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Bounds on T_c in the Eliashberg Theory of Superconductivity. II: Dispersive Phonons

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Abstract

The standard Eliashberg theory of superconductivity is studied, in which the effective electron-electron interactions are modelled as mediated by generally dispersive phonons, with Eliashberg spectral function $\alpha^2F(\omega) \geq 0$ that is $\propto \omega^2$ for small $\omega > 0$ and vanishes for large ω. The Eliashberg function also defines the electron-phonon coupling strength $\lambda := 2\int_{\mathbb{R}_+} \frac{\alpha^2F(\omega)}{\omega} d\omega$. Setting $\frac{2\alpha^2F(\omega)}{\omega} d\omega =: \lambda P(d\omega)$, formally defining a probability measure $P(d\omega)$ with compact support, and assuming as usual that the phase transition between normal and superconductivity coincides with the linear stability boundary $\mathscr{S}_c$ of the normal region in the (λ, P, T) parameter space against perturbations toward the superconducting region, it is shown that this *critical hypersurface* $\mathscr{S}_c$ is a graph of a function $\Lambda(P, T)$. This proves that the normal and the superconducting regions are simply connected. Moreover, it is shown that $\mathscr{S}_c$ is determined by a variational principle: if $(\lambda, P, T) \in \mathscr{S}_c$, then $\lambda = 1/\mathfrak{k}(P, T)$, where $\mathfrak{k}(P, T) > 0$ is the largest eigenvalue of a compact self-adjoint operator $\mathfrak{K}(P, T)$ on ℓ^2 sequences that is constructed explicitly in the paper, for all admissible P. Furthermore, given any such P, sufficient conditions on T are stated under which the map $T \mapsto \lambda = \Lambda(P, T)$ is invertible. For sufficiently large λ this yields the following: (i) the existence of a critical temperature T_c as function of λ and P; (ii) an ordered sequence of lower bounds on $T_c(\lambda, P)$ that converges to $T_c(\lambda, P)$. Also obtained is an upper bound on $T_c(\lambda, P)$. Although not optimal, it agrees with the asymptotic form $T_c(\lambda, P) \sim C\sqrt{\langle\omega^2\rangle}\sqrt{\lambda}$ valid for $\lambda \sim \infty$, given P, though with a constant C that is a factor ≈ 2.034 larger than the sharp constant; here, $\langle\omega^2\rangle := \int_{\mathbb{R}_+} \omega^2 P(d\omega)$.

Keywords Superconductivity · Eliashberg theory · Dispersive phonons · Critical temperature T_c · Rigorous results

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1 Introduction

This is our second paper in a series of papers in which we rigorously inquire into the phase transition between the normal and superconducting states as accounted for by the Eliashberg theory [1, 2, 4, 7, 8, 16–18]. Each paper uses results of the previous ones, while addressing a problem that is of interest in its own right.

In [14], the first paper of our series, to which we refer the reader also for a "master introduction" to the whole project, we studied a version of Eliashberg theory known as the γ model, introduced recently in [19] (see also [6]), which aims at describing superconductivity in systems close to a quantum phase transition, where the effective electron-electron interactions are mediated by collective bosonic excitations (fluctuations of the order parameter field) instead of phonons. The γ model contains three parameters: T, γ, and g; yet g enters only in the combination T/g, so that effectively the model contains only two parameters. We were able to characterize the critical temperature T_c in terms of a variational principle valid for all $\gamma > 0$, viz.

$$T_c(g,\gamma) = \tfrac{g}{2\pi}\big[\mathfrak{g}(\gamma)\big]^{\frac{1}{\gamma}}, \tag{1}$$

where $\mathfrak{g}(\gamma) > 0$ is the largest eigenvalue of a self-adjoint, compact operator $\mathfrak{G}(\gamma)$ constructed in [14]. Based on this variational principle we obtained a sequence of lower bounds on T_c that converges upward to T_c. The N-th lower bound is defined by the largest zero of a polynomial of degree N. The first four of these lower bounds we computed explicitly in terms of elementary functions; the bounds for $N > 4$ generically cannot be expressed in this way, as per Galois theory. Plotting our bounds jointly into a diagram suggests that for $\gamma \geq 2$ the fourth lower approximant to T_c should be accurate enough for most practical purposes. So far, however, the γ model has been applied in the theory of superconducting materials only with certain values of $\gamma < 2$ including $\gamma \in \{\frac{1}{3}, \frac{1}{2}, \frac{7}{10}, 1\}$; see [6]. For these γ values an accurate computation of $T_c(g,\gamma)$ requires a numerical approximation of $\mathfrak{g}(\gamma)$ with rank $N > 4$ approximations to $\mathfrak{G}(\gamma)$.

At $\gamma = 2$ the γ model captures the asymptotic behavior at infinite electron-phonon coupling strength λ of the standard Eliashberg theory in which the effective electron-electron interactions are mediated by generally dispersive phonons, with a spectral function[1] $\alpha^2F(\omega) \geq 0$, known as the Eliashberg function, that is $\propto \omega^2$ for small $\omega > 0$ and vanishes for large ω. The spectral function also defines the electron-phonon coupling strength $\lambda := 2\int_{\mathbb{R}_+} \frac{\alpha^2F(\omega)}{\omega}d\omega$. This standard version of Eliashberg theory has successfully explained the critical temperature T_c and other properties of most conventional superconductors [1, 2, 4, 25].

In the present paper we study this standard version of Eliashberg theory. Setting $2\alpha^2F(\omega)/\omega =: \lambda P'(\omega)$, with $P'(\omega)$ the density with respect to Lebesgue measure of a formal probability measure $P(d\omega)$ that behaves $\propto \omega$ for small $\omega > 0$, and whose support is contained in a bounded interval $[0, \overline{\Omega}(P)]$ with $0 < \overline{\Omega}(P)$, and assuming as usual that the phase transition between normal and superconductivity coincides with the linear stability boundary $\mathscr{S}_c$ of the normal region in the positive (λ, P, T) cone against perturbations toward the superconducting region, we will show that this *critical hypersurface* $\mathscr{S}_c$ is a graph of over the positive (P, T) cone. Hence the normal and the superconducting regions in the positive (λ, P, T) cone are each simply connected. Moreover, we will show that the critical surface is determined by a variational principle:

[1] We remark that α^2F is a compound symbol. This standard notation has historical roots. For further comments, see [14].

If $(\lambda, P, T) \in \mathscr{S}_c$, *then*

$$\lambda = 1\big/\mathfrak{k}(P,T), \tag{2}$$

where $\mathfrak{k}(P,T)>0$ *is the largest eigenvalue of a self-adjoint compact operator* $\mathfrak{K}(P,T)$ *on* ℓ^2 *sequences that is explicitly constructed in section 4.1 of this paper, for all admissible* P.

Approximating $\mathfrak{K}(P,T)$ with a nested sequence of finite-rank operators that converges to $\mathfrak{K}(P,T)$, an increasing sequence of rigorous lower bounds on $\mathfrak{k}(P,T)$ is obtained that converges to $\mathfrak{k}(P,T)$; these lower bounds on $\mathfrak{k}(P,T)$ translate into upper bounds on $\lambda \in \mathscr{S}_c$, given P. The first four of these can be, and are computed explicitly in closed form, involving a handful of expected values w.r.t. P. Also an explicit rigorous upper bound on $\mathfrak{k}(P,T)$ is obtained, which translates into a rigorous lower bound on $\lambda \in \mathscr{S}_c$ for each (P,T). Furthermore, conditioned on P being given, we state sufficient conditions on T for the map $T \mapsto \lambda$ and its lower N-frequency approximations to be invertible. For sufficiently large λ this yields the following:

(i) the existence of a critical temperature $T_c(\lambda, P)$;

(ii) a sequence of lower bounds on $T_c(\lambda, P)$ converging to $T_c(\lambda, P)$.

Also obtained is an upper bound on $T_c(\lambda, P)$, which is not optimal yet agrees with the asymptotic behavior $T_c(\lambda, P) \sim C\sqrt{\langle\omega^2\rangle}\sqrt{\lambda}$ for large enough λ, given P, though with $C \approx 2.034C_\infty$, where $C_\infty = 0.1827262477...$ is the sharp constant. Here, $\langle\omega^2\rangle := \int_{\mathbb{R}_+} \omega^2 P(d\omega)$.

Several results about dispersive phonons we will be able to prove by reduction to the proofs of analogous results for the γ model in [14], or by suitable adaptations of these proofs. Yet some results obtained in the present paper required totally new arguments.

The results of the present paper are for a rather general class of models, and therefore less quantitative than those in [14]. Quantitative results analogous to those obtained in [14] require a specification of $2\alpha^2F(\omega)$, something that has to be left to studies that are aimed at concrete applications. One such specific model is the non-dispersive limit of the standard Eliashberg theory, in which

$$2\alpha^2F(\omega)\Big|_{\mathrm{E}} := \lambda\Omega\delta(\omega-\Omega); \tag{3}$$

here, Ω is the Einstein frequency of the phonons. This model will be evaluated in [15].

In the remaining sections we first precisely specify the basic technical quantities of the Eliashberg theory. We employ its recent reformulation in terms of a classical Bloch spin chain model [27]. We then list our main results, followed by several sections with their proofs.

2 The Bloch Spin-Chain Model

As in [14], we work with the *condensation energy* of Eliashberg theory, the difference between the grand (Landau) potentials of the superconducting and normal states. Using units where Boltzmann's constant $k_{\mathrm{B}} = 1$ and the reduced Planck constant $\hbar = 1$, its spin chain representation reads (cf. [14], eq.(33))

$$\begin{aligned} H(\mathbf{S}|\mathbf{N}) :=& 2\pi \sum_n \omega_n \mathbf{N}_0 \cdot \big(\mathbf{N}_n - \mathbf{S}_n\big) \\ &+ \pi^2 T \sum\sum_{n\neq m} \lambda_{n,m} \left(\mathbf{N}_n\cdot\mathbf{N}_m - \mathbf{S}_n\cdot\mathbf{S}_m\right). \end{aligned} \tag{4}$$

Here, $\mathbf{S}_n \in \mathbb{S}^1 \subset \mathbb{R}^2$ with $n \in \mathbb{Z}$ denotes the n-th spin in the Bloch spin chain $\mathbf{S} \in (\mathbb{S}^1)^{\mathbb{Z}}$. The spin chain $\mathbf{N}$ is associated with the *normal state* of the Migdal–Eliashberg theory, having

n-th spin given by $\mathbf{N}_n := -\mathbf{N}_0 \in \mathbb{S}^1 \subset \mathbb{R}^2$ for $n < 0$ and $\mathbf{N}_n := \mathbf{N}_0$ for $n \geq 0$. Any other *admissible* spin chain satisfies the asymptotic conditions that, sufficiently fast, $\mathbf{S}_n \to \mathbf{N}_n$ when $n \to \infty$ and when $n \to -\infty$, where "sufficiently fast" is explained below. Moreover, admissible spin chains satisfy the symmetry relationship that for all $n \in \mathbb{Z}$, $\mathbf{N}_0 \cdot \mathbf{S}_{-n} = -\mathbf{N}_0 \cdot \mathbf{S}_{n-1}$ and $\mathbf{K}_0 \cdot \mathbf{S}_{-n} = \mathbf{K}_0 \cdot \mathbf{S}_{n-1}$, where $\mathbf{K}_0 \in \mathbb{S}^1 \subset \mathbb{R}^2$ is an arbitrary vector perpendicular to $\mathbf{N}_0$. This reduces the problem to effective spin chains $\mathbf{S} \in (\mathbb{S}^1)^{\mathbb{N}_0}$, with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The central dots between two such spins indicate Euclidean dot product, with $\mathbf{S}_n$ and $\mathbf{N}_n$ understood as vectors in $\mathbb{R}^2$ of unit length.

Furthermore, $\lambda_{n,m} \equiv \lambda(n-m)$ in (4) is a (dimensionless) positive spin-pair interaction kernel, which for the standard Eliashberg theory with dispersive phonons reads (cf. [14], eq.(7))

$$\lambda(n-m) := 2\int_0^\infty \frac{\alpha^2F(\omega)\omega}{\omega^2+(\omega_n-\omega_m)^2}d\omega, \tag{5}$$

where the $\omega_n := 2n\pi T$ are the Matsubara frequencies, and where $\omega \mapsto \alpha^2F(\omega) \in L^1_+(\mathbb{R}_+, d\omega)$ is the Eliashberg electron-phonon spectral function. It is defined as the thermodynamic limit of a finite-volume-sample sum of Dirac δ measures with positive coefficients, concentrated on a discrete set of frequencies $\Omega_k \in [\underline{\Omega}, \overline{\Omega}]$ with $0 < \underline{\Omega} \leq \overline{\Omega}$, which converges in weak* topology to a function $\alpha^2F(\omega) \propto \omega^2$ for small ω, yet still vanishing for $\omega > \overline{\Omega}$. Note that $\lambda(n-m) = \lambda(m-n)$, and that for each $k \in \mathbb{N}_0$ the expression $\lambda(k)$ is a T-dependend bounded linear functional of α^2F; the dependence on α^2F and on T of all $\lambda(\,\cdot\,)$ is not exhibited explicitly.

With this convention, the dimensionless electron-phonon coupling constant of the theory $\lambda \equiv \lambda(0)$. Explicitly,

$$\lambda = 2\int_0^\infty \frac{\alpha^2F(\omega)}{\omega}d\omega. \tag{6}$$

Note that our λ is the *standard* (renormalized) dimensionless electron-phonon coupling constant of the Eliashberg theory; cf. [1].

We have collected all the information needed to define *admissibility* of a spin chain $\mathbf{S}$ to mean, that after employing their stipulated symmetry to convert all summations over negative Matsubara frequencies in the sum over $\mathbb{Z}$ and the double sum over $\mathbb{Z}^2$ in (4) into summations over positive ones, the so rewritten (4) converges absolutely.

Not every admissible spin chain qualifies as thermal equilibrium state. For a spin chain to actually represent a thermal equilibrium state it needs to minimize the condensation energy functional (4).

Conjecture 1 *There is a critical temperature $T_c > 0$, depending on $\alpha^2F(\omega)$, such that for temperatures $T \geq T_c$, the spin chain of the normal state $\mathbf{N}$ is the unique minimizer of $H(\mathbf{S}|\mathbf{N})$, whereas at temperatures $T < T_c$ a spin chain $\mathbf{S} \neq \mathbf{N}$ for a superconducting phase minimizes $H(\mathbf{S}|\mathbf{N})$ uniquely up to an irrelevant gauge transformation (fixing of an overall phase). Moreover, the phase transition at T_c from normal to superconductivity is continuous.*

In this paper we take some steps toward the rigorous vindication of this conjecture. As in [14], we will *assume* the existence of a continuous phase transition between normal and superconductivity, so that its location in the phase diagram coincides with the linear-stability boundary of the normal state against small perturbations toward the superconducting region, from now on referred to as "superconducting perturbations." Thus we will rigorously study the Eliashberg gap equations linearized about the normal state. A confirmation of the continuous normal-to-superconductivity phase transition requires a study of the nonlinear Eliashberg gap equations, which we postpone to a later publication.

We next summarize our main results.

3 Main Results

Since it has become customary to emphasize the dependence of the critical temperature on the electron-phonon coupling constant $\lambda \equiv \lambda(0)$, we rewrite the effective electron-electron interaction mediated by generally dispersive phonons as

$$\lambda(n-m) =: \lambda \int_0^\infty \frac{\omega^2}{\omega^2 + (\omega_n - \omega_m)^2} P(d\omega), \tag{7}$$

where $P(d\omega)$ is a measure that integrates to unity, with a density $P'(\omega)$ w.r.t. Lebesgue measure that behaves $\propto \omega$ for small ω and vanishes for $\omega > \overline{\Omega}(P)$. We denote the set of these measures by $\mathcal{P}$.

Theorem 1 *The positive (λ, P, T) cone of the standard Eliashberg model consists of two simply connected regions. In one region the normal state is unstable against small superconducting perturbations, in the other region it is linearly stable. The boundary between the two regions, called the critical hypersurface $\mathscr{S}_c$, is a graph over the set $\{(P, T)\}$, i.e. $\mathscr{S}_c = \{(\lambda, P, T) : \lambda = \Lambda(P, T)\}$. The function Λ is continuous in both variables. The thermal equilibrium state at temperature T of a crystal with normalized phonon spectral density P' and electron-phonon coupling constant λ is the superconducting phase when $\lambda > \Lambda(P, T)$ and the normal (metallic) phase when $\lambda < \Lambda(P, T)$.*

Our Theorem 1 does not rule out that some lines $\mathscr{L}(\lambda, P) := \{(\lambda, P, T) : \lambda \;\&\; P \text{ fixed}\}$ could pierce $\mathscr{S}_c$ more than once, in which case the critical surface would not be a graph over the set $\{(\lambda, P)\}$ of the crystal model parameters — at odds with Conjecture 1. To rigorously confirm Conjecture 1's empirical thermodynamic narrative for the Eliashberg model, still *assuming* the existence of a continuous normal-to-superconducting phase transition, one needs to show that $\Lambda(P, T)$ depends strictly monotonically on T, given P. We have the following result.

