Integrable Matrices and Time-Dependent Integrability

Emil Yuzbashyan

RUTGERS
THE STATE UNIVERSITY OF NEW JERSEY

Happy 75th Birthday, Sriram!!

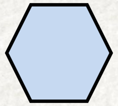
Current Trends in Strongly Correlated and Frustrated Systems
Max Planck Institute for the Physics of Complex Systems
Dresden, Germany, 10 – 14 November 2025

correlation
electrons
Shastry
-verse!
spins
frustration

mpipks

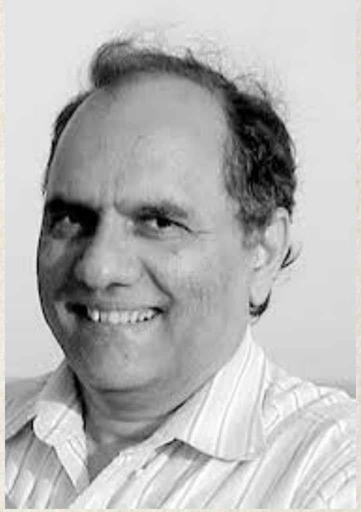
It all started in 2001 with a paper by Heilmann and Lieb:

"Violation of the noncrossing rule: The Hubbard Hamiltonian for benzene" (1971)

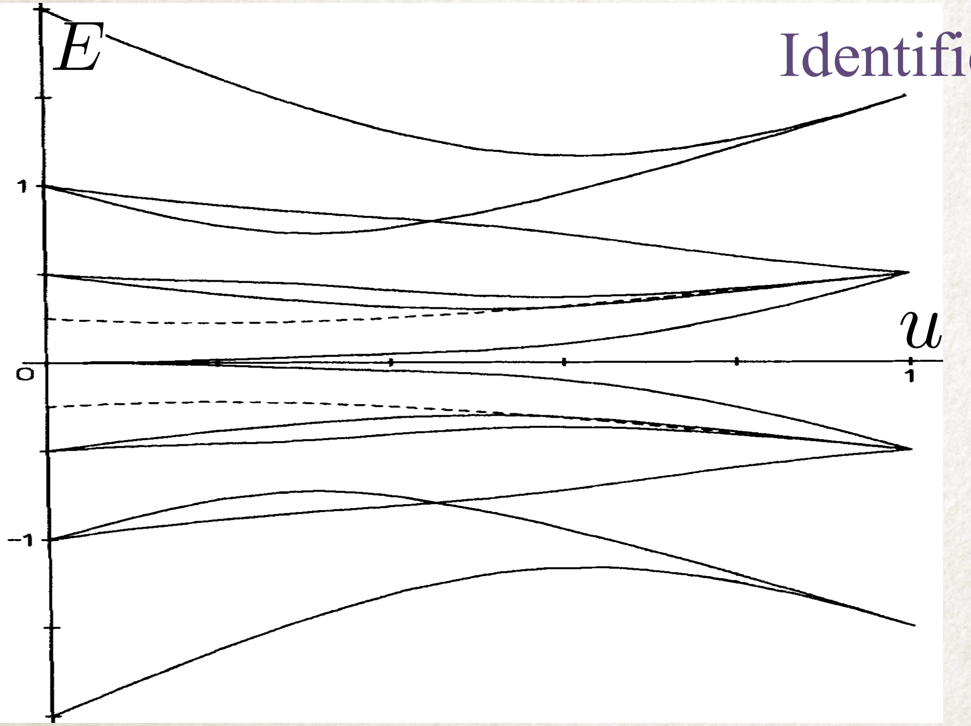
1D Hubbard model on  3 spin up & 3 spin down electrons

400 x 400 matrix of the form: $H(u) = T + uV$

E. Lieb



B. S. Shastry



Identified <u>all</u> symmetries: translation, reflection, spin, particle-hole…

Symmetry: u-independent Hermitian operator Ω
that commutes with the Hamiltonian $[\Omega, H(u)] = 0$

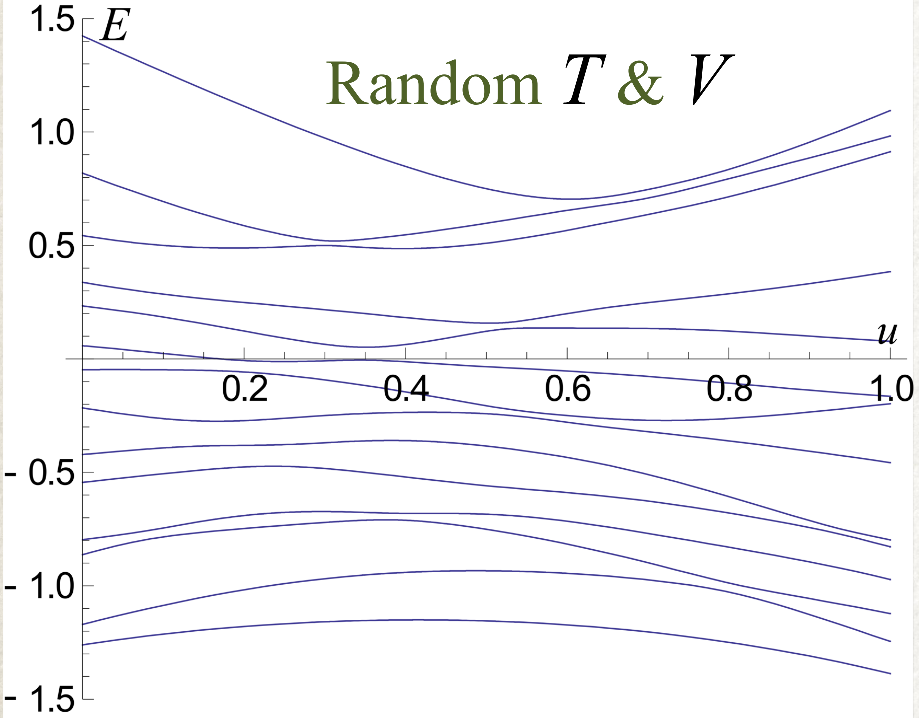
Multiple crossings of levels of the same symmetry
(same <u>complete</u> set of quantum numbers)

Noncrossing rule (theorem): “The intersection of terms [energy levels] of like symmetry is impossible” (E. Wigner and J. von Neumann 1929)

