Pure radiation stars ($w = 1/3$)

(Some of this is probably in the literature.)

Preamble

In [1]:

```
%display latex
LE = lambda latex_string: LatexExpr(latex_string);
```

In [2]:

```
from timeit import default_timer as timer
start=timer()
end=timer()
```

In [3]:

```
from mpmath import mp
from mpmath import mpf,mpc
import sage.libs.mpmath.all as a
mp.pretty = True
```

In [4]:

```
def set_precision(decimal_precision=20):
    global RealNumber,Reals,myR,sage_binary_precision,sage_decimal_precision
    mp.dps = decimal_precision
    binary_precision=mp.prec
    sage_binary_precision=binary_precision+10
    sage_decimal_precision = floor(sage_binary_precision/log(10,2))
    Reals = RealField(sage_binary_precision)
    RealNumber = Reals
    myR = Reals
    pretty_print("mp.dps = ",mp.dps,"   mp decimal precision = ", floor(mp.prec/log(10,2)),\
                 "      sage decimal precision = ", sage_decimal_precision )
set_precision(decimal_precision=20)
```

$$\texttt{mp.dps =}20 \quad \texttt{mp decimal precision =}21 \quad \texttt{sage decimal precision =}24$$

TOV equations for a perfect fluid with constant w

The equation of state is

$$p = w\rho$$

with w constant. Change variables:

$$x = 4\pi G r^2 \rho \qquad y = \frac{Gm}{r}$$

The TOV equations for m and p become

$$r\frac{dx}{dr} = 2x\frac{1-(2+c)y-cwx}{1-2y} \qquad r\frac{dy}{dr} = x - y \qquad c = \frac{1}{2}\left(1+\frac{1}{w}\right)$$

The right hand sides are independent of t. The TOV equations are the flow equations for a vector field in the x, y plane.

The fixed point

There is a fixed point at

$$x_\infty = y_\infty = \frac{2w}{w^2+6w+1}$$

The linearized flow near the fixed point is

$$x = x_\infty + \delta x \quad y = y_\infty + \delta y$$

$$r\frac{d}{dr}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = -A\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \qquad A = \begin{pmatrix}a & b\\ -1 & 1\end{pmatrix} \qquad a = \frac{2w}{1+w} \qquad b = \frac{1+5w}{(1+w)^2}$$

The eigenvalues of A are

$$\lambda_{\pm} = \frac{1 + 3w \pm i\sqrt{3 + 22w - w^2}}{2(1+w)}$$

For $w = \frac{1}{3}$

$$A = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -1 & 1 \end{pmatrix} \qquad \lambda_{\pm} = \frac{3 \pm i\sqrt{23}}{4}$$

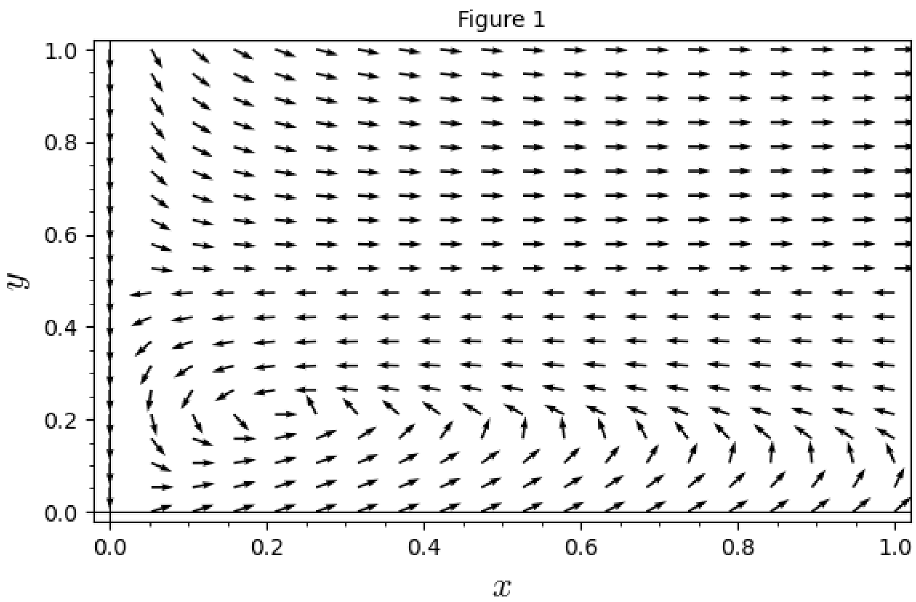
The fixed point is attractive. The flow spirals inwards.