Theorem 2 *The map $T \mapsto \Lambda(P, T)$ is strictly monotonic increasing on $[T_*, \infty)$, with $T_*(P) \leq \overline{\Omega}(P)/2\sqrt{2}\pi$. Hence, with $\lambda_*(P) := \Lambda(P, T_*(P))$, the portion of the critical surface $\mathscr{S}_c$ over the region $\{\lambda \geq \lambda_*(P)\}$ in the set of crystal parameters (λ, P) is also a graph, yielding the critical temperature $T_c(\lambda, P)$, viz.*

$$\mathscr{S}_c\big|_{\lambda \geq \lambda_*} = \big\{(\lambda, P, T) : T = T_c(\lambda, P), \lambda \geq \lambda_*(P)\big\}. \tag{8}$$

Remark 1 In [14] we characterized $T_c(g, \gamma)$ explicitly in terms of the variational principle (1). In (1), $\mathfrak{g}(\gamma) > 0$ is the largest eigenvalue of a self-adjoint, compact operator $\mathfrak{G}(\gamma)$ explicitly constructed in [14]. Based on this variational principle we were able to obtain lower bounds on $T_c(g, \gamma)$, four of these in closed form. Our sequence of lower bounds on $T_c(g, \gamma)$ is expressed in terms of the largest eigenvalues of a sequence of nested finite-rank approximations to the compact operator $\mathfrak{G}(\gamma)$ that converges monotonically to $\mathfrak{G}(\gamma)$ when the rank N is increased to ∞. With the help of Maple and, independently, Mathematica, we found that when the rank is increased beyond $N = 200$, then 10 significant decimal places of the lower approximation to $T_c(g, \gamma = 2)$ have stabilized, yielding $\frac{1}{g} T_c(g, 2) = 0.1827262477\ldots$.

For the standard Eliashberg model we have not been able to express $T_c(\lambda, P)$ as (some power) of an eigenvalue of a suitable self-adjoint operator. However, we are able to characterize $\Lambda(P, T)$ in terms of a variational principle that is mathematically similar to how we characterized $T_c(g, \gamma)$ in the γ model.

Theorem 3 *The function $\Lambda(P,T)$ is determined by the following variational principle,*

$$\Lambda(P,T) = \frac{1}{\mathfrak{k}(P,T)}, \tag{9}$$

where $\mathfrak{k}(P,T) > 0$ is the largest eigenvalue of a compact self-adjoint operator $\mathfrak{K}(P,T)$ on the Hilbert space of square-summable sequences over the non-negative integers, with $\mathfrak{K}(P,T)$ explicitly constructed in section 4.1, see (39)–(42).

Since compact operators on separable Hilbert spaces can be arbitrarily closely approximated by their finite-rank approximations obtained by a sequence of restriction to finite-dimensional subspaces that converge monotonically to the Hilbert space, our variational principle (9) furnishes a sequence of upper approximations to $\Lambda(P,T)$ that converges monotonically downward to $\Lambda(P,T)$. The first four of these can be computed in closed form, using only elementary functions.

More precisely, we have the following:

Theorem 4 *For all $N \in \mathbb{N}$, $\Lambda(P,T) < 1/\mathfrak{k}^{(N)}(P,T)$, where $\mathfrak{k}^{(N)}(P,T)$ is the largest eigenvalue of $\mathfrak{K}^{(N)}(P,T)$, the restriction of $\mathfrak{K}(P,T)$ to the first N components of $\ell^2(\mathbb{N}_0)$. The eigenvalues $\mathfrak{k}^{(N)}(P,T)$ can be explicitly computed for $N \in \{1,2,3,4\}$. They read:*

$$\mathfrak{k}^{(1)}(P,T) = \int_0^\infty \frac{\omega^2}{\omega^2 + (2\pi T)^2} P(d\omega), \tag{10}$$

which is the sole eigenvalue of $\mathfrak{K}^{(1)}(P,T)$;

$$\mathfrak{k}^{(2)}(P,T) = \tfrac{1}{2}\Big(\mathrm{tr}\,\mathfrak{K}^{(2)} + \sqrt{\big(\mathrm{tr}\,\mathfrak{K}^{(2)}\big)^2 - 4\det\mathfrak{K}^{(2)}}\Big)(P,T), \tag{11}$$

where $\mathfrak{K}^{(2)}(P,T)$ is the upper leftmost 2×2 block of the matrix $\mathfrak{K}^{(4)}(P,T)$ displayed further below;

$$\mathfrak{k}^{(3)}(P,T) = \tfrac{1}{3}\left(\mathrm{tr}\,\mathfrak{K}^{(3)} + 6\sqrt{\tfrac{p}{3}}\cos\left[\tfrac{1}{3}\arccos\left(\tfrac{q}{2}\sqrt{\left(\tfrac{3}{p}\right)^3}\right)\right]\right)(P,T), \tag{12}$$

with (temporarily suspending displaying the dependence on P,T)

$$p = \tfrac{1}{3}\big(\mathrm{tr}\,\mathfrak{K}^{(3)}\big)^2 - \mathrm{tr}\,\mathrm{adj}\,\mathfrak{K}^{(3)}, \tag{13}$$

$$q = \tfrac{2}{27}\big(\mathrm{tr}\,\mathfrak{K}^{(3)}\big)^3 - \tfrac{1}{3}\big(\mathrm{tr}\,\mathfrak{K}^{(3)}\big)\big(\mathrm{tr}\,\mathrm{adj}\,\mathfrak{K}^{(3)}\big) + \det\mathfrak{K}^{(3)}, \tag{14}$$

where $\mathfrak{K}^{(3)}(P,T)$ is the upper leftmost 3×3 block of the matrix $\mathfrak{K}^{(4)}(P,T)$ displayed further below;

$$\mathfrak{k}^{(4)}(P,T) = \left[\sqrt{\tfrac{1}{2}Z} + \sqrt{\tfrac{3}{16}A^2 - \tfrac{1}{2}B - \tfrac{1}{2}Z + \tfrac{A^3-4AB+8C}{16\sqrt{2Z}}} - \tfrac{1}{4}A\right](P,T), \tag{15}$$

where

$$Z = \tfrac{1}{3}\left[\sqrt{Y}\cos\left(\tfrac{1}{3}\arccos\tfrac{X}{2\sqrt{Y^3}}\right) - B + \tfrac{3}{8}A^2\right], \tag{16}$$

is a positive zero of the so-called resolvent cubic *associated with the characteristic polynomial* $\det\big(\eta\mathfrak{I} - \mathfrak{K}^{(4)}(P,T)\big)$, *and*

$$X = 2B^3 - 9ABC + 27C^2 + 27A^2D - 72BD, \tag{17}$$

$$Y = B^2 - 3AC + 12D, \tag{18}$$

where

$$A = -\mathrm{tr}\, \mathfrak{K}^{(4)}, \tag{19}$$

$$B = \tfrac{1}{2}\Big(\big(\mathrm{tr}\, \mathfrak{K}^{(4)}\big)^2 - \mathrm{tr}\, \big(\mathfrak{K}^{(4)}\big)^2\Big), \tag{20}$$

$$C = -\tfrac{1}{6}\Big(\big(\mathrm{tr}\, \mathfrak{K}^{(4)}\big)^3 - 3\,\mathrm{tr}\, \big(\mathfrak{K}^{(4)}\big)^2\big(\mathrm{tr}\, \mathfrak{K}^{(4)}\big) + 2\,\mathrm{tr}\, \big(\mathfrak{K}^{(4)}\big)^3\Big), \tag{21}$$

$$D = \det \mathfrak{K}^{(4)}, \tag{22}$$

with $\mathfrak{K}^{(4)}(P,T) = \int_0^\infty \mathfrak{H}^{(4)}\big(\varpi\big)P(d\omega)$, *where we introduced the abbreviation* $\varpi := \omega/2\pi T$, *and where*

$$\mathfrak{H}^{(4)} = \tag{23}$$

$$\begin{pmatrix} [\![1]\!] & \frac{1}{\sqrt{3}}\big([\![2]\!]+[\![1]\!]\big) & \frac{1}{\sqrt{5}}\big([\![3]\!]+[\![2]\!]\big) & \frac{1}{\sqrt{7}}\big([\![4]\!]+[\![3]\!]\big) \\ \frac{1}{\sqrt{3}}\big([\![2]\!]+[\![1]\!]\big) & \frac{1}{3}\big([\![3]\!]-2[\![1]\!]\big) & \frac{1}{\sqrt{15}}\big([\![4]\!]+[\![1]\!]\big) & \frac{1}{\sqrt{21}}\big([\![5]\!]+[\![2]\!]\big) \\ \frac{1}{\sqrt{5}}\big([\![3]\!]+[\![2]\!]\big) & \frac{1}{\sqrt{15}}\big([\![4]\!]+[\![1]\!]\big) & \frac{1}{5}\big([\![5]\!]-2([\![2]\!]+[\![1]\!])\big) & \frac{1}{\sqrt{35}}\big([\![6]\!]+[\![1]\!]\big) \\ \frac{1}{\sqrt{7}}\big([\![4]\!]+[\![3]\!]\big) & \frac{1}{\sqrt{21}}\big([\![5]\!]+[\![2]\!]\big) & \frac{1}{\sqrt{35}}\big([\![6]\!]+[\![1]\!]\big) & \frac{1}{7}\big([\![7]\!]-2([\![3]\!]+[\![2]\!]+[\![1]\!])\big) \end{pmatrix},$$

with $[\![n]\!]\big(\varpi\big) := \frac{\varpi^2}{n^2+\varpi^2} \equiv \frac{\omega^2}{(2n\pi T)^2+\omega^2}$ *for* $n \in \mathbb{N}$.

Remark 2 We have stated our Theorem 4 in a way that makes it obvious that the matrices $\mathfrak{K}^{(N)}(P,T)$ are averages w.r.t. $P(d\omega)$ of matrices $\mathfrak{H}^{(N)}(\varpi)$ of non-dispersive models, with $\varpi = \omega/2\pi T$. However, with the sole exception of the eigenvalue $\mathfrak{k}^{(1)}(P,T)$ of the first approximation, all eigenvalues $\mathfrak{k}^{(N)}(P,T)$ for $N > 1$ are not simply averages w.r.t. $P(d\omega)$ of the pertinent largest eigenvalues $\mathfrak{h}^{(N)}(\varpi)$ of $\mathfrak{H}^{(N)}(\varpi)$. The same is true for $\mathfrak{k}(P,T)$ in relation to $\mathfrak{h}(\varpi)$. Each $\mathfrak{k}^{(N)}(P,T)$ with $N \in \mathbb{N}$ is a function of averages w.r.t. $P(d\omega)$ of the expressions $\frac{\omega^2}{\omega^2+4n^2\pi^2T^2}$ with $n \in \{1, ..., 2N-1\}$.

It is manifestly obvious that our lower bound $\mathfrak{k}^{(1)}(P,T)$ is strictly monotonic decreasing with $T \in \mathbb{R}_+$, given P. In section 8.1 of this paper strict decrease with $T \in \mathbb{R}_+$ will be proved also for $\mathfrak{k}^{(2)}(P,T)$; see Proposition 9. While we have not succeeded in showing that all the maps $T \mapsto \mathfrak{k}^{(N)}(P,T)$ and the map $T \mapsto \mathfrak{k}(P,T)$ are strictly monotonic decreasing for all $T \in \mathbb{R}_+$, given P, we succeeded in proving decrease for sufficiently large T. More precisely, we have the following.

Proposition 1 *For* $N \in \mathbb{N}$ *the eigenvalues* $\mathfrak{k}^{(N)}(P,T)$ *decrease strictly monotonically with* $T \in [T_*(P), \infty)$. *Moreover,* $T_*(P) \leq \overline{\Omega}(P)/2\sqrt{2}\pi$.

Our Theorem 2 is a consequence of Theorems 3 and 4, and of Proposition 1.

As to the small-T behavior of our upper bounds $\Lambda^{(N)}(P,T)$ to $\Lambda(P,T)$, we have proved that in the limit $T \to 0$ the sequence of upper bounds $\Lambda^{(N)}(P,T)$ to $\Lambda(P,T)$ meets the λ axis at explicitly computable P-independent locations λ_N that converge slowly to 0 like $1/\ln N$ as $N \to \infty$. More precisely, we have:

Theorem 5 *The eigenvalues* $\mathfrak{k}^{(N)}(P,T)$ *converge when* $T \to 0$ *to* P*-independent numbers, viz.*

$$\lim_{T\to 0} \mathfrak{k}^{(N)}(P,T) = -1 + 2\sum_{n=0}^{N-1} \frac{1}{2n+1} =: \mathfrak{k}_0^{(N)}. \tag{24}$$

Thus, as $T \to 0$ *the* N*-th upper approximation* $\Lambda^{(N)}(P,T)$ *to* $\Lambda(P,T)$ *converges downward to* $1/\mathfrak{k}_0^{(N)} =: \lambda_N$. *Moreover,* $\mathfrak{k}_0^{(N)}$ *is strictly monotonically increasing with* N*, diverging* $\sim \ln N$ *to* ∞ *as* $N \to \infty$.

By Theorem 2 the critical hypersurface defines a unique critical temperature $T_c(\lambda, P)$ at least for all $\lambda > \lambda_*(P)$, and Theorem 4 in concert with Proposition 1 gives us an explicit upper bound for $\lambda_*(P)$.

Corollary 1 *For each* $P \in \mathcal{P}$ *we have the explicit upper estimate*

$$\lambda_*(P) < \frac{1}{\mathfrak{k}^{(4)}\big(P, T_*(P)\big)}. \tag{25}$$

We also obtained a rigorous lower bound on $\Lambda(P,T)$ by estimating from above the spectral radii of several operators defining $\mathfrak{K}(P,T)$.

Theorem 6 *Let* (P,T) *be given. Then* $\mathfrak{k}(P,T) \leq \mathfrak{k}^*(P,T)$*, where*

$$\mathfrak{k}^*(P,T) = \mathfrak{k}^{(1)}(P,T) + 2\sqrt{\big(2^{1+\varepsilon}-1\big)\zeta(1+\varepsilon)\zeta(5-\varepsilon)}\,\big\langle\varpi^2\big\rangle \tag{26}$$

with $\varepsilon = 0.65$*, and where* $\big\langle\varpi^2\big\rangle := \int_0^\infty \varpi^2 P(d\omega)$ *with* $\varpi = \omega/2\pi T$.

Corollary 2 *By Jensen's inequality,* $P(d\omega)$*-averaging the concave function* $\omega^2 \mapsto \frac{\omega^2}{\omega^2+4\pi^2T^2}$ *yields the weaker upper bound* $\mathfrak{k}(P,T) \leq \mathfrak{k}^\sharp(P,T)$*, where*

$$\mathfrak{k}^\sharp(P,T) = \frac{\big\langle\varpi^2\big\rangle}{\big\langle\varpi^2\big\rangle+1} + 2\sqrt{\big(2^{1+\varepsilon}-1\big)\zeta(1+\varepsilon)\zeta(5-\varepsilon)}\,\big\langle\varpi^2\big\rangle \tag{27}$$

with $\varepsilon = 0.65$*. Moreover, purging the* $\big\langle\varpi^2\big\rangle$ *contribution in the denominator of the first term at r.h.s.(27) yields a yet weaker upper bound on* $T_c(P,T)$ *that is* $\propto \big\langle\varpi^2\big\rangle = \big\langle\omega^2\big\rangle/4\pi^2T^2$.

We register that the explicit upper bounds (26) and (27) on $\mathfrak{k}(P,T)$ are averages of functions that are manifestly strictly monotone decreasing with T on $\mathbb{R}_+$, and therefore themselves strictly monotone decreasing with T on $\mathbb{R}_+$.

Moreover, since $P(d\omega)$ is compactly supported, we register also that $\mathfrak{k}^*(P,T)$, and therefore also $\mathfrak{k}(P,T)$, are bounded above by C/T^2 for all $T > 0$. Thus the largest eigenvalue of $\mathfrak{K}(P,T)$ not only overall decreases when T increases, it overall decreases to 0 at least as fast as C/T^2. This in itself does not imply monotonic decrease to zero, of course; however, together with Proposition 1, strictly monotonic decrease to zero follows for when $T > T_*(P)$.

By Theorem 1, the critical hypersurface $\mathscr{S}_c$ in the set of (λ, P, T) parameters is a graph over the set $\{(P,T)\}$. By Theorems 3, 4, and 6 in concert, that graph is sandwiched between $1/\mathfrak{k}^*(P,T)$ (explicit lower bound) and $1/\mathfrak{k}^{(N)}(P,T)$ for any $N \in \mathbb{N}$ (a decreasing sequence of upper bounds, the first four of which are explicit).

