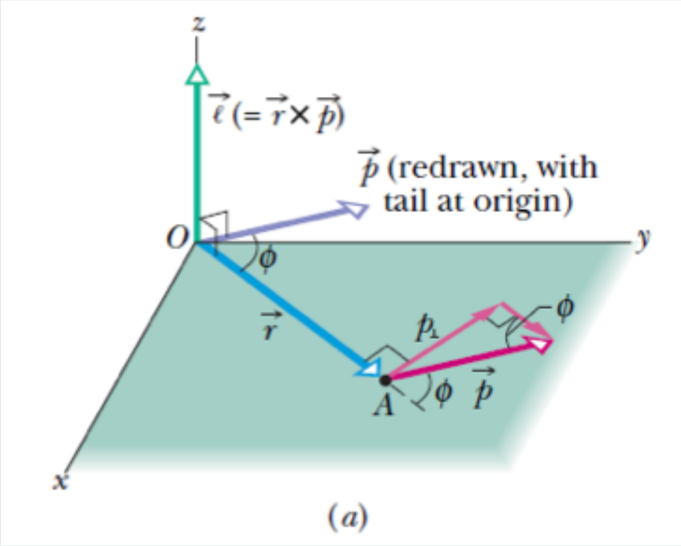
Rutgers University
Department of Physics & Astronomy

01:750:271 Honors Physics I
Fall 2015

Lecture 17

- **Angular Momentum**



Particle moving in the horizontal plane

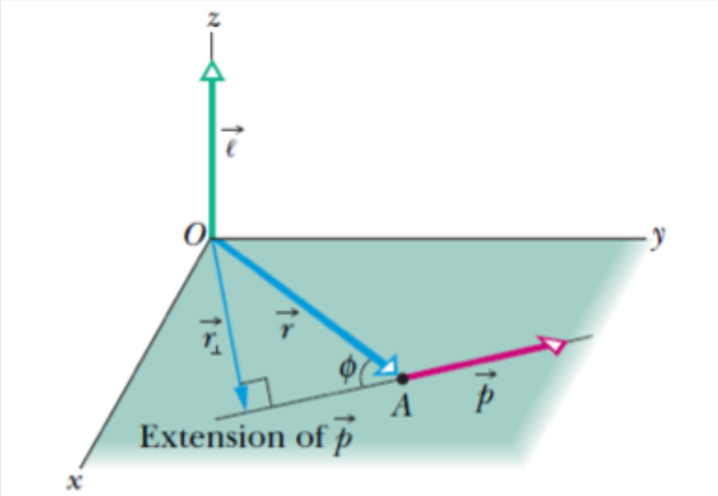
$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$$

$$\vec{r} \times \vec{F}_{\text{net}} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\frac{d\vec{r}}{dt} \times \vec{p} = 0 \;\; \Rightarrow \vec{\tau}_{\text{net}} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

Definition: $\vec{\ell} = \vec{r} \times \vec{p}$ **angular momentum**



(b)

$$\vec{\ell} = \vec{r} \times \vec{p}$$

$$\ell = rmv\sin\phi = rp_\perp = r_\perp p$$

Newton's 2nd law in angular form
(single particle)

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt}$$

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

- **System of particles**:

$$\vec{L} = \sum_i \vec{\ell}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{\ell}_i}{dt} = \sum_i \vec{\tau}_{\text{net,i}} = \vec{\tau}_{\text{net}}$$

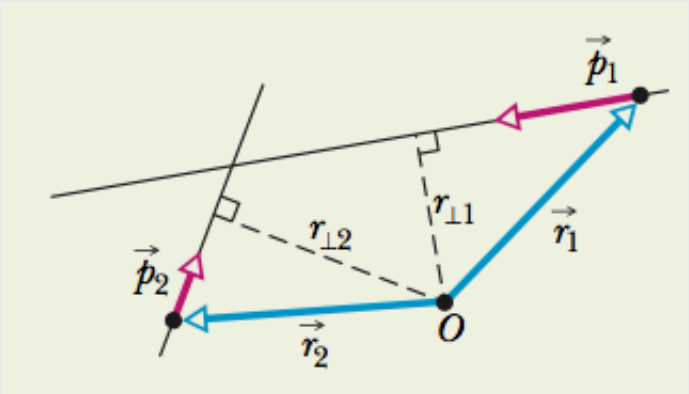
Newton's 2nd law in angular form

(system of particles)

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

The net external torque $\vec{\tau}_{\text{net}}$ acting on a system of particles is equal to the time rate of the change of the systems total angular momentum $\vec{L}$.