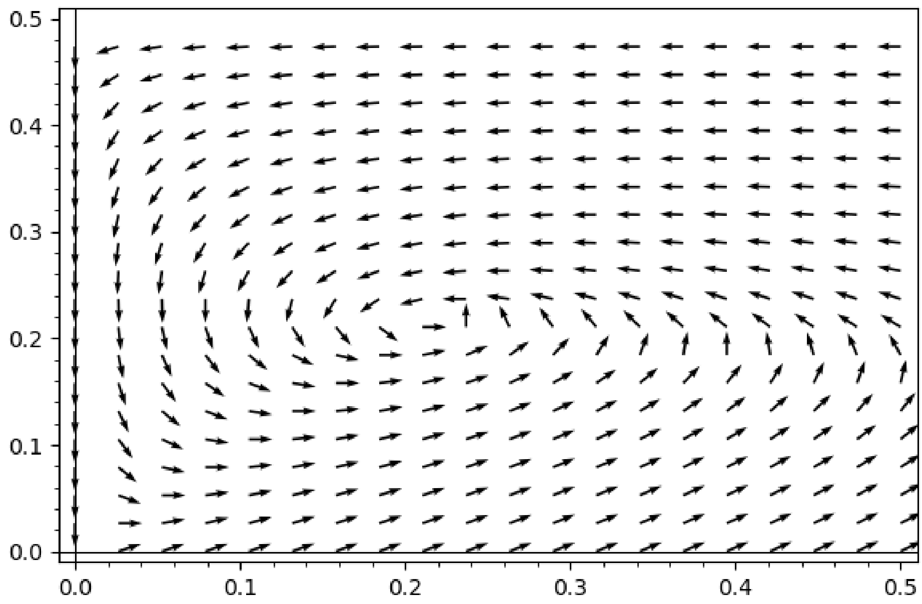
Phase portrait

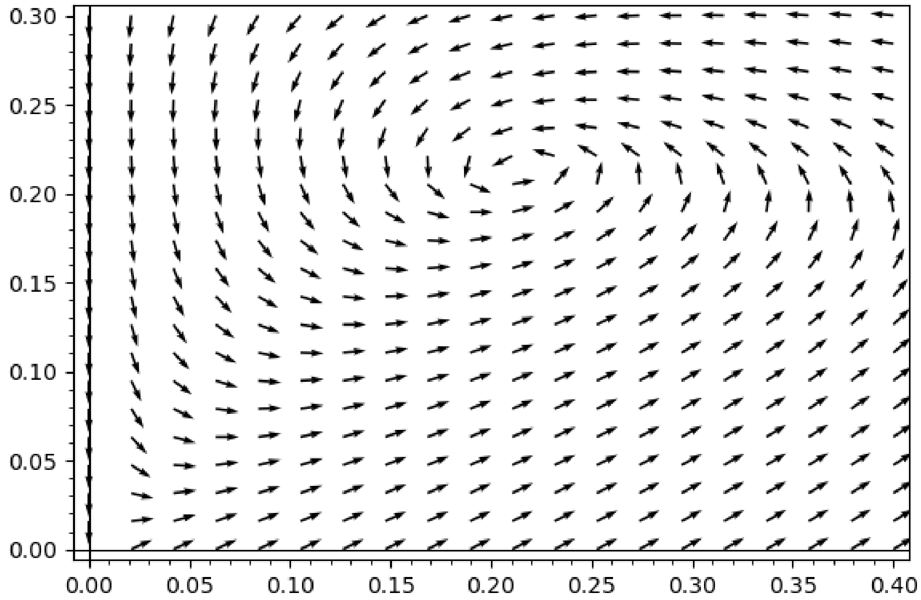
A plot of the flowlines, followed by two blowups.
The plot shows only direction. The flow vector is normalized to constant length.