By the monotonicity of $T \mapsto \Lambda(P,T)$ for $T > T_*(P)$, the inverse function of the map $T \mapsto \Lambda(P,T)$ exists for $T > T_*(P)$. Thus the critical hypersurface $\mathscr{S}_c$ is also a graph over a region of the (λ, P) crystal parameter space where $\lambda > \lambda_*(P)$, there defining the critical temperature $T_c(\lambda, P)$. From a theoretical perspective it is of course important to understand how T_c depends on λ, given P. Our upper and lower bounds on $\Lambda(P,T)$ are also monotonic in T and yield lower and upper bounds on $T_c(\lambda, P)$ that offer some insights into the behavior of $T_c(\lambda, P)$ itself.

By its strict monotonic dependence on $T \in \mathbb{R}_+$, in principle our explicit upper bound $\mathfrak{k}^*(P, T)$ on $\mathfrak{k}(P, T)$ can be inverted to yield an upper *critical-temperature bound* $T_c^*(\lambda, P)$ for all $\lambda > 0$ and all $P \in \mathcal{P}$. Furthermore, our explicit lower bounds $\mathfrak{k}^{(N)}(P, T)$, $N \in \{1, 2, 3, 4\}$, on $\mathfrak{k}(P, T)$ can in principle be inverted for $T > T_*(P)$ to yield lower *critical-temperature bounds* $T_c^{(N)}(\lambda, P)$ for $N \in \{1, 2, 3, 4\}$ when $\lambda > \max\{\lambda_*(P), \lambda_N\}$. None of these bounds on T_c can generally be expressed in closed form, though.

It is worthy of note, though, that a large T analysis of the operators $\mathfrak{K}^{(N)}(P, T)$ reveals that the large-λ asymptotics of $T_c(\lambda, P)$ can be computed explicitly in closed form; see Corollary 3 below, which is a consequence of

Theorem 7 *The eigenvalues $\mathfrak{k}^{(N)}(P, T)$ are analytic about $T = \infty$:*

$$\mathfrak{k}^{(N)}(P, T) = \frac{\mathfrak{g}^{(N)}(2)\langle\omega^2\rangle}{4\pi^2}\frac{1}{T^2} - \frac{\langle\mathfrak{G}^{(N)}(4)\rangle_2\langle\omega^4\rangle}{16\pi^4}\frac{1}{T^4} + \mathcal{O}\Big(\frac{1}{T^6}\Big); \tag{28}$$

here, $\mathfrak{g}^{(N)}(2) \geq 1$ is the largest eigenvalue for the N-Matsubara frequency approximation to the operator $\mathfrak{G}(\gamma)$ of the γ model at $\gamma = 2$, and

$$\big\langle\mathfrak{G}^{(N)}(4)\big\rangle_2 := \big\langle\Xi_N^{\mathrm{opt}}(2),\ \mathfrak{G}^{(N)}(4)\ \Xi_N^{\mathrm{opt}}(2)\big\rangle > 0 \tag{29}$$

denotes the quantum-mechanical expected value of the N-Matsubara frequency approximation to the operator $\mathfrak{G}(\gamma)$ at $\gamma = 4$, taken with the normalized N-frequency optimizer $\Xi_N^{\mathrm{opt}}(\gamma)$ of the γ model at $\gamma = 2$. Moreover, $\langle\omega^k\rangle := \int_0^\infty \omega^k P(d\omega)$ for $k \in \mathbb{N}$.

Corollary 3 *The N-Matsubara frequency approximation $\mathscr{S}_c^{(N)}$ to the critical hypersurface $\mathscr{S}_c$ in the positive (λ, P, T) cone is asymptotic to a graph over the asymptotic $\lambda \sim \infty$ region of the (λ, P) set, given by*

$$T_c^{(N)}(\lambda, P) \sim \frac{1}{\sqrt{2\pi^2\frac{\mathfrak{g}^{(N)}(2)\langle\omega^2\rangle}{\langle\mathfrak{G}^{(N)}(4)\rangle_2\langle\omega^4\rangle}\left(1 - \sqrt{1 - 4\frac{\langle\mathfrak{G}^{(N)}(4)\rangle_2\langle\omega^4\rangle}{\mathfrak{g}^{(N)}(2)^2\langle\omega^2\rangle^2}\frac{1}{\lambda}}\right)}}. \tag{30}$$

This result also holds when $N \to \infty$ (with superscripts $^{(N)}$ purged).

Since (30) is valid aysmptotically for large λ, expanding the inner square root at r.h.s.(30) to leading order in powers of $1/\lambda$ yields $T_c^{(N)}(\lambda, \Omega) \sim \frac{1}{2\pi}\sqrt{\mathfrak{g}^{(N)}(2)\langle\omega^2\rangle\lambda}$, with $N \in \mathbb{N}$. By a simple convexity estimate, r.h.s.(30)$\leq \frac{1}{2\pi}\sqrt{\mathfrak{g}^{(N)}(2)\langle\omega^2\rangle\lambda}$. And so, for large enough λ the asymptotic expression $\frac{1}{2\pi}\sqrt{\mathfrak{g}^{(N)}(2)\langle\omega^2\rangle\lambda}$ is an upper bound on $T_c^{(N)}(\lambda, P)$ that is asymptotically sharp for $\lambda \sim \infty$. Moreover, in [14] we showed that $\mathfrak{g}^{(N)}(2)$ converges upward to $\mathfrak{g}(2)$. Furthermore, as noted earlier, each $T_c^{(N)}(\lambda, P)$ vanishes for $\lambda \leq \lambda_N$, while $\sqrt{\lambda} > 0$ for all λ. All the above now suggests

Conjecture 2 *There is a critical temperature $T_c(\lambda, P) > 0$ which for all $P \in \mathcal{P}$ and $\lambda > 0$ is bounded above by $T_c(\lambda, P) < T_c^{\sim}(\lambda, P)$, with*

$$T_c^{\sim}(\lambda, P) := \frac{1}{2\pi}\sqrt{\mathfrak{g}(2)\langle\omega^2\rangle\lambda}; \tag{31}$$

here, $\mathfrak{g}(2)$ is the spectral radius of $\mathfrak{G}(\gamma)$ at $\gamma = 2$.

For the numerical approximation of $\frac{1}{2\pi}\sqrt{\mathfrak{g}(2)}$ to 10 significant decimal places, see Remark 1 after Theorem 2.

Remark 3 The asymptotic behavior $T_c(\lambda, P) \sim \frac{1}{2\pi}\sqrt{\mathfrak{g}(2)\left\langle\omega^2\right\rangle\lambda}$ was first stated in [1], see their eq.(22). Apparently they did not notice that this is presumably an upper bound on T_c for all $\lambda > 0$.

Explicit upper and lower bounds on $T_c(\lambda, P)$ valid for all $\lambda > 0$ and $P \in \mathcal{P}$ can be obtained at the expense of less accuracy. For this we can use our weaker bounds on $\Lambda(P, T)$ that involve only $\left\langle\omega^2\right\rangle$ as P-average, and $\overline{\Omega}(P)$, both of which can be treated as numerical parameters, and obtain the following explicit bounds on T_c.

Corollary 4 *Whether $T_c(\lambda, P)$ is well-defined only for $\lambda > \lambda_*(P)$ or for all $\lambda > 0$, given $P \in \mathcal{P}$, it obeys the upper bound $T_c(\lambda, P) < T_c^\sharp(\lambda, P)$, with*

$$T_c^\sharp(\lambda, P) = \frac{\left\langle\omega^2\right\rangle}{2\pi}\sqrt{\tfrac{1}{2}\left(\lambda(1+b) - 1 + \sqrt{\left(\lambda(1+b)-1\right)^2 + 4b\lambda}\right)} \tag{32}$$

with $b := 2\left(\left(2^{1+\varepsilon} - 1\right)\zeta(1+\varepsilon)\zeta(5-\varepsilon)\right)^{\frac{1}{2}}$ and $\varepsilon = 0.65$.

Furthermore, for $\lambda > \overline{\Omega}^2/\langle\omega^2\rangle$ we have the lower bound $T_c(\lambda, P) > T_c^\flat(\lambda, P)$, with

$$T_c^\flat(\lambda, P) = \frac{1}{2\pi}\left(\lambda\langle\omega^2\rangle - \overline{\Omega}^2\right)^{\frac{1}{2}}(P). \tag{33}$$

Before we next turn to the proofs of our results, we illustrate our bounds in two figures. The upper bound $T_c^\sharp(\lambda, P)$ on $T_c(\lambda, P)$, as stated in Corollary 4, and the conjectured upper bound $T_c^\sim(\lambda, P)$ of Conjecture 2, and for several choices of $\overline{\Omega}^2/\langle\omega^2\rangle$ also the lower bound $T_c^\flat(\lambda, P)$, are displayed in Fig. 1 as functions of λ. In Fig. 2 we complement Fig. 1 by displaying the large-λ behavior of the lower and upper approximations to $T_c(\lambda, P)$.

We now turn to the proofs of our results.

4 Stability Analysis of the Normal State

In this section we will show that for all $P \in \mathcal{P}$ and $T > 0$ there is a unique $\lambda = \Lambda(P, T) > 0$ such that the normal state $\mathbf{N}$ is linearly stable for $\lambda < \Lambda(P, T)$, but unstable against superconducting perturbations for $\lambda > \Lambda(P, T)$. Moreover, we will show that $\Lambda(P, T)$ is characterized by a variational principle, as indicated by (9).

Indeed, we pave the ground precisely as in [14] by first switching to a convenient parameterization of the Bloch spin chains. The symmetry relationship $\mathbf{N}_0 \cdot \mathbf{S}_{-n} = -\mathbf{N}_0 \cdot \mathbf{S}_{n-1}$ and $\mathbf{K}_0 \cdot \mathbf{S}_{-n} = \mathbf{K}_0 \cdot \mathbf{S}_{n-1}$, for all $n \in \mathbb{Z}$, allows us to work with effective spin chains $\mathbf{S} \in (\mathbb{S}^1)^{\mathbb{N}_0}$, with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All summations thus go over $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ instead of $\mathbb{Z}$.

Vectors $\mathbf{S}_n \in \mathbb{S}^1$ are represented by introducing an angle $\theta_n \in \mathbb{R}/(2\pi\mathbb{Z})$ $(= [0, 2\pi]$ with 2π and 0 identified) defined through $\mathbf{N}_0 \cdot \mathbf{S}_n =: \cos\theta_n$ for all[2] $n \in \mathbb{N}_0$. Setting $H(\mathbf{S}|\mathbf{N}) =: 4\pi^2 T K(\Theta)$ with $\Theta := (\theta_n)_{n\in\mathbb{N}_0}$ yields

[2] If one also introduces angles for spins with negative suffix by defining $\mathbf{N}_0 \cdot \mathbf{S}_n =: \cos\theta_n$ for all $n \in -\mathbb{N}$, a sequence of angles with non-negative suffix yields the angles with negative suffix as $\theta_{-1} = \pi - \theta_0$, $\theta_{-2} = \pi - \theta_1$, etc., thanks to the symmetry of $\mathbf{S} \in (\mathbb{S}^1)^{\mathbb{Z}}$ with respect to the sign switch of the Matsubara frequencies.

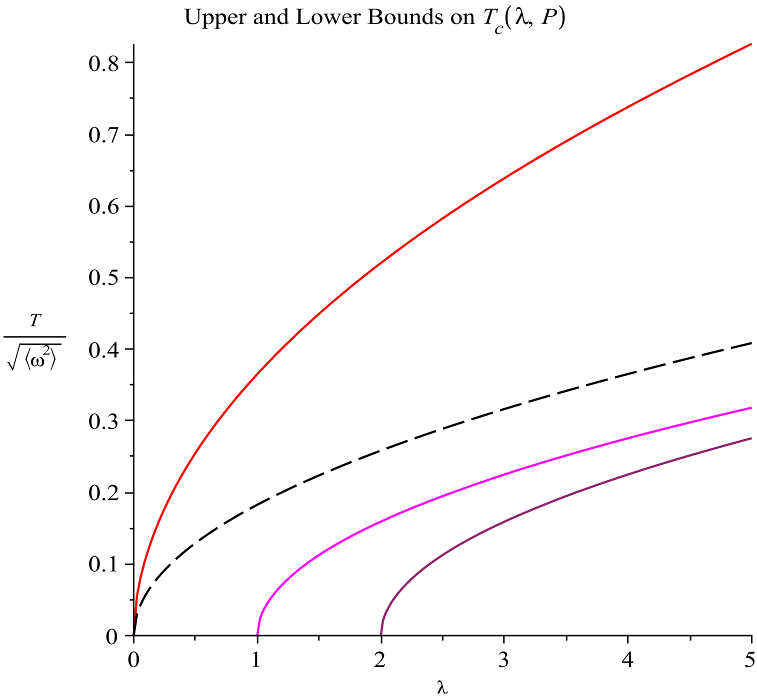

Fig. 1 Shown for the standard Eliashberg model are: the graph of the upper bound $\lambda \mapsto T_c^\sharp(\lambda, P)/\sqrt{\langle\omega^2\rangle}$ (red), the graph of the lower bound $\lambda \mapsto T_c^\flat(\lambda, P)/\sqrt{\langle\omega^2\rangle}$ for the values 1 (magenta) and 2 (maroon) of the ratio $\overline{\Omega}^2/\langle\omega^2\rangle$, and the graph of the map $\lambda \mapsto T_c^\sim(\lambda, P)/\sqrt{\langle\omega^2\rangle}$ (black, dash) which is asymptotic to $T_c(\lambda, P)/\sqrt{\langle\omega^2\rangle}$ for $\lambda \sim \infty$

$$
\begin{aligned}
K(\Theta) =& \sum_n \big(2n+1\big)\big(1-\cos\theta_n\big) \\
&+ \frac{1}{2}\sum\sum_{n\neq m}\lambda(n-m)\Big(1-\cos\big(\theta_n-\theta_m\big)\Big) \\
&- \frac{1}{2}\sum\sum_{n,m}\lambda(n+m+1)\Big(1-\cos\big(\theta_n+\theta_m\big)\Big);
\end{aligned}
\tag{34}
$$

here, the summations run over $\mathbb{N}_0$, and $\lambda(j)$ has been defined in (7)

The minimizing sequences of angles satisfy the non-linear Euler–Lagrange equation for any stationary point Θ^s of $K(\Theta)$; viz., $\forall n \in \mathbb{N}_0$:

$$
\begin{aligned}
\big(2n+1\big)\sin\theta_n^s =& \sum_{m\geq 0}\lambda(n+m+1)\sin\big(\theta_n^s+\theta_m^s\big) \\
&- \sum_{m\geq 0}\lambda(n-m)\sin\big(\theta_n^s-\theta_m^s\big).
\end{aligned}
\tag{35}
$$

In the following we shall omit the superscript ^{s} from Θ^s.

The system of equations (35) has infinitely many solutions when the θ_n are allowed to take values in $[0, 2\pi]$, restricted only by the asymptotic condition that $\theta_n \to 0$ rapidly enough when $n \to \infty$; see [27]. Here we are only interested in solutions that are putative

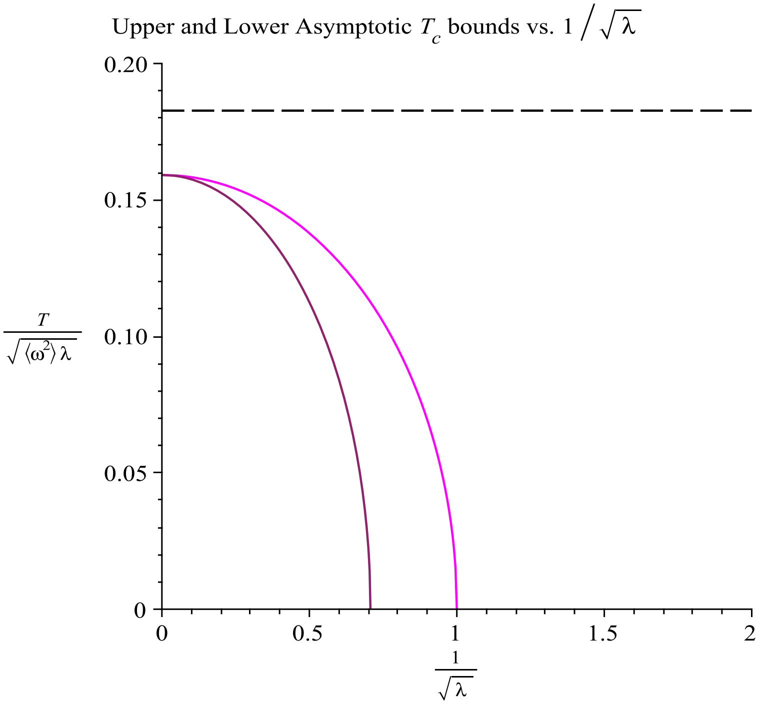

Fig. 2 Shown are the graphs of the lower bound $\sqrt{\lambda}^{-1} \mapsto T_c^{\flat}(\lambda, P)\Big/\sqrt{\langle\omega^2\rangle}\sqrt{\lambda}$ for $\overline{\Omega}^2/\langle\omega^2\rangle \in \{1, 2\}$, and the map $\sqrt{\lambda}^{-1} \mapsto T_c^{\sim}(\lambda, P)\Big/\sqrt{\langle\omega^2\rangle}\sqrt{\lambda}$. The graph of the upper bound $\sqrt{\lambda}^{-1} \mapsto T_c^{\sharp}(\lambda, P)\Big/\sqrt{\langle\omega^2\rangle}\sqrt{\lambda}$ is not shown, for it would appear as an essentially horizontal line parallel to the black dashed line, yet a factor ≈ 2 higher

minimizers of $H(\mathbf{S}|\mathbf{N})$, hence of $K(\Theta)$. In [27] it was shown that a sequence $\Theta = (\theta_n)_{n\in\mathbb{N}_0}$ that minimizes $K(\Theta)$, must have $\Theta \in [0, \frac{\pi}{2}]^{\mathbb{N}_0} =: S$; i.e., all[3] $\theta_n \in [0, \frac{\pi}{2}]$.