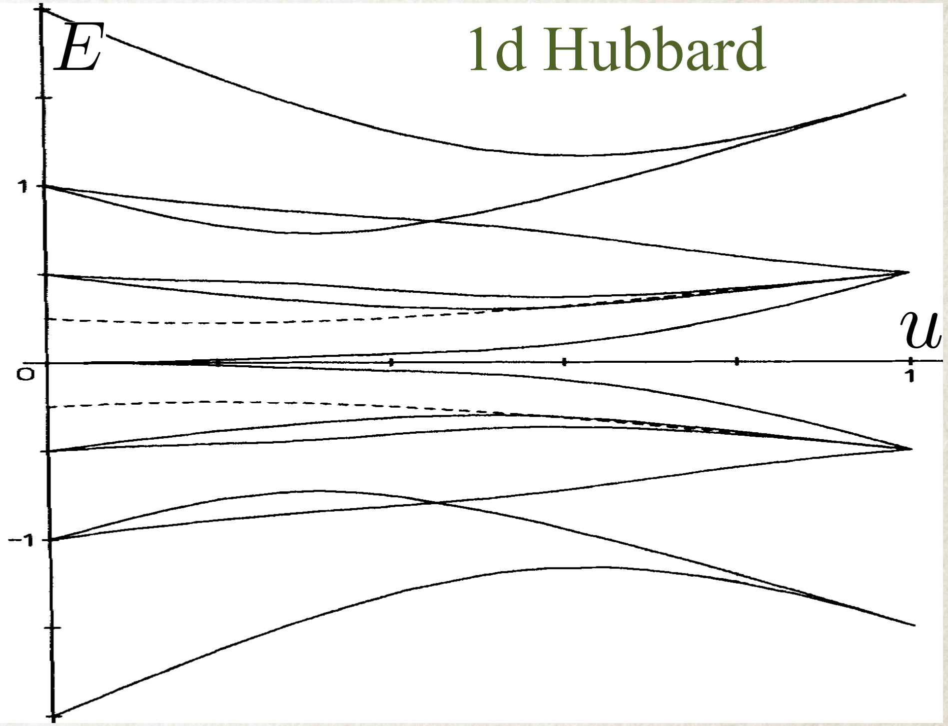

F. Hund


E. Wigner


J. von Neumann

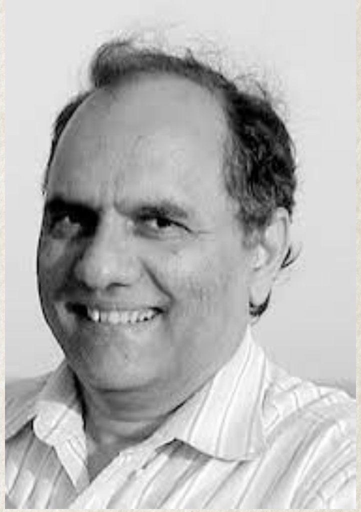


$$H(u) = T + uV$$



“The noncrossing rule is apparently violated in the case of the 1D Hubbard Hamiltonian for benzene molecule” (Heilmann and Lieb)

In addition to the usual space-time and internal space symmetries the 1D Hubbard has “dynamical” u-dependent symmetries discovered by Shastry in 1986
[B. S. Shastry, PRL 56, 1529 (1986)]



B. S. Shastry

$$\hat{H}(u) = \sum_{j,s=\uparrow\downarrow} (\hat{c}^\dagger_{js}\hat{c}_{j+1\,s} + \hat{c}^\dagger_{j+1\,s}\hat{c}_{js}) + u\sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow} \qquad \hat{n}_{j\sigma} = c^\dagger_{js}c_{j\,s}$$

$\widehat{H}_1(u)$ – linear in u $\quad$ $\widehat{H}_1(u), \widehat{H}_2(u), \widehat{H}_3(u)$ ... – higher order polynomials in u

$$\hat{H}_1(u) = -i\sum_{j=1}^{N}\sum_{s=\uparrow\downarrow}(c^\dagger_{j+2s}c_{js} - c^\dagger_{js}c_{j+2s}) - iu\sum_{j=1}^{N}\sum_{s=\uparrow\downarrow}(c^\dagger_{j+1s}c_{js} - c^\dagger_{js}c_{j+1s})(\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1)$$

This situation is generic in parameter-dependent integrable models (e.g., XXZ, BCS): they typically violate the nonxing rule and at least one symmetry is linear in the parameter

Q: Can parameter-dependent symmetries explain the level xings in integrable models?

Problem with allowing symmetries with arbitrary parameter dependence

Go to the u-dependent basis where $H(u)$ is diagonal:

$$H(u) = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & \times & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & 0 & \times & 0 \\ 0 & 0 & 0 & 0 & \times \end{pmatrix}$$

All diagonal matrices commute with H

For any H there is full set of H_k: $[H, H_k] = [H_k, H_j] = 0$

"If one allows symmetry groups that are u-dependent the 'theorem' is a mere tautology…one can always invent, *post hoc*, a u-dependent symmetry group to account for any violations" (Heilmann & Lieb, 1971). Just take a diagonal matrix with distinct eigenvalues!

All Hermitian matrices, parameter dependent or not, look the same. No way to tell one that came from an integrable system from any other.

Proposed solution: fix parameter dependence

Let $H(u) = T + uV$, u – real parameter, T, V – $N \times N$ Hermitian matrices

Suppose we require a commuting partner also linear in u: $H_1(u) = T_1 + uV_1$

$$[H(u), H_1(u)] = 0 \quad \text{for all } u$$

$$\Updownarrow$$

$$[V, V_1] = 0, \quad [T, V_1] = [T_1, V], \quad [T, T_1] = 0$$

These commutation relations severely constraint matrix elements of T. For a generic/typical $H(u)$ – no commuting partners except a linear combination of itself and identity. Now can separate generic (no integrals) from special (integrable).

B. S. Shastry, EY & collaborators: 2002 – 2025

N x N Hamiltonians linear in a parameter separate into two distinct classes

$H(u) = T + uV$ ⇨ No commuting partners linear in u other than a linear combination of itself and identity (typical) – nonintegrable, need $N^2/2$ real parameters to specify $H(u)$

⇩

Linearly independent commuting partners $H_k(u) = T_k + uV_k$ exist – integrable, turns out need less than $4N$ parameters – measure zero in the space of linear Hamiltonians

⇩

Classification by the number n of integrals of motion

$n = N - 1$ (maximum possible) – type 1 integrable matrix

$n = N - 2$ – type 2

…

$n = N - M$ – type M

…

Can obtain examples of integrable matrices from known integrable models: 1D Hubbard, XXZ, BCS. But let us abstract from all such models and ask the following questions:

Q: What follows from this definition alone, i.e., the existence of integrals of motion? In particular, can we construct such matrices generally for a given n, i.e., resolve the nonlinear commutation relations?

$$[V_i, V_j] = 0, \quad [T_i, V_j] = [T_j, V_i], \quad [T_i, T_j] = 0$$

$$i, j = 1, \dots, n$$

B. S. Shastry, EY & collaborators: 2002 – 2025

Simplest case: $n = N - 1$ (type 1 – max # of integrals)

Every type 1 family is uniquely specified by a choice of a Hermitian matrix and a vector

Hermitian matrix E Arbitrary vector $|\gamma\rangle$

⇓

N commuting $N \times N$ Hermitian matrices $H_i(u)$

General member of the commuting family: $H(u) = \sum_i d_i H_i(u) = T + uV$

To pick a typical member of the commuting family $H(u)$ pick matrix T (or V) randomly

$$[H(u)]_{km} = u\gamma_k\gamma_m \left(\frac{d_k - d_m}{\varepsilon_k - \varepsilon_m}\right), \quad [H(u)]_{mm} = d_m - u\sum_{j\neq m}\gamma_j^2\left(\frac{d_j - d_m}{\varepsilon_j - \varepsilon_m}\right)$$

ε_k – eigenvalues of E, d_k – eigenvalues of T, γ_k – components of $|\gamma\rangle$

Constructed all $n = N-1,\ N-2,\ N-3$ (types 1, 2, 3) and some for all other n

What follows from this definition of quantum integrability?