In [5]:

```
x,y,w = var('x,y,w')
w= 1/3
vx = x*(2-(1+1/w)*(y+w*x)/(1-2*y))
vy = x-y
vnorm = sqrt(vx^2+vy^2)
vxhat  = vx/vnorm
vyhat  = vy/vnorm
#
plt=plot_vector_field((vxhat,vyhat),(x,0,1),(y,0,1))
#plt.set_axes_range(ymin=0)
plt.axes_labels([r"$x$",r"$y$"])
show(plt,title="Figure 1")
#plt.save('Figure_1_vf.pdf',title="Figure 1", dpi=300)
#
plt=plot_vector_field((vxhat,vyhat),(x,0,.5),(y,0,.5))
show(plt)
#
vfplot=plot_vector_field((vxhat,vyhat),(x,0,0.4),(y,0,.3))
show(vfplot)
```

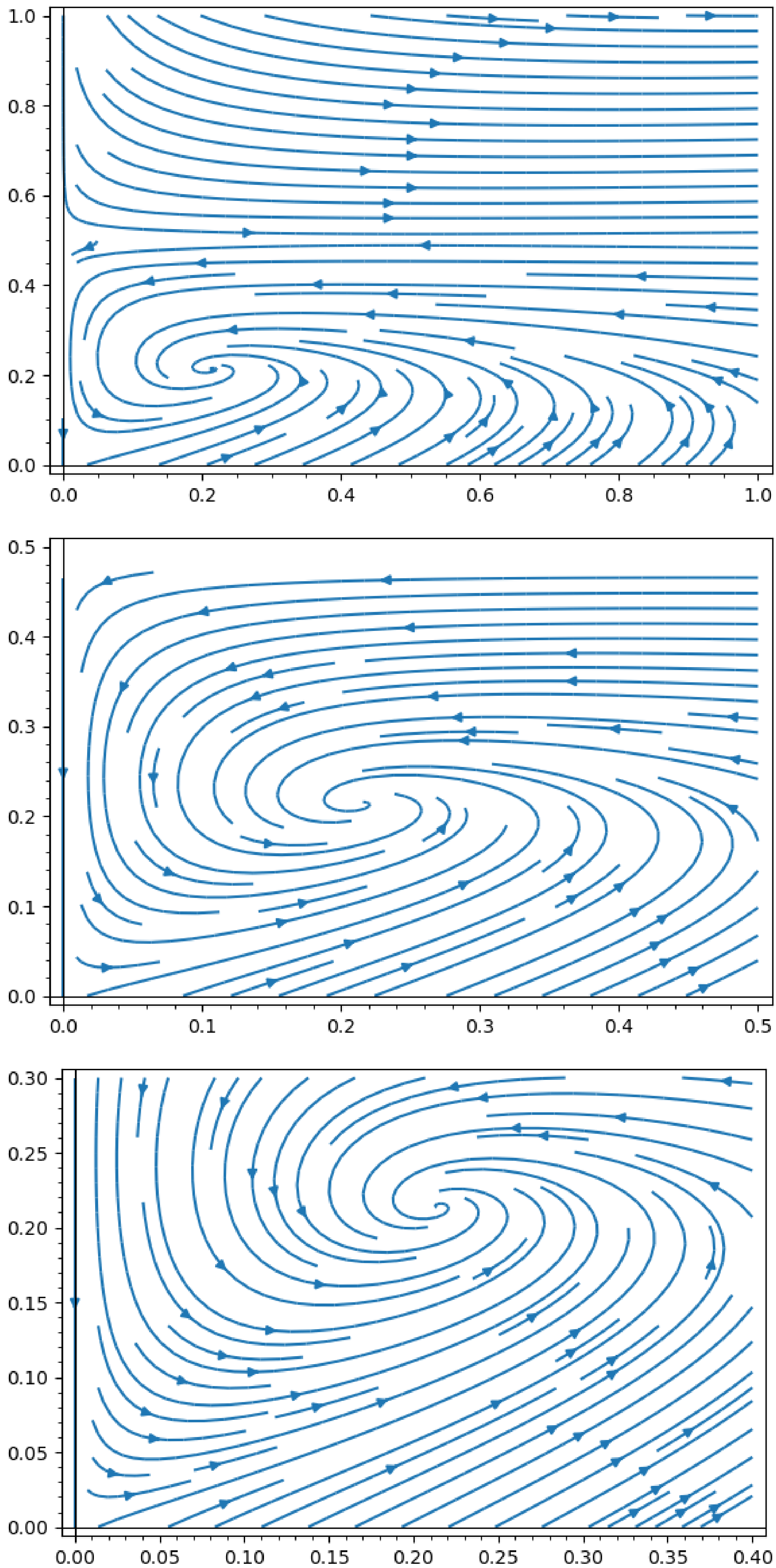






In [6]:

```
plt1 = streamline_plot((vx,vy),(x,0,1),(y,0,1))