The normal state is the sequence of angles $\underline{\Theta} := (\theta_n = 0)_{n\in\mathbb{N}_0}$. This trivial solution of (35) manifestly exists for all $\lambda > 0$ and P. However, the trivial solution $\underline{\Theta}$ representing the normal state minimizes $K(\Theta)$ only for sufficiently large T, as we will prove in a separate paper. In this paper we will only prove that $\underline{\Theta}$ is linearly stable against small superconducting perturbations $\Theta \in S$ for which $K(\Theta)$ is well-defined if and only if T is large enough.

This will prove Theorems 1 and 3.

4.1 Proof of Theorems 1 and 3

Since $K(\underline{\Theta}) = 0 = H(\mathbf{N}|\mathbf{N})$, to inquire into the question of linear stability versus instability of $\underline{\Theta}$ we expand $K(\Theta)$ about $\Theta = \underline{\Theta}$ to second order in Θ. This yields the quadratic form

[3] Alternatively, all $\theta_n \in [-\frac{\pi}{2}, 0]$; these choices are gauge equivalent.

$$K^{(2)}(\Theta) = \sum_n \left[\frac{2n+1}{2} - \frac{1}{2}\lambda(2n+1) + \sum_{k=1}^{n} \lambda(k) \right] \theta_n^2 \tag{36}$$
$$- \frac{1}{2} \sum\sum_{n \neq m} \theta_n \Big[\lambda(n-m) + \lambda(n+m+1) \Big] \theta_m,$$

which for all $\lambda > 0$ and $P \in \mathcal{P}$ is well-defined on the Hilbert space $\mathcal{H}$ of sequences that satisfy $\|\Theta\|_{\mathcal{H}}^2 := \sum_{n \geq 0}(2n+1)\theta_n^2 < \infty$. If $K^{(2)}(\Theta) \geq 0$ for all $\Theta \in \mathcal{H}$, with "$= 0$" iff $\Theta = \underline{\Theta}$, then $K(\Theta) > 0$ for all $\Theta \neq \underline{\Theta}$ in a sufficiently small neighborhood of $\underline{\Theta}$, and then the trivial sequence $\underline{\Theta}$ is a local minimizer of $K(\Theta)$ and thus linearly stable. If on the other hand there is at least one $\Theta \neq \underline{\Theta}$ in $\mathcal{H} \cap S$ for which $K^{(2)}(\Theta) < 0$, then the trivial sequence $\underline{\Theta}$ is not a local minimizer of $K(\Theta)$ in $\mathcal{H} \cap S$, and therefore unstable against "superconducting perturbations." The verdict as to linear stability versus instability depends on λ and P.

As in [14], we now recast the functional $K^{(2)}(\Theta)$ defined on $\mathcal{H}$ as a functional $Q(\Xi)$ defined on $\ell^2(\mathbb{N}_0)$. For this we note that we can take the square root of the diagonal matrix $\mathfrak{O}$ whose diagonal elements are the odd natural numbers. Its square root is also a diagonal matrix, and its action on Θ componentwise is given as

$$(\mathfrak{O}^{\frac{1}{2}}\Theta)_n = \sqrt{2n+1}\,\theta_n =: \xi_n. \tag{37}$$

Since $\Theta := (\theta_n)_{n \in \mathbb{N}_0} \subset \mathcal{H}$, the sequence $\Xi := (\xi_n)_{n \in \mathbb{N}_0} \subset \ell^2(\mathbb{N}_0)$. The map $\mathfrak{O}^{\frac{1}{2}} : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ is invertible. We set $K^{(2)}(\Theta) =: \frac{1}{2}Q(\Xi)$, viz.

$$Q(\Xi) = \sum_n \left[1 + \frac{2}{2n+1} \sum_{k=1}^{n} \lambda(k) \right] \xi_n^2 \tag{38}$$
$$- \sum\sum_{n \neq m} \xi_n \left[\frac{\lambda(n-m)}{\sqrt{2n+1}\sqrt{2m+1}} \right] \xi_m$$
$$- \sum_n \sum_m \xi_n \left[\frac{\lambda(n+m+1)}{\sqrt{2n+1}\sqrt{2m+1}} \right] \xi_m,$$

where the contributions from the first line at r.h.s.(38) are positive, those from the second and third line negative.

We now state the main properties of the functional $Q(\Xi)$, which decides the question of linear stability vs. instability of the normal state against superconducting perturbations, as a theorem. Note that this theorem restates Theorem 1 in terms of the spin chain model.

Theorem 1^+: *Let $\lambda > 0$ and $P \in \mathcal{P}$ be given. Then the functional Q in (38) has a minimum on the sphere $\{\Xi \in \ell^2(\mathbb{N}_0) : \|\Xi\|_{\ell^2} = 1\}$. The minimizing (optimizing) eigenmode Ξ^{opt} satisfies $\Xi^{\text{opt}} \in \mathbb{R}_+^{\mathbb{N}_0}$. Moreover, for each P and $T > 0$ there is a unique $\Lambda(P,T) > 0$ at which $\min\{Q(\Xi) : \|\Xi\|_{\ell^2} = 1\} = 0$, and $\min\{Q(\Xi) : \|\Xi\|_{\ell^2} = 1\} > 0$ whenever $\lambda < \Lambda(P,T)$, while $\min\{Q(\Xi) : \|\Xi\|_{\ell^2} = 1\} < 0$ when $\lambda > \Lambda(P,T)$. Furthermore, the map $(P,T) \mapsto \Lambda(P,T)$ is continuous in both variables.*

Following closely [14], we prepare the proof of Theorem 1^+ by defining several linear operators that act on $\ell^2(\mathbb{N}_0)$ and which are associated with Q. Letting $\langle \Xi, \widetilde{\Xi} \rangle$ denote the usual $\ell^2(\mathbb{N}_0)$ inner product between two ℓ^2 sequences Ξ and $\widetilde{\Xi}$, we write Q shorter thus:

$$Q(\Xi) = \left\langle \Xi\,,\left(\mathfrak{I} - \lambda\mathfrak{K}\right)\Xi\right\rangle. \tag{39}$$

Here, $\mathfrak{I}$ is the identity operator, and $\mathfrak{K} = -\mathfrak{K}_1 + \mathfrak{K}_2 + \mathfrak{K}_3$, where the $\mathfrak{K}_j = \mathfrak{K}_j(P,T)$ for $j \in \{1,2,3\}$ are given by the P averages $\int_0^\infty \mathfrak{H}_j(\varpi)P(d\omega)$ of the operators $\mathfrak{H}_j(\varpi)$ of the non-dispersive model, which componentwise act as follows:

$$(\mathfrak{H}_1(\varpi)\Xi)_n = \left[\frac{2}{2n+1}\sum_{k=1}^{n}\frac{\varpi^2}{\varpi^2+k^2}\right]\xi_n\,, \tag{40}$$

$$(\mathfrak{H}_2(\varpi)\Xi)_n = \sum_{m\neq n}\left[\frac{1}{\sqrt{2n+1}}\,\frac{\varpi^2}{\varpi^2+(n-m)^2}\,\frac{1}{\sqrt{2m+1}}\right]\xi_m\,, \tag{41}$$

$$(\mathfrak{H}_3(\varpi)\Xi)_n = \sum_{m}\left[\frac{1}{\sqrt{2n+1}}\,\frac{\varpi^2}{\varpi^2+(n+m+1)^2}\,\frac{1}{\sqrt{2m+1}}\right]\xi_m\,. \tag{42}$$

Note that $\mathfrak{H}_1$ is a diagonal operator with non-negative diagonal elements, $\mathfrak{H}_2$ is a real symmetric operator with vanishing diagonal elements and positive off-diagonal elements, and $\mathfrak{H}_3$ is a real symmetric operator with all positive elements. Therefore, also $\mathfrak{K}_1$ is a diagonal operator with non-negative diagonal elements, $\mathfrak{K}_2$ is a real symmetric operator with vanishing diagonal elements and positive off-diagonal elements, and $\mathfrak{K}_3$ is a real symmetric operator with all positive elements. Note furthermore that the operators $\mathfrak{K}_j$ do not dependend on the parameter $\lambda = \lambda(0)$.

Next, analogous to [14] we have the following result.

Proposition 2 *For $j \in \{1,2,3\}$, each $\mathfrak{K}_j \in \ell^2(\mathbb{N}_0 \times \mathbb{N}_0)$ for all $T > 0$ and $P \in \mathcal{P}$. Thus the operators $\mathfrak{K}_j = \mathfrak{K}_j(P,T)$ for $j \in \{1,2,3\}$ are Hilbert–Schmidt operators that map $\ell^2(\mathbb{N}_0)$ compactly into $\ell^2(\mathbb{N}_0)$.*

Proof of Proposition 2 We recall that $\lim_{\varpi\to 0}\frac{1}{\varpi^2}\mathfrak{H}_j(\varpi) = \mathfrak{G}_j(2)$ for $j \in \{1,2,3\}$, with $\mathfrak{G}(\gamma) = -\mathfrak{G}_1(\gamma) + \mathfrak{G}_2(\gamma) + \mathfrak{G}_3(\gamma)$ the interaction matrix of the linearized γ model. In Appendix A of [14] we proved that each $\mathfrak{G}_j(\gamma)$, $j \in \{1,2,3\}$ is a Hilbert–Schmidt operator for all $\gamma > 0$. In particular, the $\mathfrak{G}_j(2)$, $j \in \{1,2,3\}$ are Hilbert–Schmidt operators. We let $\left[\mathfrak{H}_j(\varpi)\right]_{n,m}$ denote the expressions bracketed $\left[\cdots\right]$ in (40), (41), (42). Since the maps $\varpi \mapsto \frac{1}{\varpi^2}\left[\mathfrak{H}_j(\varpi)\right]_{n,m}$ are non-negative and strictly monotonically decreasing for each $j \in \{1,2,3\}$, it follows that also the operators $\frac{1}{\varpi^2}\mathfrak{H}_j(\varpi)$ are Hilbert–Schmidt operators, hence so are the operators $\mathfrak{H}_j(\varpi)$ for all $j \in \{1,2,3\}$.

A moment of reflection reveals that this now implies that the ℓ^2 norm of the operators $\mathfrak{K}_j$ is bounded above by $\int_0^\infty \varpi^2 P(d\omega)$ times the ℓ^2 norm of the γ model operators $\mathfrak{G}_j$ at $\gamma = 2$. □

We are ready to prove Theorem 1^+.

Proof of Theorem 1^+ Having Proposition 2, the proof of our Theorem 1^+ is a straightforward adaptation of the proof of Theorem 1 in [14]. All that needs to be changed is the following:

Replace the operators $\mathfrak{G}_j(\gamma)$, $j \in \{1,2,3\}$ by operators $\mathfrak{K}_j(P,T)$, $j \in \{1,2,3\}$, and replace $\mathfrak{G}(\gamma)$ by $\mathfrak{K}(P,T)$.

Similarly, replace the eigenvalues $\mathfrak{g}_j(\gamma)$, $j \in \{1,2,3\}$ by the eigenvalues $\mathfrak{k}_j(P,T)$, $j \in \{1,2,3\}$, and replace $\mathfrak{g}(\gamma)$ by $\mathfrak{k}(P,T)$, β^γ by λ.

Here it shall suffice to point out that (39) makes it plain that $\min\left\{Q(\Xi) : \|\Xi\|_{\ell^2} = 1\right\} = 0$ if and only if $\lambda\mathfrak{k}(P,T) = 1$, with $\mathfrak{k}(P,T) > 0$ denoting the largest eigenvalue of $\mathfrak{K}(P,T)$.

Precisely when $\lambda = \Lambda(P, T)$, with

$$\Lambda(P, T) = \frac{1}{\mathfrak{k}(P,T)}, \tag{43}$$

then the pertinent eigenvalue problem for the minimizing mode $\Xi^{\rm opt}$ of $Q(\Xi)$ reads

$$\big(\mathfrak{I} - \Lambda(P, T)\mathfrak{K}\big)\,\Xi^{\rm opt} = 0, \tag{44}$$

which, since $\mathfrak{k} = 1/\Lambda(P, T)$, is equivalent to

$$\mathfrak{C}\big(\mathfrak{k}(P, T)\big)\,\Xi^{\rm opt} = \Xi^{\rm opt} \tag{45}$$

where here

$$\mathfrak{C}\big(\eta\big) := \big(\eta\mathfrak{I} + \mathfrak{K}_1\big)^{-1}\big(\mathfrak{K}_2 + \mathfrak{K}_3\big). \tag{46}$$

As in the proof of Theorem 1 in [14] one shows that $\mathfrak{C}(\eta)$ for $\eta > 0$ is a compact operator that maps the positive cone $\ell^2_{\geq 0}(\mathbb{N}_0)$ into itself, in fact mapping any non-zero element of $\ell^2_{\geq 0}(\mathbb{N}_0)$ into the interior of $\ell^2_{\geq 0}(\mathbb{N}_0)$, and that the spectral radius of $\mathfrak{C}\big(\mathfrak{h}\big)$ equals 1. Thus the Krein–Rutman theorem applies and guarantees that the non-trivial solution $\Xi^{\rm opt}$ of (45) is in the positive cone $\ell^2_{\geq 0}(\mathbb{N}_0)$ (after at most choosing the overall sign), hence a superconducting perturbation of the normal state $\underline{\Xi}$. □

The proof of Theorem 1^+ also proves our Theorem 1. □

The proof of Theorem 1^+ also proves our Theorem 3. □

As in [14], it is useful to add the following non-obvious fact about the spectrum of the operator $\mathfrak{K}(P, T)$.

Proposition 3 *Let $T > 0$ and $P \in \mathcal{P}$ be given. Then the largest eigenvalue $\mathfrak{k}(P, T)$ of $\mathfrak{K}(P, T)$ is also the spectral radius $\rho\big(\mathfrak{K}(P, T)\big)$.*

Proof of Proposition 3 The proof is a straightforward adaptation from the proof of Proposition 1 in [14], with the same replacements needed as stated in the proof of our Theorem 1^+. □

Proposition 3 allows us to characterize $\Lambda(\varpi)$ as follows:

$$\Lambda(P, T) = \frac{1}{\rho(\mathfrak{K})(P,T)}. \tag{47}$$

Each of (47) and (43) offer their own advantages to estimate Λ.

5 Upper Bounds on $\Lambda(P, T)$

5.1 Upper Bounds $\Lambda^{(N)}(P, T)$, $N \in \{1, 2, 3, 4\}$

We here prove Theorem 4.

Proof Theorem 1^+, or rather its proof established that $\Lambda(P, T) = \frac{1}{\mathfrak{k}(P,T)}$, with $\mathfrak{k}(P, T) > 0$ the largest eigenvalue of $\mathfrak{K}(P, T)$. More explicitly,

$$\Lambda(P, T) := \frac{1}{\max_{\Xi} \frac{\big\langle \Xi\, ,\, \mathfrak{K}(P,T)\, \Xi \big\rangle}{\big\langle \Xi\, ,\, \Xi \big\rangle}}, \tag{48}$$

where the maximum is taken over non-vanishing $\Xi \in \ell^2(\mathbb{N}_0)$.