- Exact solution through a single algebraic equation for all types (cf. Bethe's Ansatz)

(type 1) $$\sum_{k=1}^{N}\frac{|\gamma_k|^2}{\lambda-\varepsilon_k}=0,\quad E(\lambda)=\sum_k\frac{d_k|\gamma_k|^2}{\lambda-\varepsilon_k},\quad |\lambda\rangle=\sum_k\frac{d_k\gamma_k}{\lambda-\varepsilon_k}$$

$d_k, \varepsilon_k, \gamma_k$ – given, solve for λ

- Number of level crossings as a function of type, i.e. the number (n) of integrals of motion

$$\#\text{ of crossings} = (N^2-5N+2)/2+n-2k,\quad k=1,2,\ldots$$

Typically $\approx N^2/2$ crossings.

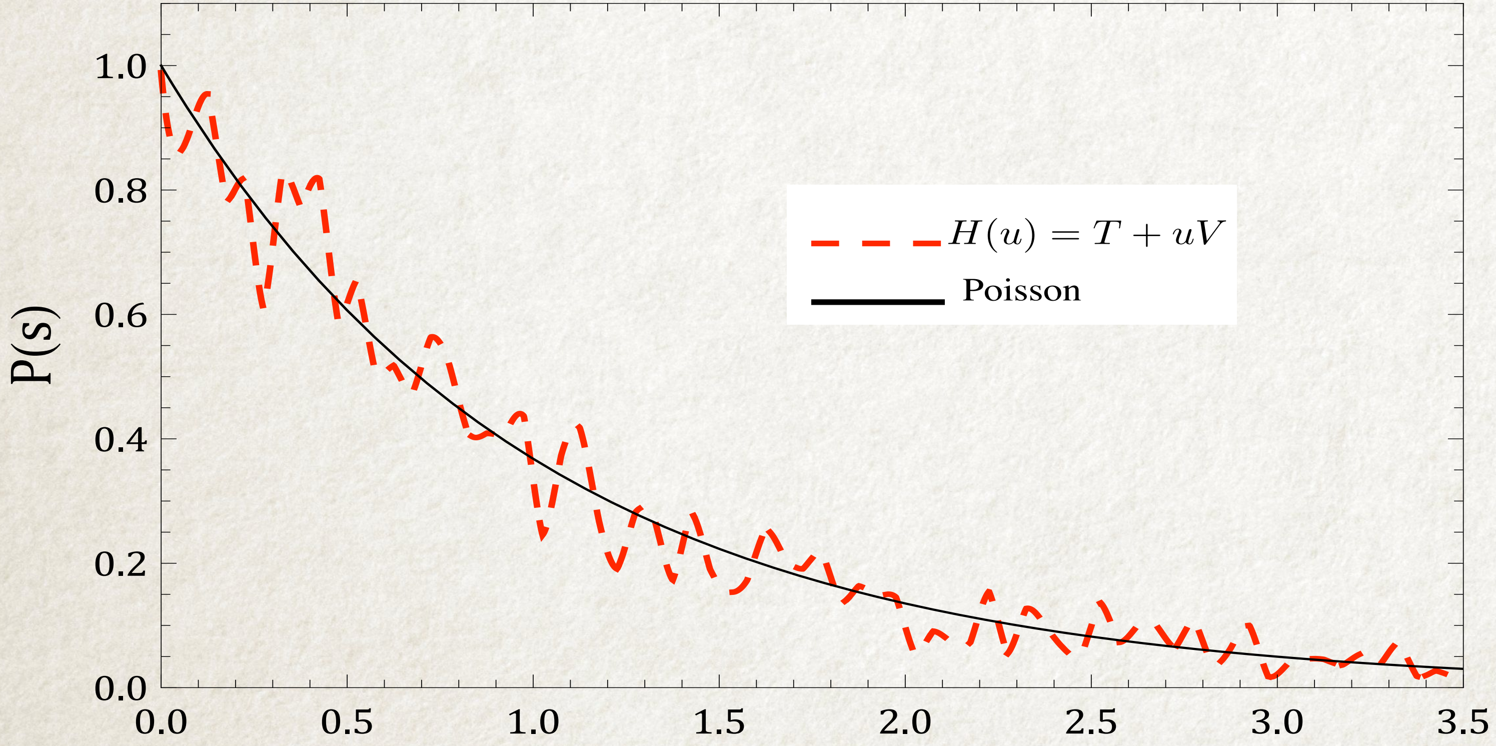
Any type 1 Hamiltonian has at least one crossing.

But for higher types it is also possible to have no crossings.

Integrable Matrix Theory (IMT) – ensemble theory of quantum integrability

Now can study ensembles of integrable matrices and obtain integrable counterparts of the RMT results as opposed to only a spectral statistics of isolated integrable models

I. Statistics are typically Poisson as long as the number of integrals (= size-type) isn't too small



Integrable matrices are ergodic

Nearest neighbor level spacing distribution for a 4000×4000 time reversal invariant integrable Hamiltonian $H(u) = T + uV$ at $u = 1$

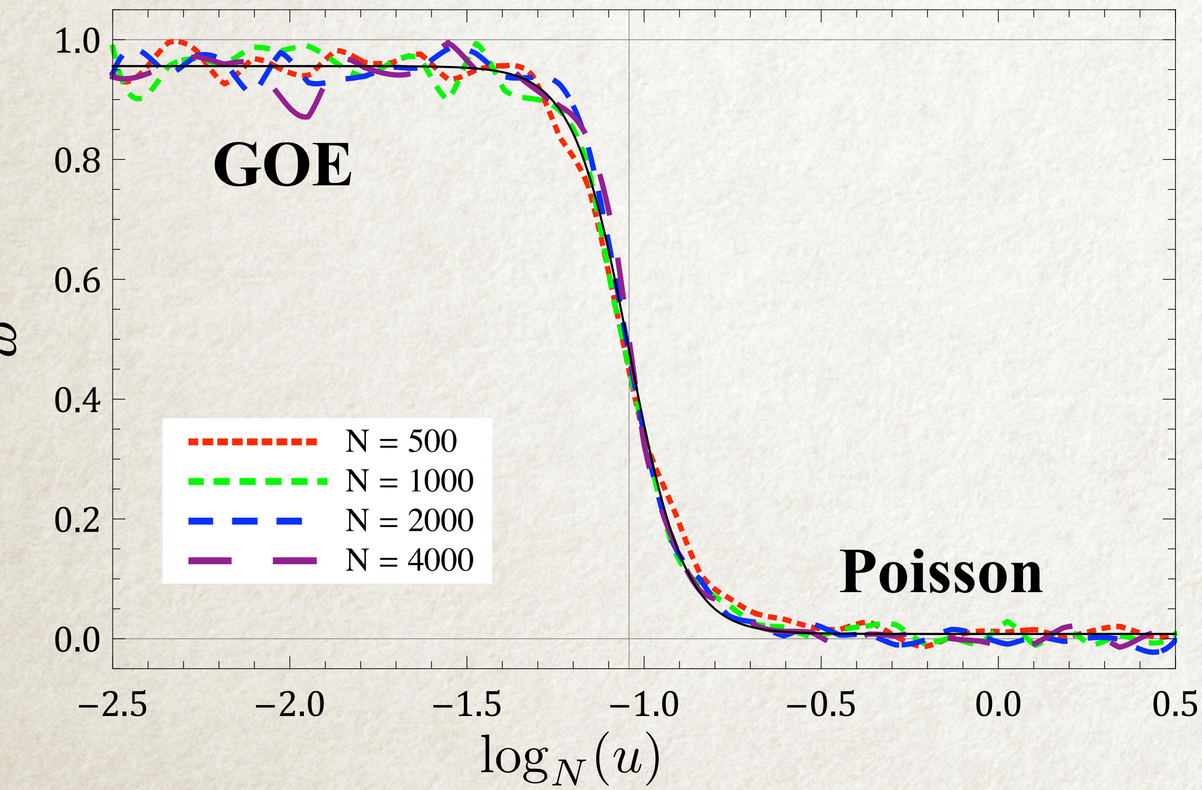
Integrable Matrix Theory, Level Statistics

