begin{gathered}P_{\text{gauge}}\mathcal{V} = (W_{\text{gauge}})^{\perp\Omega} \qquad \mathcal{V} = \mathbb{C}\tilde{\mathcal{W}}_{\text{gauge}} \oplus (W_{\text{gauge}})^{\perp\Omega} \\ P_{\text{phys}}\mathcal{V} = \mathcal{V}_{\text{phys}} \qquad (W_{\text{gauge}})^{\perp\Omega} = \mathbb{C}\mathcal{W}_{\text{gauge}} \oplus \mathcal{V}_{\text{phys}} \\ \mathcal{V} = \mathbb{C}\tilde{\mathcal{W}}_{\text{gauge}} \oplus \mathbb{C}\mathcal{W}_{\text{gauge}} \oplus \mathcal{V}_{\text{phys}}\end{gathered} \tag{18.18}$$

$$\begin{gathered}i\Omega\mathcal{P}(i\Omega)^t = \tilde{\mathcal{W}}_{\text{gauge}}(\mathcal{W}_{\text{gauge}}^t\mathcal{W}_{\text{gauge}})^{-1}\tilde{\mathcal{W}}_{\text{gauge}}^t = \tilde{\mathcal{P}} \\ \Omega\mathcal{P}\Omega = \tilde{\mathcal{P}} \qquad \Omega\mathcal{P} = \tilde{\mathcal{P}}\Omega \qquad \Omega = \tilde{\mathcal{P}}\Omega\mathcal{P} + \mathcal{P}\Omega\tilde{\mathcal{P}}\end{gathered} \tag{18.19}$$

References

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