Since $\mathfrak{K}(P,T)$ is compact, in principle one can get arbitrarily accurate upper approximations to $\Lambda(P,T)$ by restricting $\mathfrak{K}(P,T)$ to suitably chosen finite-dimensional subspaces of $\ell^2(\mathbb{N}_0)$. A sequence of decreasing rigorous upper bounds on $\Lambda(P,T)$ that converges to $\Lambda(P,T)$ is obtained by restricting the variational principle to a sequence of subspaces of $\ell^2(\mathbb{N}_0)$ consisting of vectors $\Xi_N := (\xi_0, \ldots, \xi_{N-1}, 0, \ldots)$, with $\xi_j > 0$ for $j \in \{0, ..., N-1\}$ and $N \in \mathbb{N}$. Evaluating (48) with Ξ_N in place of Ξ^{opt} yields a strictly monotonically decreasing sequence of upper bounds $\Lambda^{(N)}(P,T)$ on $\Lambda(P,T)$, viz.

$$\Lambda^{(N)}(P,T) := \frac{1}{\max_{\Xi_N} \frac{\left\langle \Xi_N\,,\,\mathfrak{K}(P,T)\,\Xi_N \right\rangle}{\left\langle \Xi_N,\,\Xi_N \right\rangle}}. \tag{49}$$

The evaluation of (49) is equivalent to finding the largest eigenvalue of a real symmetric matrix $N \times N$ matrix $\mathfrak{M}$, i.e. the largest zero of the associated degree-N characteristic polynomial of $\mathfrak{M}$. As noted in [14], the coefficients c_k of the characteristic polynomial $\det\left(\mu\mathcal{I} - \mathfrak{M}\right) =: \sum_{k=0}^{N} c_k \mu^k$ are explicitly known polynomials of degree $N-k$ in $\operatorname{tr}\mathfrak{M}^j$, $j \in \{1, ..., N\}$. When $N \in \{1, 2, 3, 4\}$ the zeros of the characteristic polynomial can be computed algebraically in closed form. For general real symmetric $N \times N$ matrices $\mathfrak{M}$ these spectral formulas have been listed in [14] and need not be repeated here.

The task that remains is to substitute $\mathfrak{K}^{(N)}$, $N \in \{1, 2, 3, 4\}$, for $\mathfrak{M}$ and to select the largest eigenvalue for each N from these spectra. This is trivial for $N = 1$, yielding (10), and it is straightforward for $N = 2$ and $N = 4$, yielding (11) and (15), respectively. The only case that is not quite so straightforward is $N = 3$. In this case we argue as follows.

We recall that $\lim_{\varpi \to 0} \frac{1}{\varpi^2}\mathfrak{H}^{(N)}(\varpi) = \mathfrak{G}^{(N)}(2)$, the interaction matrix of the linearized γ model at $\gamma = 2$, truncated to N Matsubara frequencies. In [14] we determined the largest eigenvalue of $\mathfrak{G}^{(3)}(2)$, which equals its spectral radius, then used a continuity argument, combined with the Perron–Frobenius theorem that establishes the non-degeneracy of the spectral radius, to extend the determination of the largest eigenvalue of $\mathfrak{G}^{(3)}(2)$ to that of $\mathfrak{G}^{(3)}(\gamma)$ for all $\gamma > 0$. By the same type of reasoning we now can extend the determination of the largest eigenvalue of $\mathfrak{G}^{(3)}(2)$ to that of $\frac{1}{\varpi^2}\mathfrak{H}^{(3)}(\varpi)$ for all $\varpi > 0$. Multiplication by ϖ^2 for $\varpi > 0$ does not change the ordering of the eigenvalues, and this yields (12) for when P is a Dirac delta measure, i.e. for the non-dispersive model.

At last, since the interaction kernels $\lambda(n)$ are bounded linear functionals of P, they are continuous in P, and manifestly they are continuous in $T > 0$. Therefore by the same type of reasoning we now can extend the determination of the largest eigenvalue of $\mathfrak{H}^{(3)}(\varpi)$ to that of the largest eigenvalue of $\mathfrak{K}^{(3)}(P,T)$, yielding (12).

This proves Theorem 4. □

For $N > 4$ a numerical approximation of $\mathfrak{k}^{(N)}(P,T)$ is necessary for any choices of P and T that are of interest.

5.2 Upper Bounds $\Lambda^{(N)}(P,T)$ as $T \to 0$, $N \in \mathbb{N}$

We here prove Theorem 5 by evaluating $\Lambda^{(N)}(P,T)$ in the limit $T \to 0$, for all $N \in \mathbb{N}$.

Proof of Theorem 5 Note that $T \to 0$ means $\varpi \to \infty$ for all $\omega > 0$. For each $N \in \mathbb{N}$ one easily finds

$$\lim_{\varpi \to \infty} \mathfrak{H}^{(N)}(\varpi) = -\mathfrak{I}^{(N)} + 2\,\Xi_N^* \otimes \Xi_N^*, \tag{50}$$

with $\mathfrak{I}^{(N)}$ the $N \times N$ identity matrix, and where $\Xi_N^* = (\xi_0^*, \xi_1^*, ..., \xi_{N-1}^*)$ has components $\xi_n^* = \frac{1}{\sqrt{2n+1}}$. Since r.h.s.(50) is independent of ω, also

$$\lim_{T\to 0} \mathfrak{K}^{(N)}(P, T) = -\mathfrak{I}^{(N)} + 2\, \Xi_N^* \otimes \Xi_N^*, \tag{51}$$

independently of P. The limiting matrix at r.h.s.(51) has an $N-1$-dimensional eigenspace for the eigenvalue -1, consisting of the orthogonal complement of Ξ_N^*. The remaining eigenspace is $\{t\,\Xi_N^*; t \in \mathbb{R}\}$, associated with the eigenvalue $-1+2\sum_{n=0}^{N-1}\frac{1}{2n+1} \geq 1$. Manifestly this eigenvalue is the largest eigenvalue of the matrix at r.h.s.(51); it also is its spectral radius. □

We note that r.h.s.(24) diverges to ∞ when $N \to \infty$, essentially like $\ln N$. Thus, $\lim_{T\to 0} \Lambda^{(N)}(P, T) = \frac{1}{\mathfrak{k}_0^{(N)}} =: \lambda_N \to 0$ as $N \to \infty$, as claimed in the introduction.

5.3 Upper Bounds $\Lambda^{(N)}(P, T)$ at $T \gg \overline{\Omega}(P)$, $N \in \mathbb{N}$

We here prove Theorem 7.

Proof of Theorem 7 Since P is supported on $[0, \overline{\Omega}(P)]$, the parameter regime $T \gg \overline{\Omega}(P)$ implies that $\varpi \ll 1$ uniformly in all integrals w.r.t. P. Since all kernels of $\mathfrak{H}_j(\varpi)$, $j \in \{1, 2, 3\}$, are of the type $\frac{\varpi^2}{\varpi^2+k^2} > 0$, with $k \in \mathbb{N}$, the operators $\mathfrak{H}_j(\varpi)$, $j \in \{1, 2, 3\}$, and their N-frequency truncations, are real analytic in ϖ, and in ϖ^2, about $\varpi = 0 = \varpi^2$. Maclaurin expansion in powers of ϖ^2 about $\varpi^2 = 0$ yields

$$\frac{\varpi^2}{\varpi^2+k^2} = \varpi^2\frac{1}{k^2} - \varpi^4\frac{1}{k^4} \pm \cdots, \tag{52}$$

which implies that

$$\mathfrak{H}_j(\varpi) = \varpi^2\mathfrak{G}_j(2) - \varpi^4\mathfrak{G}_j(4) \pm \cdots, \quad j \in \{1, 2, 3\}, \tag{53}$$

and the analog holds for their N-frequency truncations. Averaging over P then yields the asymptotic large T expansion

$$\mathfrak{K}_j(P, T) = \frac{\mathfrak{G}_j(2)\langle\omega^2\rangle}{4\pi^2}\frac{1}{T^2} - \frac{\mathfrak{G}_j(4)\langle\omega^4\rangle}{16\pi^4}\frac{1}{T^4} \pm \cdots, \; j \in \{1, 2, 3\}, \tag{54}$$

and analogously for their N-frequency truncations.

The claim (28) of the theorem now follows from first-order perturbation theory [13].

It remains to show that $\forall\, N \in \mathbb{N}$ we have $\langle\mathfrak{G}^{(N)}(4)\rangle_2 > 0$, where

$$\langle\mathfrak{G}^{(N)}(4)\rangle_2 := \frac{\left\langle \Xi_N^{\text{opt}}(2), \mathfrak{G}^{(N)}(4)\, \Xi_N^{\text{opt}}(2)\right\rangle}{\left\langle \Xi_N^{\text{opt}}(2), \Xi_N^{\text{opt}}(2)\right\rangle}; \tag{55}$$

here, $\Xi_N^{\text{opt}}(2)$ is a not necessarily normalized positive eigenvector for the top eigenvalue $\mathfrak{g}^{(N)}(2)$ of $\mathfrak{G}^{(N)}(2)$.

We prove a stronger result that implies that $\langle\mathfrak{G}^{(N)}(4)\rangle_2 > 0$.

Proposition 4 *Let $\gamma > 0$ be given. Then for all $\gamma' > 0$ and $N \in \mathbb{N}_0$,*

$$\langle \mathfrak{G}^{(N)}(\gamma') \rangle_\gamma := \frac{\left\langle \Xi_N^{\mathrm{opt}}(\gamma), \mathfrak{G}^{(N)}(\gamma')\, \Xi_N^{\mathrm{opt}}(\gamma) \right\rangle}{\left\langle \Xi_N^{\mathrm{opt}}(\gamma), \Xi_N^{\mathrm{opt}}(\gamma) \right\rangle} > 0, \tag{56}$$

with $\Xi_N^{\mathrm{opt}}(\gamma)$ any eigenvector of the top eigenvalue $\mathfrak{g}^{(N)}(\gamma)$ of $\mathfrak{G}^{(N)}(\gamma)$.

Proof of Proposition 4 We begin by noting that the proof is trivial when $N = 1$. It remains to prove the proposition for when $N > 1$.

In [14] we already proved that $\langle \mathfrak{G}^{(N)}(\gamma) \rangle_\gamma = \mathfrak{g}^{(N)}(\gamma) \geq 1$, with "=" holding iff $N = 1$. Thus we see that Proposition 4 is true when restricted to $\gamma = \gamma'$. To see that Proposition 4 is true also when $\gamma \neq \gamma'$, we need the input of another result.

Proposition 5 *Let $\gamma > 0$ be given. Then for all $N \in \mathbb{N}_0$, the map $n \mapsto \frac{1}{\sqrt{2n+1}} \left(\Xi_N^{\mathrm{opt}}(\gamma) \right)_n$ is positive and decreasing.*

Proof of Proposition 5 Also the proof of Proposition 5 is trivial when $N = 1$. Henceforth $N > 1$, therefore.

We first prove the positivity. The eigenvector $\Xi_N^{\mathrm{opt}}(\gamma)$ of the largest eigenvalue $\mathfrak{g}^{(N)}(\gamma)$ of $\mathfrak{G}^{(N)}(\gamma)$ solves the linear eigenvalue equation

$$\mathfrak{G}^{(N)}(\gamma)\Xi = \mathfrak{g}^{(N)}(\gamma)\Xi, \tag{57}$$

where

$$\mathfrak{G}^{(N)}(\gamma) = -\mathfrak{G}_1^{(N)}(\gamma) + \mathfrak{G}_2^{(N)}(\gamma) + \mathfrak{G}_3^{(N)}(\gamma) \tag{58}$$

is the projection of $\mathfrak{G}(\gamma) = -\mathfrak{G}_1(\gamma) + \mathfrak{G}_2(\gamma) + \mathfrak{G}_3(\gamma)$ onto the subspace spanned by the first N positive Matsubara frequencies. The operators $\mathfrak{G}_j(\gamma)$ act componentwise as follows (cf. [14]),

$$(\mathfrak{G}_1(\gamma)\Xi)_n = \left[\frac{1}{2n+1} \sum_{k=1}^{n} \frac{2}{k^\gamma} \right] \xi_n\,, \tag{59}$$

$$(\mathfrak{G}_2(\gamma)\Xi)_n = \sum_m \left[\frac{1}{\sqrt{2n+1}} \frac{1-\delta_{n,m}}{|n-m|^\gamma} \frac{1}{\sqrt{2m+1}} \right] \xi_m\,, \tag{60}$$

$$(\mathfrak{G}_3(\gamma)\Xi)_n = \sum_m \left[\frac{1}{\sqrt{2n+1}} \frac{1}{(n+m+1)^\gamma} \frac{1}{\sqrt{2m+1}} \right] \xi_m\,, \tag{61}$$

where it is understood that $\frac{1-\delta_{n,m}}{|n-m|^\gamma} := 0$ when $n = m$. Analogously their N-Matsubara frequency truncations are defined by limiting n and m to the set $\{0, ..., N-1\}$ in (59), (60), and (61).

Analogously to (45), we can rewrite (57) as the fixed point problem

$$\mathfrak{C}_\gamma^{(N)}\big[\mathfrak{g}(\gamma)\big]\Xi_N = \Xi_N \tag{62}$$

where $\mathfrak{C}_\gamma^{(N)}[\eta]$, with $\eta > 0$, is the restriction of

$$\mathfrak{C}_\gamma[\eta] := \big(\eta\mathfrak{I} + \mathfrak{G}_1(\gamma)\big)^{-1}\big(\mathfrak{G}_2(\gamma) + \mathfrak{G}_3(\gamma)\big) \tag{63}$$

to the subspace of $\ell^2(\mathbb{N}_0)$ spanned by the first N positive Matsubara frequencies. The subscript $_\gamma$ is meant as a reminder to distinguish the operator $\mathfrak{C}_\gamma[\eta]$ of the γ model from the operator $\mathfrak{C}[\eta]$ of the standard Eliashberg model that we defined in (46). As shown in [14], and analogously in the proof of Theorem 1^+ in this paper, the operator $\mathfrak{C}_\gamma\big[\eta\big]$ maps $\ell^2_{\geq 0}(\mathbb{N}_0)$ compactly into itself. Any non-zero element of $\ell^2_{\geq 0}(\mathbb{N}_0)$ is mapped into the interior of $\ell^2_{\geq 0}(\mathbb{N}_0)$. Iff $\eta = \mathfrak{g}(\gamma)$ its spectral radius equals 1. It is non-degenerate and the associated normalized eigenmode $\Xi^{\rm opt}(\gamma) \in \ell^2_{\geq 0}(\mathbb{N}_0)$ has only positive components, by the Krein–Rutman theorem. The analogous statement holds for all its finite-rank, N-frequency approximations $\Xi^{\rm opt}_N(\gamma)$, by the Perron–Frobenius theorem.

Having established the positivity of $\Xi^{\rm opt}_N(\gamma)$ for all $N \in \mathbb{N}$, the positivity of the map $n \mapsto \frac{1}{\sqrt{2n+1}}\big(\Xi^{\rm opt}_N(\gamma)\big)_n$ follows trivially.

We turn to the decrease of the map $n \mapsto \frac{1}{\sqrt{2n+1}}\big(\Xi^{\rm opt}_N(\gamma)\big)_n$. Recall that $\Xi^{\rm opt}_N(\gamma)$ is a positive maximizer of $\big\langle \Xi_N\,, \mathfrak{G}^{(N)}(\gamma)\;\Xi_N\big\rangle/\big\langle \Xi_N\;,\;\;\Xi_N\big\rangle$ on $\mathbb{R}^N\backslash\{0_N\}$, where 0_N is the vanishing vector. Now recall that $\frac{1}{\sqrt{2n+1}}\big(\Xi_N(\gamma)\big)_n = \big(\Theta_N(\gamma)\big)_n$, where by Θ_N we denote a truncation to the first N positive Matsubara frequencies of the sequence Θ of angles $(\theta_n)_{n\in\mathbb{N}_0}$ defined in section 4, except that for the linearized problem the restriction of the θ_n to $[0, 2\pi]$ can be dropped. Defining $\mathfrak{D}$ to be the diagonal matrix whose n-th diagonal entry is the n-th odd positive integer, we can more compactly write $\Xi = \sqrt{\mathfrak{D}}\Theta$ and $\Xi_N = \sqrt{\mathfrak{D}}^{(N)}\Theta_N$. Thus we have

$$\max_{\mathbb{R}^N\backslash\{0_N\}} \frac{\big\langle \Xi_N, \mathfrak{G}^{(N)}(\gamma)\;\Xi_N\big\rangle}{\big\langle \Xi_N, \;\Xi_N\big\rangle} \equiv \max_{\mathbb{R}^N\backslash\{0_N\}} \frac{\big\langle \Theta_N, \widehat{\mathfrak{G}}^{(N)}(\gamma)\;\Theta_N\big\rangle}{\big\langle \Theta_N, \mathfrak{D}^{(N)}\Theta_N\big\rangle}, \tag{64}$$

where $\widehat{\mathfrak{G}}^{(N)}(\gamma)$ is the truncation to the first N Matsubara frequencies of $\widehat{\mathfrak{G}}(\gamma) := \sqrt{\mathfrak{D}}\mathfrak{G}(\gamma)\sqrt{\mathfrak{D}}$. Written explictly in terms of the θ_n, the quadratic forms at r.h.s.(64) read

$$\big\langle \Theta, \mathfrak{D}\Theta\big\rangle := \sum_n (2n+1)\theta_n^2 \tag{65}$$

and

$$\begin{aligned}&\big\langle \Theta, \widehat{\mathfrak{G}}(\gamma)\;\Theta\big\rangle := \\ &-\sum_n\Bigg(\sum_{k=1}^n \frac{2}{k^\gamma}\Bigg)\theta_n^2 + \sum_n\sum_m \theta_n\left[\frac{1-\delta_{n,m}}{|n-m|^\gamma} + \frac{1}{(n+m+1)^\gamma}\right]\theta_m,\end{aligned} \tag{66}$$

where $\sum_n(\cdots)$ means summation over $\mathbb{N}_0$, and where again it is understood that $\frac{1-\delta_{n,m}}{|n-m|^\gamma} := 0$ when $n = m$. The N-Matsubara frequency truncations of (66) and (65) are defined by limiting n and m to the set $\{0, ..., N-1\}$. We have completed our preparations for showing that any positive maximizer $\Theta^{\rm opt}_N(\gamma)$ of r.h.s.(64) is decreasing in n.