I. Statistics are typically Poisson as long as the number of integrals (= size-type) isn't too small

II. There are two exceptions to Poisson statistics

A. There are isolated values of the coupling $u = u_0$ where the level statistics of $H(u) = T + uV$ are Wigner-Dyson (here $u_0 = 0$)

But it reverts to Poisson already at $(u - u_0) \propto 1/N$



Brody parameter ω as a function of $\log_N(u)$

Brody distribution: $P(s,\omega) = as^{\omega}e^{-bs^{\omega+1}}$

$P(s,1) = \frac{\pi}{2}se^{-\frac{\pi}{4}s^2}$ – Wigner – Dyson

$P(s,0) = e^{-s}$ – Poisson

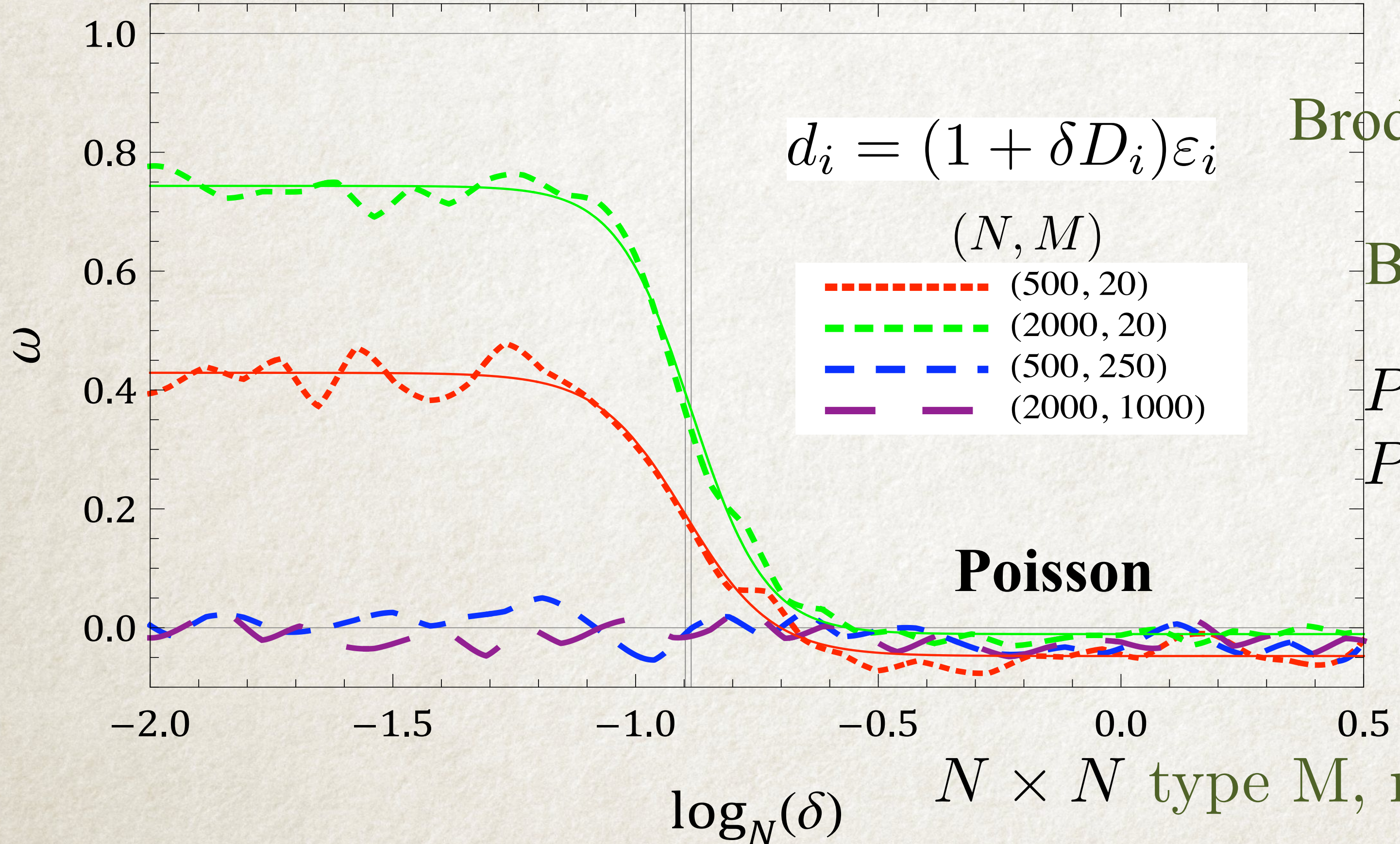
$N \times N$ type 1, number of integrals $= N - 1$

Exceptions to Poisson Statistics in IMT

A. There are isolated values of the coupling $u = u_0$ where the level statistics of $H(u) = T + uV$ are Wigner-Dyson

T, E – random matrices, e.g. from GOE, $|\gamma\rangle$ – random vector

B. Statistics are non-Poisson when normally uncorrelated parameters (matrices T and E) become correlated (atypical integrable model, e.g., BCS has $T = E$)