plt2 = streamline_plot((vx,vy),(x,0,.5),(y,0,.5))
plt3 = streamline_plot((vx,vy),(x,0,0.4),(y,0,.3))
show(plt1)
show(plt2)
show(plt3)
```



The solutions regular at $r = 0$

Suppose $\rho = \rho_0$ at $r = 0$. Then in the limit $r \to 0$

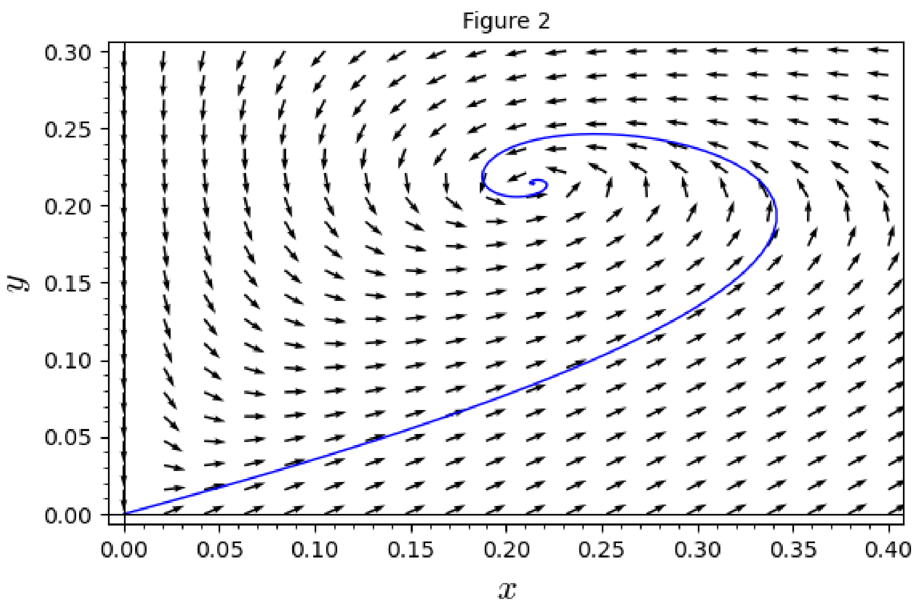
$$m \to \frac{4}{3}\pi r^3 \rho_0 \qquad x \to 4\pi G r^2 \rho_0 \qquad y \to \frac{1}{3}x$$

So all the solutions regular at $r = 0$ are on the trajectory that leaves $(0, 0)$ along the line $y = \frac{1}{3}x$.
Changing the initial condition $\rho(0) \to e^s \rho(0)$ is expressed by "time" translation along the flow, $\ln r \to \ln r + \frac{1}{2}s$

In [7]:

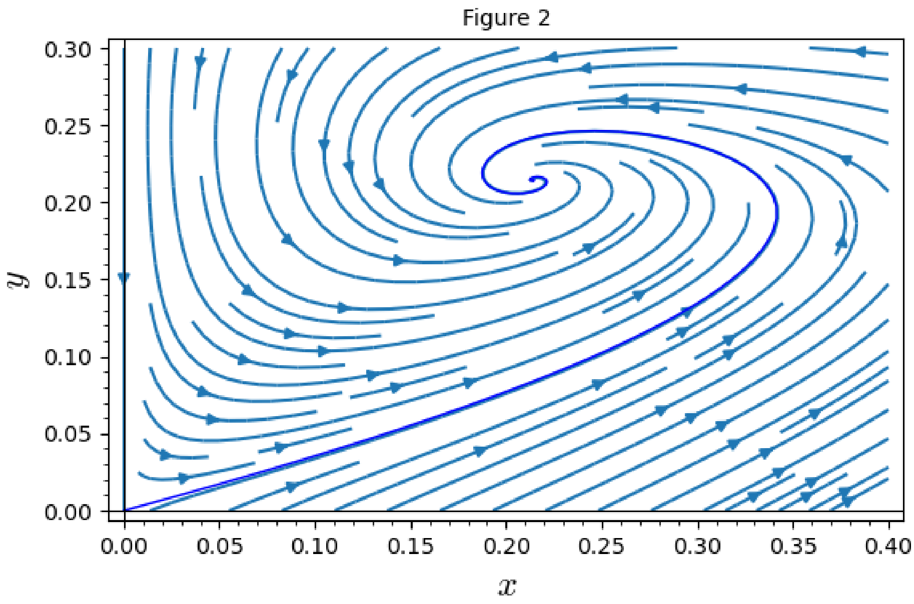
```
set_precision(20)
ode_tol = None
ode_degree = None
#
t = var('t')
xdot = lambda x,y: x*(2-(1+1/w)*(y+w*x)/(1-2*y))
ydot = lambda x,y: x-y
F = lambda t,y: [xdot(y[0],y[1]), ydot(y[0],y[1])]

t0 = mpf(-7)
F0 =mpf(1)
y0 = mp.exp(2*t0)*F0
y1 = y0/3
traj = mp.odefun(F,t0,[y0,y1],tol=ode_tol,degree=ode_degree)
#
trajlist0 = [traj(t) for t in mp.linspace(t0,-5,100)]
trajlist1 = [traj(t) for t in mp.linspace(-5,-3.5,100)]
trajlist2 = [traj(t) for t in mp.linspace(-3.5,0,100)]
trajlist3 = [traj(t) for t in mp.linspace(0,5,100)]
trajlist4 = [traj(t) for t in mp.linspace(5,10,100)]
trajplot=list_plot(trajlist0+trajlist1+trajlist2+trajlist3+trajlist4,marker=None,plotjoined=True)
#list_plot(trajlist0+trajlist1+trajlist2+trajlist3+trajlist4,marker='.')
plt=trajplot+vfplot
plt.axes_labels([r"$x$",r"$y$"])
show(plt,title="Figure 2")
#plt.save('Figure_2_vf.pdf',title="Figure 2", dpi=300)
```

```
mp.dps =20    mp decimal precision =21     sage decimal precision =24
```



In [8]:

```
plt=trajplot+plt3
plt.axes_labels([r"$x$",r"$y$"])
show(plt,title="Figure 2")
#plt.save('Figure_2_vf.pdf',title="Figure 2", dpi=300)
```



M-R star curve

As the flow approaches the fixed point, $r \to \infty$ with

$$\rho \to \frac{x_\infty}{4\pi G r^2} \qquad m \to \frac{y_\infty r}{G}$$

The radius and mass are infinite because there is no cutoff at low density.

So introduce a cutoff at ρ_{min}. That is, ρ decreases to ρ_{min} with $p = w\rho$, then for $\rho < \rho_{min}$, $p = 0$.

$$p = \begin{cases} w\rho & \rho > \rho_{min} \\ 0 & \rho \le \rho_{min} \end{cases}$$

The stellar radius R is given by $\rho(R) = \rho_{min}$.