We will show that with the help of decreasing rearrangement $^\sharp\Theta_N$ (by suitable permutation of the entries of Θ_N) we can increase the ratio of the two quadratic forms at r.h.s.(64) if Θ_N was not already decreasing everywhere. While decreasing rearrangements generally do not preserve the normalization, this is not a problem since the ratio of the two quadratic forms at r.h.s.(64) is homogeneous of degree zero under scaling of Θ_N, so that after a decreasing rearrangement the stipulated normalization can be restored with a simple overall scaling, without changing the value of the ratio of the two quadratic forms.

First, since the map $n \mapsto \sum\limits_{k=1}^n \frac{1}{k^\gamma}$ is strictly increasing on $\mathbb{N}_0$, given any $\Theta_N \in \mathbb{R}^N_+$ that is locally strictly increasing somewhere, its decreasing rearrangement $^\sharp\Theta_N$ obeys the inequality $\sum\limits_n\Big(\sum\limits_{k=1}^n \frac{1}{k^\gamma}\Big)^\sharp\theta_n^2 < \sum\limits_n\Big(\sum\limits_{k=1}^n \frac{1}{k^\gamma}\Big)\theta_n^2$, where n here runs from 0 to $N-1$.

Second, we note that

$$\sum_n\sum_m \theta_n\left[\frac{1-\delta_{n,m}}{|n-m|^\gamma}+\frac{1}{(n+m+1)^\gamma}\right]\theta_m \equiv 2^{\gamma-1}\sum\sum_{i\neq j}\tilde\theta_i\frac{1}{|i-j|^\gamma}\tilde\theta_j, \tag{67}$$

where as before $\frac{1-\delta_{n,m}}{|n-m|^\gamma} := 0$ if $n = m$, where i and j are odd integers running from $-2N+1$ to $2N-1$, and where the truncated sequence $i \mapsto \tilde\theta_i$ is symmetric, i.e. $\tilde\theta_{-i} = \tilde\theta_i$, with $\tilde\theta_{2n+1} \equiv \theta_n$ for $n \in \{0, ..., N-1\}$. By the re-arrangement inequality of Hardy–Littlewood–Pólya [11], see Theorem 3.71 in [12], if the symmetric map $i \mapsto \tilde\theta_i$ is not symmetric decreasing then

$$\sum_i\sum_j {}^\sharp\tilde\theta_i\frac{1}{\delta_{i,j}+|i-j|^\gamma}{}^\sharp\tilde\theta_j > \sum_i\sum_j \tilde\theta_i\frac{1}{\delta_{i,j}+|i-j'|^\gamma}\tilde\theta_j, \tag{68}$$

where $i \mapsto {}^\sharp\tilde\theta_i$ is the symmetric decreasing rearrangement of $i \mapsto \tilde\theta_i$. Note that the diagonal terms (from when $i = j$) contribute $\sum_i {}^\sharp\tilde\theta_i^2$ to the double sum at l.h.s.(68), and $\sum_i \tilde\theta_i^2$ to the double sum at r.h.s.(68). Note furthermore that ℓ^2 norms are invariant under permutations of the elements in the sequence, hence under symmetric decreasing rearrangements, so that these diagonal contributions can be purged from (68). Thus, if the symmetric map $i \mapsto \tilde\theta_i$ is not symmetric decreasing then

$$\sum\sum_{i\neq j} {}^\sharp\tilde\theta_i\frac{1}{|i-j|^\gamma}{}^\sharp\tilde\theta_j, > \sum\sum_{i\neq j}\tilde\theta_i\frac{1}{|i-j|^\gamma}\tilde\theta_j. \tag{69}$$

Third, since $n \mapsto 2n+1$ is strictly increasing on $\mathbb{N}_0$, and $N > 1$, also the inequality $\langle{}^\sharp\Theta_N, \mathfrak{D}^{(N)}\,{}^\sharp\Theta_N\rangle < \langle\Theta_N, \mathfrak{D}^{(N)}\Theta_N\rangle$ holds (under the same hypothesis on Θ_N). And since we know that the maximum is positive, we only need to consider Θ_N for which $\langle\Theta_N, \widehat{\mathfrak{G}}^{(N)}(\gamma)\,\Theta_N\rangle > 0$, so that $\langle{}^\sharp\Theta_N, \widehat{\mathfrak{G}}^{(N)}(\gamma)\,{}^\sharp\Theta_N\rangle > 0$ as well. For any non-negative sequence Θ_N that yields a positive result for the quadratic form (66), but is locally strictly increasing somewhere, it then follows that

$$\frac{\langle\Theta_N, \widehat{\mathfrak{G}}^{(N)}(\gamma)\,\Theta_N\rangle}{\langle\Theta_N, \mathfrak{D}^{(N)}\Theta_N\rangle} < \frac{\langle{}^\sharp\Theta_N, \widehat{\mathfrak{G}}^{(N)}(\gamma)\,{}^\sharp\Theta_N\rangle}{\langle{}^\sharp\Theta_N, \mathfrak{D}^{(N)}\,{}^\sharp\Theta_N\rangle}. \tag{70}$$

The three rearrangement inequalities imply that $\Theta_N^{\mathrm{opt}}(\gamma) = {}^\sharp\Theta_N^{\mathrm{opt}}(\gamma)$.
This completes the proof of Proposition 5. □

Remark 4 The Hardy–Littlewood–Pólya rearrangement inequality for finite sequences on symmetric intervals of integers has been generalized to ℓ^2 sequences on $\mathbb{Z}$ by Pruss [20]. A continuum version for non-negative $L^2(\mathbb{R})$ functions is due to F. Riesz [21]; see Theorem 379 in [12]. □

Remark 5 Completely analogously to Proposition 5 it can be shown with the help of Pruss' discretized Riesz inequality that $\Theta^{\mathrm{opt}}(\gamma) = {}^\sharp\Theta^{\mathrm{opt}}(\gamma)$. Since in $\ell^2_{\geq 0}(\mathbb{N}_0)$ a constant sequence Θ has to vanish identically, and thus cannot satisfy the normalization of Θ, it follows that the completely positive sequence $\Theta^{\mathrm{opt}}(\gamma)$ is not only also a decreasing sequence, it is strictly decreasing infinitely often. □

Remark 6 Since $(n, m) \mapsto \frac{1}{(n+m+1)^\gamma}$ with $(n, m) \in \mathbb{N}_0^2$ is strictly decreasing, separately and jointly so, given any $\Theta \in \ell^2_{\geq 0}(\mathbb{N}_0)$ that is locally strictly increasing somewhere its decreasing rearrangement ${}^\sharp\Theta$ obeys the inequality

$$\sum_n\sum_m {}^\sharp\theta_n\frac{1}{(n+m+1)^\gamma}{}^\sharp\theta_m > \sum_n\sum_m \theta_n\frac{1}{(n+m+1)^\gamma}\theta_m. \tag{71}$$

The analogous conclusion does not hold for the form in which $\frac{1}{(n+m+1)^\gamma}$ is replaced by the Riesz kernel $|n-m|^{-\gamma}$, which is the reason for why we had to work with the symmetric extension $\widetilde{\Theta}_N$ of Θ_N. □

Armed with Proposition 5 we return to the proof of Proposition 4 for when $N > 1$ and $\gamma \neq \gamma'$. We first note that since $\Xi^{\text{opt}}(\gamma)$ is not the vanishing sequence, the denominator of l.h.s.(56) is > 0, and thus to prove Proposition 4 for when $N > 1$ and $\gamma \neq \gamma'$ it suffices to demonstrate that the numerator of l.h.s.(70) is > 0 for *non-vanishing*, non-negative, decreasing $\Theta_N \in \mathbb{R}^N$; for this we may drop the $'$ from γ'.

To demonstrate the positivity of the numerator of l.h.s.(70) we expand it into a Dirichlet series, viz

$$-\sum_{n=0}^{N-1}\left(\sum_{k=1}^{n}\frac{2}{k^\gamma}\right)\theta_n^2 + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\theta_n\left[\frac{1-\delta_{n,m}}{|n-m|^\gamma} + \frac{1}{(n+m+1)^\gamma}\right]\theta_m = \sum_{k=1}^{2N-1}\frac{c_k(\Theta)}{k^\gamma}, \tag{72}$$

where the coefficients $c_k(\Theta)$ are given by

$$\begin{aligned} c_k(\Theta) = \mathbf{1}_{\{k\le N-1\}}\sum_{n=k}^{N-1}2\big(\theta_{n-k}-\theta_n\big)\theta_n \\ + \mathbf{1}_{\{2\le k\le N\}}\sum_{n=0}^{k-2}2\theta_{k-1-n}\theta_n \\ + \mathbf{1}_{\{k\ \text{odd}\}}\,\theta^2_{\frac{k-1}{2}}, \end{aligned} \tag{73}$$

with $\mathbf{1}_A$ the indicator function of the set A. Since the $\theta_n \geq 0$, all contributions from the second and third line at r.h.s.(73) are manifestly non-negative, and since the sequence Θ_N is not only non-negative but also decreasing, it follows that also the contribution from the first line at r.h.s.(73) is manifestly non-negative; lastly, since the sequence Θ_N is not only non-negative and decreasing, but also non-vanishing, it follows that $\theta_0^2 > 0$, which is contributed by the $k = 1$ term in the third line at r.h.s.(73). Thus the numerator of l.h.s.(70) is > 0 on the set of non-vanishing, non-negative, decreasing Θ_N.

The proof of Proposition 4 is complete. □

The proof of Theorem 7 is complete. □

Remark 7 For our purposes the proof of the positivity of l.h.s.(70) on the set of non-vanishing, non-negative, decreasing sequences suffices. An interesting question in its own right is whether there is a positive lower bound to l.h.s.(70) on this set. We proved that the family of constant non-vanishing sequences is a local minimizer of l.h.s.(70) on this set. We suspect that these constant sequences in fact are global minimizers on this set, which is also supported by numerical comparisons of the evaluations of l.h.s.(70) for a constant sequence with those for a handful of other pertinent sequences. Since the contribution from the first line at r.h.s.(73) vanishes if Θ_N is a constant sequence, and since those from the second and third line reduce to simple numerical multiples of θ_0^2, and since θ_0^2 cancels at l.h.s.(70), our suspicion becomes

Conjecture 3 *On the set of non-vanishing, non-negative, decreasing sequences, the l.h.s.(70) is bounded below through*

$$\frac{\left\langle \Theta_N \widehat{\mathfrak{G}}^{(N)}(\gamma)\, \Theta_N \right\rangle}{\left\langle \Theta_N, \mathfrak{D}^{(N)} \Theta_N \right\rangle} \geq \frac{1}{N^2} \sum_{k=1}^{2N-1} \frac{\min\{k, 2N-k\}}{k^\gamma}, \tag{74}$$

with equality holding iff Θ_N is a constant sequence. □

6 A Rigorous Lower Bound on $\Lambda(P, T)$

We here prove Theorem 6.

Proof of Theorem 6 When $\lambda < \Lambda(P, T)$, then the largest eigenvalue κ of the linear operator $\mathfrak{I} - \lambda\mathfrak{K}$ satisfies $\kappa = 1 - \lambda\mathfrak{k} > 0$. In that case the normal state $\underline{\Xi} \equiv 0$ is linearly stable, i.e. $Q(\Xi) \geq 0$ on all of $\ell^2(\mathbb{N}_0)$, with "$= 0$" if and only if $\Xi = \underline{\Xi}$.

Now recall that in the proof of Theorem 1^+ we noted that when $\lambda = \Lambda(P, T)$ then the eigenvalue problem $(\mathfrak{I} - \lambda\mathfrak{K})\Xi_{\text{opt}}$ is equivalent to the fixed point problem $\mathfrak{C}\big(\frac{1}{\Lambda(P,T)}\big)\Xi_{\text{opt}} = \Xi_{\text{opt}}$, and the compact operator $\mathfrak{C}\big(\frac{1}{\Lambda(P,T)}\big)$ has spectral radius $\rho\big(\mathfrak{C}\big(\frac{1}{\Lambda(P,T)}\big)\big) = 1$.

The two observations in concert imply that when $\lambda < \Lambda(P, T)$, then $\rho\big(\mathfrak{C}\big(\frac{1}{\lambda}\big)\big) < 1$. And since $\mathfrak{C}(\eta)$ leaves $\ell^2_{\geq 0}(\mathbb{N}_0)$ invariant for any $\eta > 0$, this now means that when $\lambda < \Lambda(P, T)$ then $\mathfrak{C}\big(\frac{1}{\lambda}\big)$ is a contraction mapping on $\ell^2_{\geq 0}$, with $\Xi = 0$ as the only fixed point. Put differently, if $\lambda < \Lambda(P, T)$, then

$$\big\|\mathfrak{C}\big(\tfrac{1}{\lambda}\big)(\Xi - \Xi')\big\| \leq L[\lambda]\big\|\Xi - \Xi'\big\|, \tag{75}$$

with $L[\lambda] := \rho\big(\mathfrak{C}\big(\frac{1}{\lambda}\big)\big) < 1$ the *Lipschitz constant* for the linear map $\mathfrak{C}\big(\frac{1}{\lambda}\big) : \ell^2_{\geq 0} \to \ell^2_{\geq 0}$.

Remark 8 The previous paragraph can be rephrased by saying that linear stability of the normal state $\underline{\Xi}$ against superconducting perturbations is equivalent to $\mathfrak{C}\big(\frac{1}{\lambda}\big)$ being a contraction mapping on $\ell^2_{\geq 0}$. □

We next construct a lower bound $\Lambda^*(P, T)$ on $\Lambda(P, T)$, for any $T > 0$ and $P \in \mathcal{P}$, by showing that $\rho\big(\mathfrak{C}\big(\frac{1}{\Lambda^*}\big)\big) < 1$. We accomplish this by arguing as in [14], invoking two well-known lemmas. The first one is

Lemma 1 *Let $T > 0$ and P be given. Then*

$$\rho(\mathfrak{C}(\eta)) \leq \rho\Big(\big(\eta\mathfrak{I} + \mathfrak{K}_1\big)^{-1}\Big)\rho\big(\mathfrak{K}_2 + \mathfrak{K}_3\big). \tag{76}$$

Recall that $\big(\eta\mathfrak{I} + \mathfrak{K}_1\big)^{-1}$ is a diagonal operator with all positive entries, and that $\mathfrak{K}_1$ has non-negative entries including 0. Hence,

$$\rho\Big(\big(\tfrac{1}{\lambda}\mathfrak{I} + \mathfrak{K}_1\big)^{-1}\Big) = \max_{n\geq 0}\left[\frac{1}{\lambda} + \frac{2}{2n+1}\sum_{k=1}^{n}\int_0^\infty \frac{\varpi^2}{\varpi^2 + k^2}P(d\omega)\right]^{-1} = \lambda. \tag{77}$$

Since the operator $\mathfrak{K}_2 + \mathfrak{K}_3$ is independent of λ, we now arrive at the following conclusion.

Proposition 6 *Let $T > 0$ and $P \in \mathcal{P}$ be given. Suppose*

$$\lambda \leq \frac{1}{\rho\big(\mathfrak{K}_2 + \mathfrak{K}_3\big)}. \tag{78}$$

Then $\lambda \leq \Lambda(P, T)$.