Brody parameter ω as a function of $\log_N(u)$

Brody distribution: $P(s, \omega) = a s^{\omega} e^{-b s^{\omega+1}}$

$P(s,1) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}$ – Wigner – Dyson

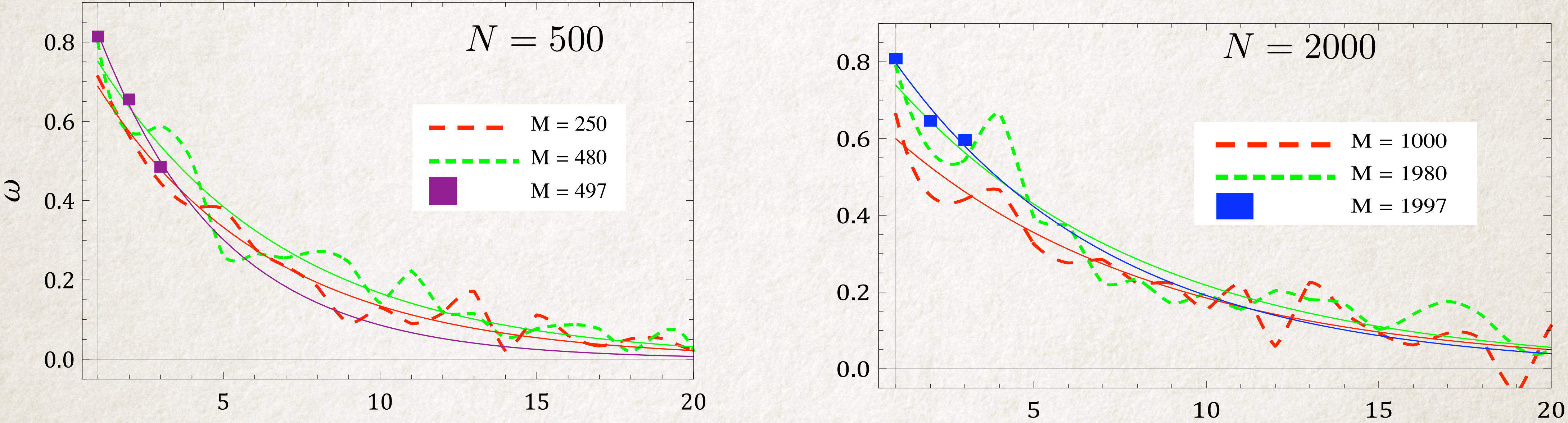
$P(s,0) = e^{-s}$ – Poisson

$N \times N$ type M, number of integrals $= N - M$, $u = 1$

Q: How many nontrivial integrals of motion must a system have so that its level statistics are Poisson?

# of nontrivial integrals = Size – Type = N - M

$$H(u) = \sum_{i=1}^{k} d_i H_i(u), \quad k \leq N - M$$



Brody parameter ω as a function of k for $N \times N$ type M matrices.
Fit: $a \exp(-bk/\ln N)$. $b = (1.13, 1.04; 0.99, 1.03)$ for $M = (250, 480; 1000, 1980)$

# of integrals needed $\approx \ln N =$ log of Hilbert space dim $\propto$ particle #

Connection to multi-level Landau-Zener problem

$$H(t) = A + Bt$$ Again, Hermitian matrix linear in a parameter (now parameter = time)

A, B – $N \times N$ time-independent Hermitian matrices

$$i\frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi \qquad \Psi(t \to -\infty) = |\text{in}\rangle, \quad \Psi(t \to +\infty) = S|\text{in}\rangle$$

S – scattering matrix = ?

Transition probabilities: $p_{i \to k} = |S_{ik}|^2$

$$N = 2$$

Landau, Zener, Majorana, Stuckelberg (1932)

$$H(t) = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} t$$

Survival probability (Landau-Zener formula): $p_{0 \to 0} = 1 - e^{-\frac{\pi g^2}{\lambda}}$

Only two new nontrivial exactly solvable multi-level Landau-Zener problems have been found from 1932 to 2018

And both of them are type 1 integrable matrices!

$$H_{\mathrm{DO}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & a_N \end{pmatrix} + t \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

To obtain the DO model set:

$$u = \lambda t, \quad d_1 = 1, \quad \varepsilon_1 = 0, \quad \gamma_1 = 1$$

$$d_{m>1} = 0, \quad \varepsilon_{m>1} = \frac{1}{a_m}, \quad \gamma_{m>1} = \frac{g_m}{a_m}$$

The bow-tie model obtains from DO model by a variable change: $a_m = (\lambda_m + \lambda)t$

$$H_{\mathrm{bt}}(t) = \begin{pmatrix} 0 & g_2 & \cdots & g_N \\ g_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \cdots & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

$\Downarrow$

$$H_{\mathrm{bt}}(t) = H_{\mathrm{DO}}(t) - \lambda t\, \mathbb{1}$$

Many-body time-dependent integrability

Example: BCS model of superconductivity: $\hat{H}_{\rm BCS} = \sum_k 2\epsilon_k \hat{s}_k^z - \frac{1}{2B} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$

Relatively simple integrable model, satisfies classical Yang-Baxter equation

$\hat{\boldsymbol{s}}_k$ – spin−1/2 operators

Integrals of motion: Gaduin magnets: $\hat{H}_k = 2B\hat{s}_k^z - \sum_{j\neq k} \frac{\hat{\boldsymbol{s}}_k \cdot \hat{\boldsymbol{s}}_j}{\varepsilon_k - \varepsilon_j}$

$$\left[\hat{H}_k, \hat{H}_j\right] = \left[\hat{H}_k, \hat{H}_{\rm BCS}\right] = 0$$

Suppose we make the superconducting coupling a function of time: $B \to B(t)$

This immediately breaks usual integrability, e.g., commuting partners are no longer conserved

$$\frac{d\hat{H}_k}{dt} = i\left[\hat{H}_{\rm BCS}, \hat{H}_k\right] + \frac{\partial \hat{H}_k}{\partial t} = 2\frac{\partial B}{\partial t}\,\hat{s}_k^z \neq 0$$

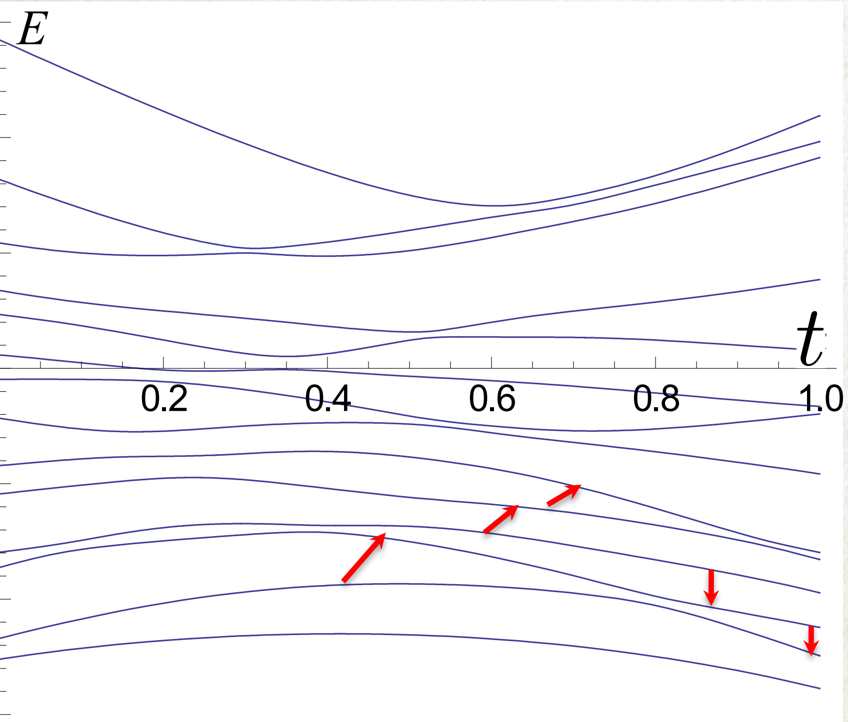
Many-body time-dependent integrability

Example: BCS model of superconductivity: $\hat{H}_{\rm BCS} = \sum_k 2\epsilon_k \hat{s}_k^z - \frac{1}{2B} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$

Suppose we make the superconducting coupling a function of time: $B \to B(t)$

Conventional exact solution – instantaneous (adiabatic) eigenstates. Not enough due to Landau-Zener tunneling between them.