Parametrize the regular trajectory by t with

$$r\frac{d}{dr} = \frac{d}{dt} \qquad \frac{dx}{dt} = 2x\frac{1-(2+c)y-cwx}{1-2y} \qquad \frac{dy}{dt} = x - y \qquad c = \frac{1}{2}\left(1+\frac{1}{w}\right)$$

Let $x(t), y(t)$ be the solution of the ode with initial condition

$$t \to -\infty \qquad x(t) \to e^{2t} \qquad y(t) \to \frac{1}{3}e^{2t}$$

Write the original change of variables as

$$\sqrt{4\pi G\rho}\, r = x^{1/2} \qquad \sqrt{4\pi G^3 \rho}\, m = x^{1/2} y$$

and define dimensionless variables

$$\hat{r} = \sqrt{4\pi G\rho_{min}}\, r \qquad \hat{m} = \sqrt{4\pi G^3\rho_{min}}\, m$$

There is a solution with radius $\hat{R}$ and mass $\hat{M}$ iff for some t

$$\hat{R} = x(t)^{1/2} \qquad \hat{M} = x(t)^{1/2} y(t)$$

So the star curve in the $\hat{M}$-$\hat{R}$ plane is

$$t \mapsto \hat{M}(t), \hat{R}(t) = x(t)^{1/2} y(t),\ x(t)^{1/2}$$

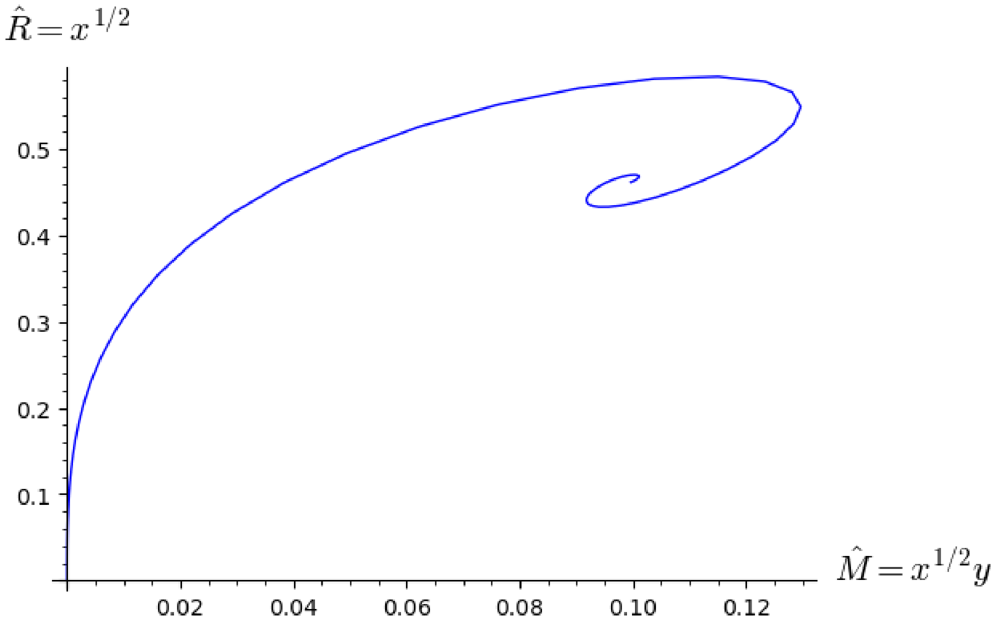
The central density $\rho(0)$ is a function of t. In the limit $t' \to -\infty$,

$$r = Re^{t'-t} \qquad \sqrt{4\pi G\rho} r = x(t')^{1/2} \qquad \sqrt{4\pi G\rho} Re^{t'-t} \to e^{t'} \qquad \sqrt{4\pi G\rho(0)} \frac{\hat{R}(t)}{\sqrt{4\pi G\rho_{min}}} e^{-t} = 1$$

$$\rho(0) = \rho_{min} \frac{e^{2t}}{\hat{R}(t)^2}$$

In [9]:

```
def MR(t):
    [x,y]=traj(t)
    xsqrt = sqrt(x)
    return [xsqrt*y,xsqrt]
MRlist = [MR(t) for t in mp.linspace(t0,5,100)]
plt = list_plot(MRlist,marker=None,plotjoined=True)
plt.axes_labels([r"$\hat M=x^{1/2} y$",r"$\hat R=x^{1/2}$"])
show(plt)
```



The curve near the fixed point is universal, independent of the low density cutoff.
Away from the fixed point, the curve depends on the cutoff.