Proof By Lemma 1 and by (77), and by the hypothesis (78) of the Proposition, we have that

$$\rho(\mathfrak{C}(\tfrac{1}{\lambda})) \leq \lambda\rho\big(\mathfrak{K}_2 + \mathfrak{K}_3\big) \leq 1 = \rho\big(\mathfrak{C}(\tfrac{1}{\Lambda(P,T)})\big). \tag{79}$$

Since $\eta \mapsto \rho(\mathfrak{C}(\eta))$ is strictly monotonically decreasing, the claim of the Proposition follows. □

Corollary 5 *Under the hypotheses of Proposition 6, we have*

$$\Lambda(P,T) \geq \frac{1}{\rho\big(\mathfrak{K}_2+\mathfrak{K}_3\big)}. \tag{80}$$

The second well-known fact is

Lemma 2 *Let $T > 0$ and $P \in \mathcal{P}$ be given. Then*

$$\rho\big(\mathfrak{K}_2 + \mathfrak{K}_3\big) \leq \rho\big(\mathfrak{K}_2\big) + \rho\big(\mathfrak{K}_3\big). \tag{81}$$

Corollary 6 *Under the hypotheses of Proposition 6, we have*

$$\Lambda(\varpi) \geq \frac{1}{\rho\big(\mathfrak{K}_2\big)+\rho\big(\mathfrak{K}_3\big)}. \tag{82}$$

We now estimate the two spectral radii at r.h.s.(81) from above in terms of the spectral radii of $\mathfrak{G}_2(2)$ and $\mathfrak{G}_3(2)$ that we estimated in [14].

Proposition 7 *Let $T > 0$ and $P \in \mathcal{P}$ be given. Then, with $\varpi = \omega/2\pi T$, for $j \in \{2,3\}$ we have*

$$\rho\big(\mathfrak{K}_j(P,T)\big) \leq \rho\big(\mathfrak{G}_j(2)\big)\int_0^\infty \varpi^2 P(d\omega). \tag{83}$$

Proof All kernels of the operators $\mathfrak{K}_j(P,T)$, $j \in \{2,3\}$, are of the type $\int_0^\infty \frac{\varpi^2}{\varpi^2+k^2}P(d\omega) > 0$ for $k \in \mathbb{N}$, while all kernels of $\mathfrak{G}_j(\gamma)$, $j \in \{2,3\}$, are of the type $\frac{1}{k^\gamma} > 0$ for $k \in \mathbb{N}$. So the spectral radii of all these operators are achieved on the positive cone $\ell^2_{\geq 0}(\mathbb{N}_0)$. Since $\varpi^2 + k^2 > k^2$, the claim of the proposition follows. □

Now we recall that $\rho\big(\mathfrak{G}_j(2)\big)$ for $j \in \{2,3\}$ has been estimated from above in [14], see their Propositions 5 and 6. These estimates prove the weaker upper estimate of $T_c(P,T)$ registered after the statement of Theorem 6, where the term $\mathfrak{k}^{(1)}(P,T) = \int_0^\infty \frac{\varpi^2}{1+\varpi^2}P(d\omega)$ in (26) is replaced by the weaker $\int_0^\infty \varpi^2 P(d\omega)$. This almost proves our Theorem 6.

Finally, closer inspection of the proof of Proposition 6 in [14] reveals that the estimate $\varpi^2 + 1 > 1$, used in the proof of Proposition 7 above, is not needed to arrive at an analog of inequality (100) in [14], and avoiding the unnecessary estimate $\varpi^2+1 > 1$ for this particular contribution to the upper bound on $\rho\big(\mathfrak{K}_3\big)$ gives the bound stated in Theorem 6. □

7 From $\Lambda(P,T)$ to $T_c(\lambda,P)$

We here prove Proposition 1, and then Theorem 2 and Corollary 1.

Proof of Proposition 1 By its definition, $\mathfrak{k}^{(N)}(P,T)$ is given by the variational principle

$$\mathfrak{k}^{(N)}(P,T) := \max_{\Xi_N} \frac{\big\langle \Xi_N, \mathfrak{K}^{(N)}(P,T)\,\Xi_N\big\rangle}{\big\langle \Xi_N, \Xi_N\big\rangle}. \tag{84}$$

With the normalization $\big\langle \Xi_N, \Xi_N\big\rangle = 1$, and with the optimizer denoted $\Xi_N^{\rm opt}(P,T)$, we thus have

$$\mathfrak{k}^{(N)}(P,T) = \big\langle \Xi_N^{\rm opt}(P,T), \mathfrak{K}^{(N)}(P,T)\,\Xi_N^{\rm opt}(P,T)\big\rangle. \tag{85}$$

Since all kernels of $\mathfrak{K}^{(N)}(P,T)$ depend on T only through analytic functions of T^2, by first order perturbation theory [13] we have

$$\tfrac{\partial}{\partial T^2}\mathfrak{k}^{(N)}(P,T) = \big\langle \Xi_N^{\rm opt}(P,T), \big[\tfrac{\partial}{\partial T^2}\mathfrak{K}^{(N)}(P,T)\big]\,\Xi_N^{\rm opt}(P,T)\big\rangle. \tag{86}$$

We want to show that r.h.s.(86)$<$ 0. So far we can show this only if $T \geq T_*(P)$ with $T_*(P) \leq \frac{\overline{\Omega}(P)}{2\sqrt{2}\pi}$. To do so we recycle the strategy of proving Proposition 4.

As in the proof of Proposition 4, we need the analog of Proposition 5, now for the standard Eliashberg model.

Proposition 8 *For all $N \in \mathbb{N}_0$, the map $n \mapsto \frac{1}{\sqrt{2n+1}}\big(\Xi_N^{\rm opt}(P,T)\big)_n$ is positive and decreasing.*

Proof The proof of Proposition 8 is essentially verbatim to the proof of Proposition 5 and does not need to be stated in detail. It should suffice to recall that $\mathfrak{K}^{(N)} = -\mathfrak{K}_1^{(N)} + \mathfrak{K}_2^{(N)} + \mathfrak{K}_3^{(N)}$ in complete analogy to $\mathfrak{G}^{(N)} = -\mathfrak{G}_1^{(N)} + \mathfrak{G}_2^{(N)} + \mathfrak{G}_3^{(N)}$, with kernels of the type $\int_0^\infty \frac{\varpi^2}{\varpi^2+j^2}P(d\omega)$ taking over the role of the kernels of the type $\frac{1}{j^\gamma}$. Note that both types of kernels are positive, and they decrease when the index $j \in \mathbb{N}$ is increased.

This suffices to prove Proposition 8. □

Next we register the componentwise actions of $\frac{\partial}{\partial T^2}\mathfrak{K}_j^{(N)}(P,T)$ as N-frequency truncations of the componentwise actions of $\frac{\partial}{\partial T^2}\mathfrak{K}_j(P,T)$, given by the P-averages of $-\omega^2/4\pi^2T^4$ times the N-frequency truncations of the componentwise actions

$$\left(\frac{d}{d\varpi^2}\mathfrak{H}_1(\varpi)\Xi\right)_n = \left[\frac{2}{2n+1}\sum_{k=1}^{n}\frac{k^2}{(\varpi^2+k^2)^2}\right]\xi_n\,, \tag{87}$$

$$\left(\frac{d}{d\varpi^2}\mathfrak{H}_2(\varpi)\Xi\right)_n = \sum_m\left[\frac{1}{\sqrt{2n+1}}\,\frac{(n-m)^2}{(\varpi^2+(n-m)^2)^2}\,\frac{1}{\sqrt{2m+1}}\right]\xi_m, \tag{88}$$

$$\left(\frac{d}{d\varpi^2}\mathfrak{H}_3(\varpi)\Xi\right)_n = \sum_m\left[\frac{1}{\sqrt{2n+1}}\,\frac{(n+m+1)^2}{(\varpi^2+(n+m+1)^2)^2}\,\frac{1}{\sqrt{2m+1}}\right]\xi_m\,. \tag{89}$$

Lemma 3 *Let $\varpi > 0$. Then the map $j \mapsto G_\varpi(j) := \frac{j^2}{(\varpi^2+j^2)^2}$, $j \in \mathbb{N}$, is decreasing iff $\varpi \leq \sqrt{2}$.*

Proof Elementary calculus reveals that the rational function $x \mapsto \frac{x^2}{(\varpi^2+x^2)^2}$ on $\mathbb{R}_+$ increases strictly monotonically for $x \in [0,\varpi]$, reaches a unique maximum at $x = \varpi$, and decreases strictly monotonically to 0 for $x \in [\varpi,\infty)$. This already implies that $G_\varpi(j)$, $j \in \mathbb{N}$, decreases with increasing j if $\varpi \leq 1$. Yet, since $j \in \mathbb{N}$, we can increase ϖ a little bit beyond 1 so long as $G_\varpi(2) \leq G_\varpi(1)$. This is only feasible when the maximum of $G_\varpi(x)$, $x \in \mathbb{R}_+$, is located at some x below $\sqrt{2}$. The limiting situation occurs for $\varpi = \sqrt{2}$, with $G_{\sqrt{2}}(2) = G_{\sqrt{2}}(1)$. □

We note that when $\varpi = \sqrt{2}$, so that $G_{\sqrt{2}}(2) = G_{\sqrt{2}}(1)$, one still has $G_{\sqrt{2}}(j+1) < G_{\sqrt{2}}(j)$ for all $j > 1$.

Corollary 7 *Let $T \geq \overline{\Omega}(P)/2\sqrt{2}\pi$. Then r.h.s.(86)$< 0$.*

Proof Since by general hypothesis on P its support is contained in $[0, \overline{\Omega}(P)]$, the hypothesis of Corollary 7 that $T \geq \overline{\Omega}/2\sqrt{2}\pi$ implies that $\omega/2\pi T = \varpi \leq \sqrt{2}$ for all $\omega \leq \overline{\Omega}(P)$.

Thus, having Proposition 8 and Lemma 3, we can follow the strategy of the proof of Proposition 4, recall the normalization $\langle \Xi_N^{\text{opt}}, \Xi_N^{\text{opt}} \rangle = 1$ with $\Xi_N^{\text{opt}} \equiv \Xi_N^{\text{opt}}(P,T)$, and conclude that for $\varpi \in [0, \sqrt{2}]$ we have

$$\left\langle \Xi_N^{\text{opt}}, \left[\tfrac{d}{d\varpi^2}\mathfrak{H}^{(N)}(\varpi)\right] \Xi_N^{\text{opt}} \right\rangle > 0. \tag{90}$$

Now multiplying inequality (90) by $-\omega^2/4\pi^2 T^4$, then averaging over $P(d\omega)$, reverses the inequality in (90).

This proves Corollary 7. □

The proof of Proposition 1 is complete. □

Remark 9 Following up on Remark 8, we suspect that

$$\left\langle \Xi_N^{\text{opt}}, \left[\tfrac{d}{d\varpi^2}\mathfrak{H}^{(N)}(\varpi)\right] \Xi_N^{\text{opt}} \right\rangle \geq \left\langle \overline{\Theta}_N, \left[\tfrac{d}{d\varpi^2}\widehat{\mathfrak{H}}^{(N)}(\varpi)\right] \overline{\Theta}_N \right\rangle, \tag{91}$$

where $\widehat{\mathfrak{H}}^{(N)}(\varpi)$ is obtained from $\mathfrak{H}^{(N)}(\varpi)$ analogously to how $\widehat{\mathfrak{G}}^{(N)}(\gamma)$ is obtained from $\mathfrak{G}^{(N)}(\gamma)$ in the proof of Proposition 4, and where $\overline{\Theta}_N$ is the constant vector $(\frac{1}{N}, \ldots, \frac{1}{N})$; recall that $\langle \overline{\Theta}_N, \mathfrak{D}^{(N)} \overline{\Theta}_N \rangle = 1$. Note that equality in (91) holds iff $N = 1$. R.h.s.(91) is a weighted sum of the kernels $\frac{k^2}{(k^2+\varpi^2)^2}$ with $k \in \{1, \ldots, 2N-1\}$, yielding

$$\left\langle \overline{\Theta}_N, \left[\frac{d}{d\varpi^2}\widehat{\mathfrak{H}}^{(N)}(\varpi)\right] \overline{\Theta}_N \right\rangle = \tag{92}$$

$$\frac{1}{N^2}\sum_{k=1}^{2N-1} \frac{k^2}{(k^2+\varpi^2)^2}\min\{k, 2N-k\} > 0. \tag{93}$$

□

Proof of Theorem 2 The strict monotonicity of $T \mapsto \Lambda^{(N)}(P,T)$, given P, for $T \geq \overline{\Omega}(P)/2\sqrt{2}\pi$ and all $N \in \mathbb{N}$, implies via the convergence $\Lambda^{(N)}(P,T) \to \Lambda(P,T)$ as $N \to \infty$, also the monotonicity of $T \mapsto \Lambda(P,T)$ if $T \geq \overline{\Omega}(P)/2\sqrt{2}\pi$. Since $T \mapsto \Lambda(P,T)$ is analytic for each $P \in \mathcal{P}$, it cannot have a constant piece, for this would violate its lower bound $\Lambda^*(P,T)$; hence $T \mapsto \Lambda(P,T)$ is strictly monotonic increasing, too. Yet $\overline{\Omega}(P)/2\sqrt{2}\pi$ is merely the lower limit of T values for which our reasoning establishes the negativity of $\frac{\partial}{\partial T^2}\mathfrak{k}(P,T)$. Our argument does not reveal "how negative" $\frac{\partial}{\partial T^2}\mathfrak{k}(P,T_*(P))$ is. Yet it is < 0, and so by continuity $\frac{\partial}{\partial T^2}\mathfrak{k}(P,T)) < 0$ also in some left neighborhood of $\overline{\Omega}(P)/2\sqrt{2}\pi$.

The proof of Theorem 2 is complete. □

Proof of Corollary 1 Having proved Proposition 1 and also Theorem 4, the explicit bound (25) on the quantity $\lambda_*(P)$ defined in Theorem 2 follows right away. □

We close this section with some comments on practical matters. While the bound (25) is explicit, its evaluation still requires the P-averages of $\frac{\omega^2}{\omega^2+4\pi^2T^2j^2}$ for $j \in \{1, ..., 7\}$, which in general will require numerical quadratures for each T value of interest. A weaker but more user-friendly estimate on $\lambda_*(P)$ is

$$\lambda_*(P) < \frac{1}{\mathfrak{k}^{(1)}\big(P, T_*(P)\big)}. \tag{94}$$

Since P is compactly supported in $[0, \overline{\Omega}(P)]$, we furthermore have the estimate

$$\mathfrak{k}^{(1)}(P, T) \geq \frac{1}{\overline{\Omega}(P)^2+(2\pi T)^2} \int_0^\infty \omega^2 P(d\omega). \tag{95}$$

Now substituting $T_*(P)$ for T yields the upper estimate

$$\lambda_*(P) \leq \frac{3}{2}\frac{\overline{\Omega}(P)^2}{\langle\omega^2\rangle}, \tag{96}$$

which requires only knowledge of $\overline{\Omega}(P)$ and the computation of $\langle\omega^2\rangle$.

8 Upper and Lower Bounds on $T_c(\lambda, \Omega)$

The validity of Corollary 4 is obvious, so nothing needs to be added here about these explicit but relatively weak bounds on $T_c(\lambda, P)$.

Our better bounds on $T_c(\lambda, P)$ are not explicitly given as functions of λ and P. However, since their inverse functions $\lambda = \Lambda^{(N)}(P, T)$, $N \in \{1, 2, 3, 4\}$, respectively $\lambda = \Lambda^*(P, T)$, are explicitly given in section 3, for any P, one can conveniently visualize the bounds on $T_c(\lambda, P)$ that we proved to exist for $T > T_*(P)$ (and which reasonably are expected to exist for all $T > 0$). All that needs to be done is to graph the $\Lambda^{(N)}(P, T)$ and $\Lambda^*(P, T)$ in a (λ, T) diagram, given a desirable choice of P. For each P this still requires one or more T-dependent quadratures to be carried out, in most cases by numerical algorithm unless by good fortune those P integrals can be carried out in terms of known functions. One such fortunate case is the non-dispersive limit, with $P(d\omega) = \delta(\omega - \Omega)d\omega$, for fixed $\Omega > 0$; this case merits a discussion of its own and will be featured in paper III of our series.

In the remainder of this section we confine ourselves to adding a mathematical result for general P that we announced in section 3.

8.1 The Lower Bound $T_c^{(2)}(\lambda, P)$

As announced in section 3, we here supply the proof of monotonicity for the map $T \mapsto \Lambda^{(2)}(P, T)$, given P, which implies its invertibility and the monotonicity of its inverse $\lambda \mapsto T_c^{(2)}(\lambda, P)$, a better lower bound on $T_c(\lambda, P)$ than $T^\flat(\lambda, P)$. While it does not seem to have a closed form expression in known functions, the lower bound $T_c^{(2)}(\lambda, P)$ is defined for all $\lambda \geq \lambda_2$, as we will see now.

For the 2×2 matrix given by the P-average of the upper leftmost 2×2 block of r.h.s.(23), the largest eigenvalue $\mathfrak{k}^{(2)}(P, T)$ is given by (11), with

$$\text{tr}\,\mathfrak{K}^{(2)} = \frac{1}{3}\big\langle [\![1]\!] + [\![3]\!] \big\rangle \tag{97}$$

and

$$\det \mathfrak{K}^{(2)} = -\frac{1}{3}\left(\left\langle[\![1]\!]+[\![2]\!]\right\rangle^2+\left\langle[\![1]\!]\right\rangle\left\langle 2[\![1]\!]-[\![3]\!]\right\rangle\right), \tag{98}$$

where $[\![n]\!](\varpi) := \frac{\varpi^2}{n^2+\varpi^2}$ for $n \in \mathbb{N}$, with $\varpi = \omega/2\pi T$, and where the angular brackets indicate $P(d\omega)$-averages.