Conventional Bethe ansatz: $\hat{H}_{\rm BCS}(t)\,\Phi_n(t) = E_n(t)\,\Phi_n(t)$



We want to solve: $i\frac{\partial \Psi}{\partial t} = \hat{H}_{\rm BCS}(t)\Psi$

For this, we need the Landau-Zenner tunneling dynamics on top of adiabatic eigenstates to be integrable

Turns out that for certain special choices of $B(t)$ it is indeed integrable, i.e., the non-stationary Schrodinger equation is exactly solvable!

Embedding into generalized Knizhnik-Zamolodchikov equations

$$\begin{cases} i\nu \dfrac{\partial \Psi}{\partial \varepsilon_k} = \hat{H}_k \Psi \\ i\nu \dfrac{\partial \Psi}{\partial B} = \hat{H}_{\mathrm{BCS}} \Psi \end{cases}$$

$$\hat{H}_k = 2B\hat{s}_k^z - \sum_{j\neq k} \frac{\hat{\boldsymbol{s}}_k \cdot \hat{\boldsymbol{s}}_j}{\varepsilon_k - \varepsilon_j}$$ – Gaudin magnets

$$\hat{H}_{\mathrm{BCS}} = \sum_k 2\epsilon_k \hat{s}_k^z - \frac{1}{2B} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$$

Original KZ equations: $B = 0$ and no equation for $\widehat{H}_{\mathrm{BCS}}$. Describe N-point correlation function $\Psi(\varepsilon_1, \ldots, \varepsilon_N)$ in $SU(2)$ Wess-Zumino-Witten CFT

Set $B = \nu t$. The last equation becomes the time-dependent Schrodinger equation for $\widehat{H}_{\mathrm{BCS}}(t)$

$$\hat{H}_{\mathrm{BCS}}(t) = \sum_k 2\epsilon_k \hat{s}_k^z - \frac{1}{2\nu t} \sum_{j,k} \hat{s}_j^+ \hat{s}_k^-$$

EY, Ann. Phys. (2018)

Off-shell Bethe ansatz solution of generalized KZ equations

Off-shell Bethe states: $$\Phi(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = \prod_{\alpha=1}^{M} \hat{L}^{+}(\lambda_\alpha)|0\rangle, \quad \hat{L}^{+}(\lambda) = \sum_{j=1}^{N} \frac{\hat{s}_j^{+}}{\lambda - \varepsilon_j}$$

Yang-Yang action: $$\mathcal{S}(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) = -2B \sum_{j} \varepsilon_j s_j + 2B \sum_{\alpha} \lambda_\alpha - \frac{1}{2} \sum_{j} \sum_{k \neq j} s_j s_k \ln(\varepsilon_j - \varepsilon_k) + \sum_{j} \sum_{\alpha} s_j \ln(\varepsilon_j - \lambda_\alpha) - \frac{1}{2} \sum_{\alpha} \sum_{\beta \neq \alpha} \ln(\lambda_\beta - \lambda_\alpha)$$

Babujian, J. Phys. A (1993)

Solution of KZ eqs: $$\Psi_{\mathrm{KZ}}(B, \boldsymbol{\varepsilon}) = \oint_{\gamma} d\boldsymbol{\lambda} \exp\left[-\frac{i\mathcal{S}(\boldsymbol{\lambda}, \boldsymbol{\varepsilon})}{\nu}\right] \Phi(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}), \quad d\boldsymbol{\lambda} = \prod_{\alpha=1}^{M} d\lambda_\alpha$$

To obtain the solution for the time-dependent Schrodinger equation for $\widehat{H}_{\mathrm{BCS}}(t)$ set $B = \nu t$ and fix the magnitude of all spins to $s_k = s = 1/2$

$$\hat{H}_{\mathrm{BCS}} = \sum_j 2\varepsilon_j \hat{n}_j - \frac{1}{2\nu t} \sum_{j,k} \hat{c}^\dagger_{j\uparrow} \hat{c}^\dagger_{j\downarrow} \hat{c}_{k\downarrow} \hat{c}_{k\uparrow}$$

Start in the ground state at $t = 0^+$ and evolve to $t \to +\infty$

1) Exact asymptotic wavefunction for $\widehat{H}_{\mathrm{BCS}}(t)$ [spins $s = 1/2$]:

$$\Psi(t \to +\infty) = C \sum_{\{\alpha\}} e^{i\Lambda_{\{\alpha\}}} \prod_\alpha \left[e^{-2it\varepsilon_\alpha} e^{-\frac{\pi\alpha}{\nu}} e^{-i\theta_\alpha} \right] |\{\alpha\}\rangle \equiv |M\rangle_\infty$$

$|\{\alpha\}\rangle$ – the state where energy levels $\{\alpha\} = \{\alpha_1, \alpha_2, \dots \alpha_M\}$ are doubly occupied (spin up) and the remaining levels are empty (spin down), M – number of Cooper pairs

$$\theta_\alpha = \frac{1}{\nu} \sum_{j \neq \alpha} \ln |\varepsilon_j - \varepsilon_\alpha|, \qquad \Lambda_{\{\alpha\}} = \frac{1}{\nu} \sum_{\beta \neq \alpha} \ln |\varepsilon_\beta - \varepsilon_\alpha|$$

This is the exact answer for any N single-particle levels $\varepsilon_1, \dots, \varepsilon_N$ and arbitrary number M of fermion pairs [Zabalo, Wu, Pixley & EY, PRB (2022)]

2) Exact mean-field (classical) solution [spin magnitudes $s \to \infty$]:

$$\Psi_{\rm mf}(t \to +\infty) = \prod_{k=1}^{N} \left(u_k + v_k \hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{k\downarrow} \right) |0\rangle$$

$$u_k = \frac{e^{\frac{\zeta_k - i\varphi_k}{2}}}{\sqrt{2\cosh\zeta_k}}, \quad v_k = \frac{e^{-2i\varepsilon_k t} e^{\frac{-\zeta_k + i\varphi_k}{2}}}{\sqrt{2\cosh\zeta_k}}, \quad \varphi_k = -\frac{1}{\nu}\sum_{j\neq k} \tanh\zeta_j \ln|\varepsilon_j - \varepsilon_k|, \quad \zeta_k = \frac{\pi(k-\mu)}{\nu}$$

$$\mu = \frac{N+1}{2} + \frac{N}{2\pi\eta} \ln\left\{ \frac{\sinh\left[\frac{\pi\eta M}{N}\right]}{\sinh\left[\pi\eta - \frac{\pi\eta M}{N}\right]} \right\} \qquad \eta = \frac{N}{\nu}$$

Similar probabilities, but the total particle # is not fixed and phases are different

3) Mean field is <u>exact</u> for local observables in the thermodynamic limit!

Consider the most general product of n operators with nonzero expectation value:

$$\hat{O} = \hat{o}_{k_1} \dots \hat{o}_{k_n} \qquad k_1, \dots, k_n \text{ – } n \text{ distinct energy levels}$$

$\hat{o}_k$ is pair creation $\hat{s}_k^+ = \hat{c}_{k\uparrow}^\dagger \hat{c}_{k\downarrow}^\dagger$ or annihilation $\hat{s}_k^- = \hat{c}_{k\downarrow} \hat{c}_{k\uparrow}$ or level occupancy $\hat{n}_k$ operator

We say that $\hat{O}$ is local if and only if $\dfrac{n}{N} \to 0$ when $N \to \infty$

Averages of local operators in the exact asymptotic state coincide with their expectation values in the mean-field wavefunction in the thermodynamic limit:

$$\langle M + l | \hat{O} | M \rangle_\infty = \langle \hat{O} \rangle_{\mathrm{mf}} = \langle \hat{o}_{k_1} \rangle_{\mathrm{mf}} \dots \langle \hat{o}_{k_n} \rangle_{\mathrm{mf}}$$

Corrections to mean field are of order $\dfrac{n}{N}$

Moreover, we know all local asymptotic averages explicitly ⇒

$$\langle \hat{c}_{k\uparrow}^\dagger \hat{c}_{k\downarrow}^\dagger \rangle_{\mathrm{mf}} = u_k v_k^*,$$
$$\langle \hat{c}_{k\downarrow} \hat{c}_{k\uparrow} \rangle_{\mathrm{mf}} = u_k^* v_k,$$
$$\langle \hat{n}_k \rangle_{\mathrm{mf}} = 2|v_k|^2.$$

$$\hat{H}(t \to +\infty) = \sum_{k=1}^{N} \varepsilon_k \hat{n}_k$$

For both the quantum solution in $N \to \infty$ limit & the classical solution:

LZ transition probabilities: $P_{0 \to \{n_k\}} = e^{-\sum_k \zeta_k n_k}$ $\qquad \zeta_k = \dfrac{\pi(k-\mu)}{\nu}$

The asymptotic state is a gapless superconductor:

$$\langle \hat{c}_{k\downarrow} \hat{c}_{k\uparrow} \rangle = \frac{e^{-2i\varepsilon_k t}}{2\cosh \zeta_k} e^{-\frac{i\varphi_k}{2}} \neq 0 \qquad \Delta(t) \propto \sum_k \langle \hat{c}_{k\downarrow} \hat{c}_{k\uparrow} \rangle \to 0$$

It is nonthermal but conforms to <u>emergent</u> Generalized Gibbs Ensemble:

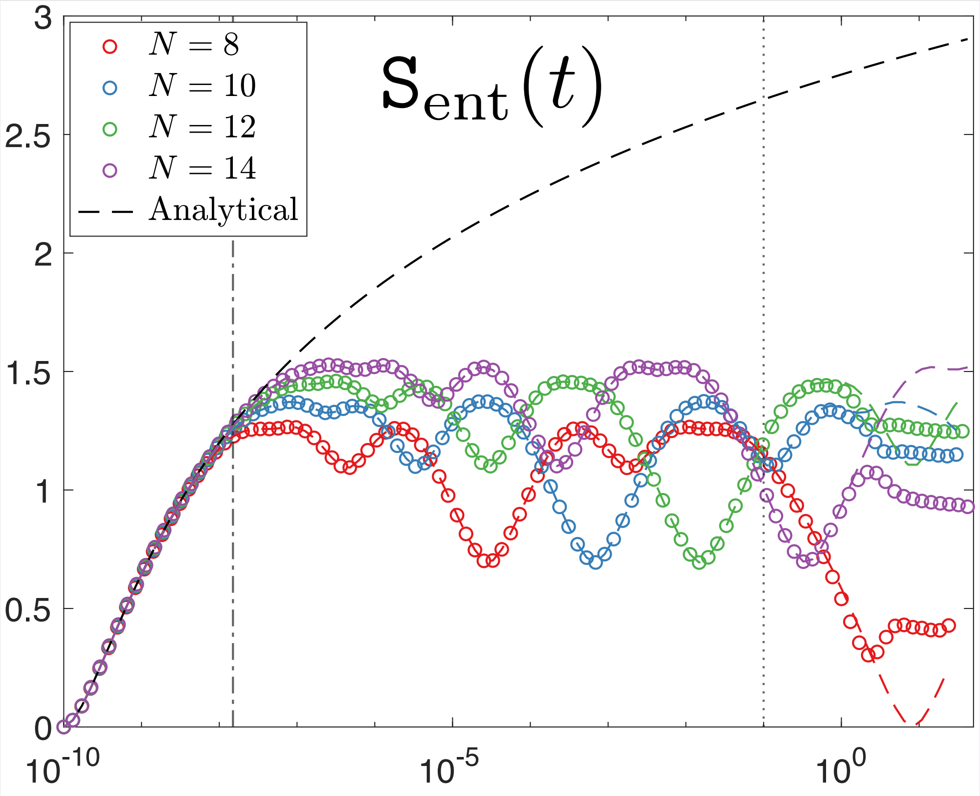
$$\hat{\rho}_{\mathrm{GGE}} = e^{-\sum_k \zeta_k \hat{n}_k}$$

Need only N numbers to specify $\hat{\rho}$ instead of 2^N required generally

4) Mean field breaks down for global observables

Example #1: Entanglement. The mean-field wavefunction is unentangled (a product state), while the exact ground state and asymptotic wavefunctions are entangled

$$\Psi_{\rm mf} = \prod \left(u_k + v_k \hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{k\downarrow} \right) |0\rangle \quad \Longrightarrow \quad \text{Entanglement entropy } S_{\rm ent} = 0$$



In contrast, in exact quantum dynamics starting from the unentangled mean-field ground state, the entropy monotonically grows as

$$S_{\rm ent} = \sqrt{1+\frac{\tau^2}{4}}\coth^{-1}\left[\sqrt{1+\frac{\tau^2}{4}}\right] + \ln\frac{\tau}{4}, \quad \tau = \eta \ln\frac{t}{t_0}$$

and saturates at $\tau \sim \sqrt{N}$ at $S_{\rm ent} \sim \ln N$

In exact ground state $S_{\rm ent} \sim \ln N$

4) Mean field breaks down for global observables

Example #2: Loschmidt echo (return amplitude): $\mathcal{Z}(t) = \langle \Psi_i | e^{-i\hat{H}t} | \Psi_i \rangle$

Mean-field analysis: Numerous singularities (DQPTs) in $\mathcal{Z}(t)$ for quench dynamics of s-wave BCS superconductors and none for p-wave.
Rylands, EY, Gurarie, Zabalo, Galitski, Ann. Phys. (2021)

Quantum analysis: no singularities for s-wave and periodic singularities for the topological p-wave superconductor, for the evolution starting from a quantum critical point separating the topological and non-topological phases.
Gaur, Gurarie, EY, Phys. Rev. (2022)

Reason: mean field fails because it determines the bulk of the time-dependent system wavefunction, while the Loschmidt is determined by the exponentially small tails of the wavefunction

Classical Yang-Baxter equation and Knizhnik-Zamolodchikov eqs

The connection between BCS-Gaudin models and KZ eqs is not accidental. For every model satisfying classical Yang-Baxter equation there are corresponding generalized KZ eqs.