Note that (97) reveals that $\operatorname{tr}\mathfrak{K}^{(2)} > 0$; note furthermore that $n \mapsto \frac{\varpi^2}{n^2+\varpi^2} > 0$ is strictly decreasing with increasing $n \in \mathbb{N}$, and so (98) reveals that $\det \mathfrak{K}^{(2)} < 0$. This shows that (11) is well-defined; incidentally, it also vindicates the choice of the $+$ sign in front of the $\surd$ term in (11), for choosing a $-$ sign instead would not produce a positive eigenvalue.

Taking the reciprocal of (11) yields the upper bound $\Lambda^{(2)}(P,T)$ on $\Lambda(P,T)$, explicitly

$$\Lambda^{(2)} = \tag{99}$$

$$\frac{6}{\left\langle[\![1]\!]+[\![3]\!]\right\rangle+\sqrt{\left\langle[\![1]\!]+[\![3]\!]\right\rangle^2+12\left(\left\langle[\![1]\!]+[\![2]\!]\right\rangle^2+\left\langle[\![1]\!]\right\rangle\left\langle 2[\![1]\!]-[\![3]\!]\right\rangle\right)}}.$$

The map $(P,T) \mapsto \Lambda^{(2)}(P,T)$ is readily seen to be jointly continuous. We now show that it is also strictly increasing when $T > 0$ increases from 0 to ∞, given any $P \in \mathcal{P}$.

Indeed, with $\varpi = \omega/2\pi T$ it is manifest that for any $n \in \mathbb{N}$, the map $T \mapsto [\![n]\!](\varpi) \equiv \frac{\omega^2}{4n^2\pi^2T^2+\omega^2}$ is continuous and monotonically decreasing from 1 to 0 as T runs from 0 to ∞, for each $\omega > 0$. Thus $\langle[\![1]\!]+[\![2]\!]\rangle$ and $\langle[\![1]\!]+[\![3]\!]\rangle$ and their squares are bounded, continuous, and monotonically decreasing as T runs from 0 to ∞. The only term that could cause problems is the $-\langle[\![3]\!]\rangle$ contribution at r.h.s.(99), which increases with T.

After investigating first the T dependence of $\langle[\![1]\!]\rangle\langle 2[\![1]\!]-[\![3]\!]\rangle$ and finding that it is not monotonic, taking into account also the additive term $\left\langle[\![1]\!]+[\![2]\!]\right\rangle^2$ yields success. Indeed, rewriting $\left\langle[\![1]\!]+[\![2]\!]\right\rangle^2+\left\langle[\![1]\!]\right\rangle\left\langle 2[\![1]\!]-[\![3]\!]\right\rangle = \left\langle[\![2]\!]\right\rangle^2+\left\langle[\![1]\!]\right\rangle\Big(3\left\langle[\![1]\!]\right\rangle+2\left\langle[\![2]\!]\right\rangle-\left\langle[\![3]\!]\right\rangle\Big)$ and noting that $\left\langle[\![2]\!]\right\rangle^2$ and $\left\langle[\![1]\!]\right\rangle$ are monotonically strictly decreasing with T, given P, and that the term between big round parentheses that multiplies $\left\langle[\![1]\!]\right\rangle$ is positive (by the decrease of $n^2 \mapsto \frac{\varpi^2}{n^2+\varpi^2}$), it remains to check the T dependence of the term between big round parentheses that multiplies $\left\langle[\![1]\!]\right\rangle$. With the help of Maple one finds (with $C_n \in \mathbb{N}$, $n \in \{1, ..., 5\}$)[4] that

$$\tfrac{\partial}{\partial T^2}\Big(3\left\langle[\![1]\!]\right\rangle+2\left\langle[\![2]\!]\right\rangle-\left\langle[\![3]\!]\right\rangle\Big)(P,T) = \tag{100}$$

$$-\int_0^\infty \frac{\sum_{n=1}^5 C_n\omega^{2n}(4\pi^2T^2)^{5-n}}{\prod_{j=1}^3\left(4j^2\pi^2T^2+\omega^2\right)^2}P(d\omega) < 0, \tag{101}$$

which was to be shown.

We summarize our result in

Proposition 9 *Given any $P \in \mathcal{P}$, the map $T \mapsto \lambda = \Lambda^{(2)}(P,T)$ is invertible, as announced in section 3. The inverse function is the lower critical-temperature bound $T_c^{(2)}(\lambda,P)$ that continuously and monotonically increases in λ on its domain of definition $\lambda \in [\lambda_2,\infty)$, with λ_2 given by (24) for $N = 2$; viz.*

$$\lambda_2 = \tfrac{3}{5} = 0.6\,. \tag{102}$$

4 For the sake of completeness, we state the numerical values of the C_n explicitly here: $C_1 = 4392$, $C_2 = 3888$, $C_3 = 1370$, $C_4 = 148$, and $C_5 = 2$.

We remark that even in the simplest choice, $P(d\omega) = \delta(\omega - \Omega)d\omega$, the inversion of $T \mapsto \lambda = \Lambda^{(2)}(P, T)$ is not expressible in closed algebraic form.

9 Summary and Outlook

9.1 Summary

In this paper we rigorously studied the phase transition between normal and superconductivity in the standard Eliashberg theory in which the effective electron-electron interactions are mediated by generally dispersive phonons, with Eliashberg spectral density function $\alpha^2 F(\omega)$ that is $\propto \omega^2$ for small ω and vanishes for sufficiently large $\omega > \overline{\Omega}$; it also defines the standard electron-phonon coupling strength $\lambda := 2\int_{\mathbb{R}_+} \alpha^2 F(\omega)\frac{d\omega}{\omega}$.

After a suitable rescaling with λ the standard Eliashberg model is asymptotic to the γ model at $\gamma = 2$ when $\lambda \to \infty$. The γ model was studied in our previous paper [14], and the results obtained there were convenient ingredients to prove several results of the present paper, too. Several other results that we proved in the present paper are based on entirely new arguments, though.

Defining a formal probability measure P through $2\alpha^2 F(\omega)\frac{d\omega}{\omega} =: \lambda P(d\omega)$, we in this paper proved that the normal and the superconducting regions in the positive (λ, P, T) cone are both simply connected, and separated by a critical surface $\mathscr{S}_c$ that is a graph over the positive (P, T) cone, given by a function $\lambda = \Lambda(P, T)$. This is the content of our Theorem 1. We furthermore proved that $\Lambda(P, T) = 1/\mathfrak{k}(P, T)$, where $\mathfrak{k}(P, T) > 0$ is the largest eigenvalue of an explicitly constructed compact operator $\mathfrak{K}(P, T)$ on $\ell^2(\mathbb{N}_0)$, where $\mathbb{N}_0$ is the set of non-negative integers that enumerates the positive Matsubara frequencies. This is stated in Theorem 2.

Since a compact operator on a separable Hilbert space can be arbitrarily closely approximated by truncating it to finite-dimensional subspaces, in this case spanned by the first N positive Matsubara frequencies, we obtained from our variational principle a strictly monotonically decreasing sequence of rigorous upper bounds on $\Lambda(P, T)$, the first four of which we have computed explicitly in closed form, though still involving quadratures of $\frac{\omega^2}{\omega^2+(2n\pi T)^2}$ w.r.t. $P(d\omega)$ for $n \in \{1, ..., 2N - 1\}$. This is the content of our Theorem 4.

In Theorem 5 we explicitly stated the bounds $\Lambda^{(N)}(P, T)$ in the limit $T \to 0$. Interestingly these limits are independent of the choice of P.

Through spectral estimates of $\mathfrak{k}(P, T)$ from above we also rigorously obtained an explicit lower bound on $\Lambda(P, T)$, involving only the quadratures of ω^2 and of $\frac{\omega^2}{\omega^2+(2\pi T)^2}$ w.r.t. $P(d\omega)$; see our Theorem 6. At the expense of less accuracy we also obtained weaker upper and lower bounds on $\Lambda(P, T)$ involving only the quadrature of ω^2 w.r.t. $P(d\omega)$, and the maximum frequency $\overline{\Omega}$, as a corollary to Theorem 6.

Physical intuition, based on empirical evidence, suggests that the phase transition can be characterized in terms of a *critical temperature* $T_c(\lambda, P)$, which is equivalent to saying the critical surface $\mathscr{S}_c$ is a graph over the positive (λ, P) cone. This in turn is equivalent to saying that, given P, the map $T \mapsto \Lambda(P, T)$ is strictly monotone, hence invertible to yield $T = \Lambda^{-1}(\lambda, P) \equiv T_c(\lambda, P)$. In this paper we were able to prove that all our upper approximations to the map $T \mapsto \Lambda(P, T)$, and that map itself, are strictly monotone decreasing for $T \in [T_*(P), \infty]$, with $T_*(P) \leq \frac{\overline{\Omega}(P)}{2\sqrt{2\pi}}$. This is the content of Theorem 2.

The restriction of the monotonicity result to $T \in [T_*(P), \infty]$ is due to the limitations of our technique of proof and presumably not intrinsic to the model. In this vein, we do expect that the map $T \mapsto \Lambda(P, T)$ is strictly monotone increasing for all $T \in [0, \infty)$, given any $P \in \mathcal{P}$. To prove this is a worthy goal for future works.

Since the explicit fourth upper bound on $\Lambda(P, T)$ yields $\lambda_*(P) < \frac{1}{\mathfrak{k}^{(4)}\left(P, T_*(P)\right)}$, what we just wrote proves that a unique critical temperature $T_c(\lambda, P)$ in the standard Eliashberg model is mathematically well-defined in terms of the untruncated linearized Eliashberg gap equations whenever $\lambda > \lambda_*(P)$. A weaker but rather explicit estimate is obtained with the weaker first upper bound on $\Lambda(P, T)$, further estimated itself in terms of the expected value of ω^2 w.r.t. $P(d\omega)$ and in terms of $\overline{\Omega}$. This has yielded the estimate $\lambda_*(P) \leq \frac{3}{2}\frac{\overline{\Omega}(P)^2}{\langle\omega^2\rangle}$.

Also λ_* is not a sharp boundary but a consequence of our method of proof. While mathematically desirable to prove the existence of a unique $T_c(\lambda, P)$ in the Eliashberg model for all $\lambda > 0$, from a theoretical physics perspective the range $\lambda > \frac{3}{2}\frac{\overline{\Omega}(P)^2}{\langle\omega^2\rangle}$ would seem to cover all cases of interest so far.

Furthermore we proved that asymptotically for large λ one has $T_c(\lambda, P) \sim C\sqrt{\langle\omega^2\rangle\lambda}$, with $C = \frac{1}{2\pi}\sqrt{\mathfrak{g}(2)} = 0.1827262477...$, where $\mathfrak{g}(2)$ is the spectral radius of a compact operator $\mathfrak{G}(2)$ associated with the γ model for $\gamma = 2$. This is expressed in Theorem 7 and its corollary.

With the existence of a unique critical temperature secured for most situations of interest, we have the following interesting application of our results.

It is known that if one measures $\alpha^2 F(\omega)$ reasonably accurately in a laboratory experiment, then λ is obtained by a single numerical quadrature via (6). Yet suppose that only $P(d\omega)$ can reasonably accurately be measured for certain materials featuring a superconductivity transition temperature T_c, and suppose that also T_c has been measured in the laboratory. Then numerical quadrature of ω^2 and of $\frac{\omega^2}{\omega^2+4n^2\pi^2T_c^2}$ for $n \in \{1, ..., 7\}$ over $P(d\omega)$ yields explicit upper and lower bounds on the electron-phonon coupling constant λ, given by $1/\mathfrak{k}^{(4)}(P, T_c)$ and $1/\mathfrak{k}^*(P, T_c)$, respectively.

Our T_c bounds accomplish the following regarding the existing literature, where traditionally T_c has been investigated through a linear stability analysis of the normal state, involving some heuristic truncation to a finite-dimensional matrix problem of the linearized Eliashberg gap equation in their original formulation. First of all, our T_c bounds are obtained rigorously, which in itself is a novelty in this area of research. Second, the closed form lower bounds for $N \in \{3, 4\}$, and our explicit upper bound are new. Third, the closed form lower bounds for $N \in \{1, 2\}$ vindicate the T_c bounds claimed in [1], as discussed next.

Our lower bound (33) on T_c obtained with $N = 1$, and given as inverse function of the lower bound (10) on $\mathfrak{k}(P, T)$, agrees with the lower bound stated in [1] in their formula (19) when their $\mu^*(N) = 0$. It is also mentioned in [2] and [4]. Of course, this is unsurprising, for a truncation of the linearized Eliashberg gap equation to the first positive Matsubara frequency inevitably yields this result, no matter which formulation one uses for the full theory. Allen and Dynes start from eq.(7) of [3].

Our second-lowest bound $T_c^{(2)}(\lambda, P)$ on $T_c(\lambda, P)$, given as inverse function (w.r.t. T) of the lower bound (11) on $\mathfrak{k}(P, T)$, and which is > 0 on the domain (λ_2, ∞), agrees with what one gets by setting the expression for ρ_1 (and $\mu^*(N)$) in formula (23) of [1] equal to zero, demanding the vanishing of this largest eigenvalue of an auxiliary 2×2 matrix. However, unlike Allen and Dynes we actually proved that the resulting expression is a lower bound to T_c, by proving the invertibility of $T \mapsto \mathfrak{k}^{(2)}(P, T)$, see Proposition 9 in section 8.

We also found that it is not correct when [1] claim that from their (23) "one can generate an explicit lower bound to T_c analogous to ... (21)" [that] "is complicated and uninstructive" [and therefore not displayed] (the quotes are from [1], fleshed out here in a manner that captures the apparent spirit of their statement). The explicit lower bound (21) in [1] is for the non-dispersive limit of the Eliashberg model, when their inequality (19) can indeed be inverted to yield an explicit lower bound for $T_c(\lambda, \Omega)$ (shorthand for $T_c(\lambda, P)$ when $P(d\omega) = \delta(\omega - \Omega)d\omega$). However, as we noted in section 8, although invertible, even for the simpler non-dispersive model the inverse of the map $T \mapsto \Lambda_{\mathrm{E}}(\Omega, T)$ cannot be computed explicitly in closed algebraic form "analogous[ly] to (21)," contrary to what is claimed in [1].

Our third-lowest and fourth-lowest bounds on $T_c(\lambda, P)$, based on the largest eigenvalue of a pertinent 3×3, respectively 4×4 matrix, have not been stated before — as far as we are able to tell. They are given in terms of their inverse functions $T \mapsto \Lambda^{(N)}(P, T)$ in Theorem 4, whose inverses we showed to exist for when $T > T_*(P)$; see Proposition 1. Once a choice of P has been made, one can carry out the quadratures — numerically, if necessary — and plot these bounds on λ as functions of T_c and visually obtain lower bounds on $T_c(\lambda, P)$.

Moreover, while Allen and Dynes [1] claimed that their increasing sequence of auxiliary eigenvalues yields an increasing sequence of lower bounds on T_c, they did not supply any compelling argument that T_c actually exists, and why their sequence would converge to it monotonically. Their narrative seems to have been largely informed by their numerical studies of the truncated Eliashberg gap equations for the non-dispersive model, based on which they made sweeping, much more general statements. By contrast, we proved that our sequence of lower bounds $T_c^{(N)}(\lambda, P)$ converges upward to $T_c(\lambda, P)$, for each $P \in \mathcal{P}$.

Finally, we emphasize that our asymptotically (for $\lambda \sim \infty$) exact upper bound (31) on $T_c(\lambda, P)$, that we conjecture to be an upper bound on $T_c(\lambda, P)$ for all $\lambda > 0$, has previously played an important role in the derivation of a physically fundamental upper limit on T_c arising from considerations beyond the Eliashberg theory [10, 22–24]. By contrast, we in our series of papers obtain bounds on T_c from within this theory, without considering whether or not it by itself provides a valid description of any actual physical system. For such considerations, see [5, 9, 10, 22–24, 26, 28, 29].

9.2 Outlook

Any more detailed evaluations necessitate the specification of the measure $P(d\omega)$. In our next paper, part III of our series, we will specialize the analysis of the present paper to the non-dispersive limit, which operates with $P(d\omega) = \delta(\omega - \Omega)d\omega$, with Einstein phonon frequency $\Omega > 0$. In this case all of the quadratures involving P can be carried out explicitly and much more detailed information can be obtained by evaluation of our general formulas obtained in the present paper. Having the detailed information for the non-dispersive model one also will be able to compare with empirical results on T_c when the electron-phonon interactions in superconductors are mediated by optical phonons.

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Data Availability No data have been produced for this paper.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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