Classical Yang-Baxter equation:

$$[r_{ij}(z_i, z_j), r_{ik}(z_i, z_k)] + [r_{ij}(z_i, z_j), r_{jk}(z_j, z_k)] + [r_{kj}(z_k, z_j), r_{ik}(z_i, z_k)] = 0$$

Corresponding generalized KZ eqs: $$i\nu \frac{\partial \Psi(\boldsymbol{z})}{\partial z_i} = H_i \Psi(\boldsymbol{z}) \qquad H_i = {\sum_{j=1}^{n}}' r_{ij}(z_i, z_j)$$

$$i = 1, \ldots, n$$

These multi-time Schrodinger equations for $\Psi(\boldsymbol{z})$ are compatible if and only if $r_{ij}(z_i, z_j)$ satisfy the classical Yang-Baxter equation [Cherednik, Dokl. Math. 40, 43 (1990)]

For Gaudin model: $$r_{ij}(z_i, z_j) = \frac{\hat{\boldsymbol{s}}_k \cdot \hat{\boldsymbol{s}}_j}{z_i - z_j}$$

Most other quantum integrable models (XXZ, 1D Hubbard, Kondo etc.) satisfy *quantum* Yang-Baxter equation. **Q:** What can we do for them?

Quantum YBE: $R^{ij}(z_i, z_j)\, R^{ik}(z_i, z_k)\, R^{jk}(z_j, z_k) = R^{jk}(z_j, z_k)\, R^{ik}(z_i, z_k)\, R^{ij}(z_i, z_j)$

Classical YBE obtains in $\hbar \to 0$ limit: $R^{ij}(z_i, z_j) = 1 + \hbar\, r_{ij}(z_i, z_j) + O(\hbar^2)$

Example: Kondo model with time-dependent coupling:

$$\hat{H}(t) = -i \int_{-L/2}^{L/2} \hat{\psi}_s^\dagger(x)\, \partial_x \hat{\psi}_s(x)\, dx + J(t)\, \hat{\psi}_s^\dagger(0)\, \vec{\sigma}_{ss'}\, \hat{\psi}_{s'}(0) \cdot \vec{S}$$

Integrable for: $J(t) = \lambda t + p(t) \pm \sqrt{[\lambda t + p(t)]^2 + \frac{4}{3}}$ $\quad p(t)$ – arbitrary function with period L

Maps to *quantum* Knizhnik-Zamolodchikov equations. Exact solution of the non-stationary Schrodinger equation via off-shell Bethe Ansatz as before.

Pasnoori & EY, arXiv:2509.05640

Quantum Knizhnik-Zamolodchikov equations are a set of finite difference equations: $\varphi(y_0,\ldots,y_j+\kappa,\ldots,y_N) = M_j(y_0,\ldots,y_N)\,\varphi(y_0,\ldots,y_j,\ldots,y_N)$

Transport operator:

$$M_j(y_0,\ldots,y_N) = R^{j+1\,j}(y_{j+1}-\kappa,y_j)\cdots R^{N\,j}(y_N-\kappa,y_j)\,R^{0\,j}(y_0,y_j)\cdots R^{j-1\,j}(y_{j-1},y_j)$$

Turn into usual KZ equations in $\hbar\to 0$ limit

Solution of non-stationary Schrodinger eq for Kondo Hamiltonian with $J(t) = \lambda t \pm \sqrt{\lambda^2 t^2 + \frac{4}{3}}$

$$f^{N\ldots 10}(z_1,\ldots,z_N) = \sum_{\{u_j\}} \prod_{i=0}^{N}\prod_{j=1}^{M} \frac{\Gamma\,(\nu z_i - u_j + 1 - ic)}{\Gamma\,(\nu z_i - u_j + 1)} \prod_{1\le i<j\le M} \frac{(u_i-u_j)\,\Gamma(u_i-u_j+ic)}{\Gamma(u_i-u_j-ic+1)} \prod_{j=1}^{M} B(\{\nu z_i\},u_j)\,|\Omega\rangle$$

$z_i = x_i - t$ – light-cone coordinates

Expect the answer to simplify greatly at late times as in the case of $\widehat{H}_{\mathrm{BCS}}(t)$. Take the plus sign and let $\lambda>0$. This corresponds to switching on the Kondo coupling from zero at $t\to-\infty$. Suppose initially $S^z_{\mathrm{imp}} = +\frac{1}{2}$ and electrons are in their ground state. Determine the following observables: $\langle S^z_{\mathrm{imp}}\rangle, \quad \langle S^z_{\mathrm{el}}(x)\rangle, \quad \langle S^z_{\mathrm{el}}(x)\,S^z_{\mathrm{imp}}\rangle$

Happy 75th Birthday, Sriram!!

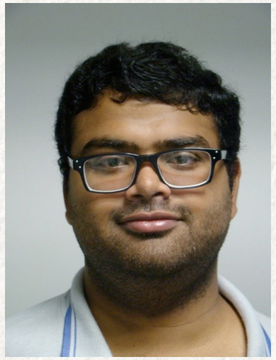
Integrable Matrix Theory:

Sriram Shastry
UC Santa Cruz

Haile Owusu
Rutgers

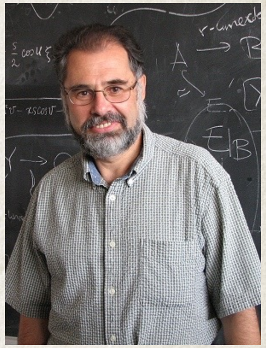
Jasen Scaramazza
Rutgers

Time-dependent integrability:

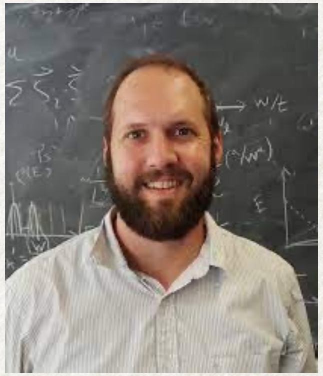


Aniket Patra
PCS, IBS

Nikolai Sinitsyn
LANL

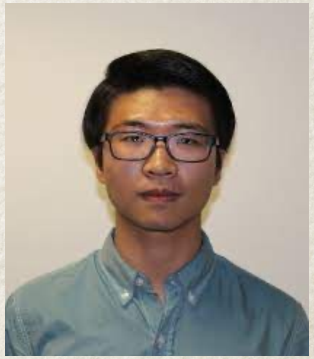


Vladimir Chernyak
Wayne State



Jed Pixley
Rutgers

Aidan Zabalo
Rutgers



Angkun Wu
Rutgers