- **Example**



- Particle 1:

$$p_1 = 5.0\,\mathrm{kg}\cdot\mathrm{m/s} \qquad r_{\perp 1} = 2.0\mathrm{m}$$

- Particle 2:

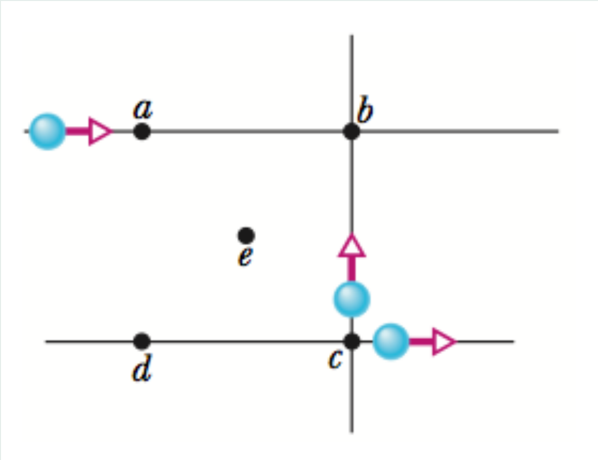
$$p_2 = 2.0\,\mathrm{kg}\cdot\mathrm{m/s} \qquad r_{\perp 2} = 4.0\mathrm{m}$$

- What is $\vec{L}$?

$$\vec{\ell}_1 = \vec{r}_1 \times \vec{p}_1 = r_{\perp 1} p_1\,\vec{k} \qquad \text{out of the page}$$
$$\vec{\ell}_2 = \vec{r}_2 \times \vec{p}_2 = -r_{\perp 2} p_2\,\vec{k} \qquad \text{into the page}$$

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 = (r_{\perp 1}p_1 - r_{\perp 2}p_2)\,\vec{k} = (2\,\mathrm{kg}\cdot\mathrm{m}^2/s)\,\vec{k} \text{ out of the page}$$

i-Clicker



Three particles of the same mass m and the same constant speed v moving as indicated by the velocity vectors. Points a, b, c, and d form a square of size h with point e at the center. What is $\vec{L}$ with respect to point e?

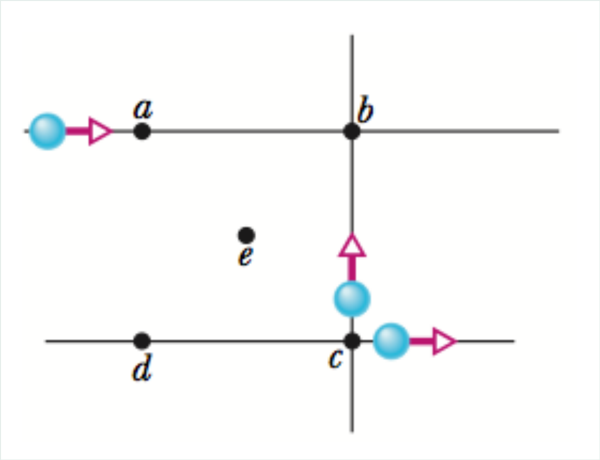
$\vec{k}$ points out of the page

$A)\ \vec{L} = (mvh)\,\vec{k}$

$B)\ \vec{L} = (mvh/2)\,\vec{k}$

$C)\ \vec{L} = -mvh\,\vec{k}$

$D)\ \vec{L} = -(mvh/2)\,\vec{k}$



Three particles of the same mass m and the same constant speed v moving as indicated by the velocity vectors. Points a, b, c, and d form a square of size h with point e at the center. What is $\vec{L}$ with respect to point e?

$\vec{k}$ points out of the page

$$\vec{\ell} = -(mvh/2)\,\vec{k} + (mvh/2)\,\vec{k} + (mvh/2)\,\vec{k}$$

A) $\vec{L} = (mvh)\,\vec{k}$

B) $\vec{L} = (mvh/2)\,\vec{k}$

C) $\vec{L} = -mvh\,\vec{k}$

D) $\vec{L} = -(mvh/2)\,\vec{k}$

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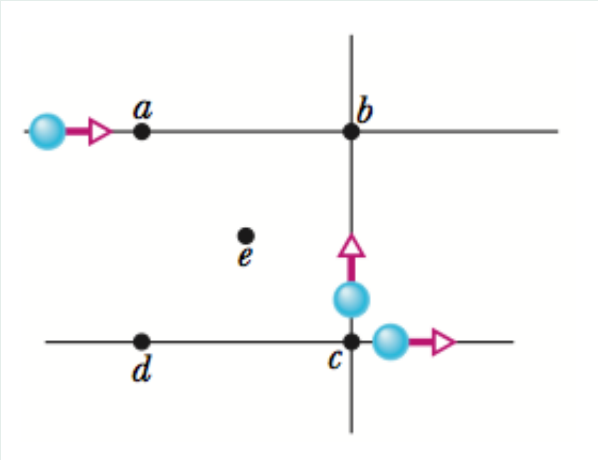

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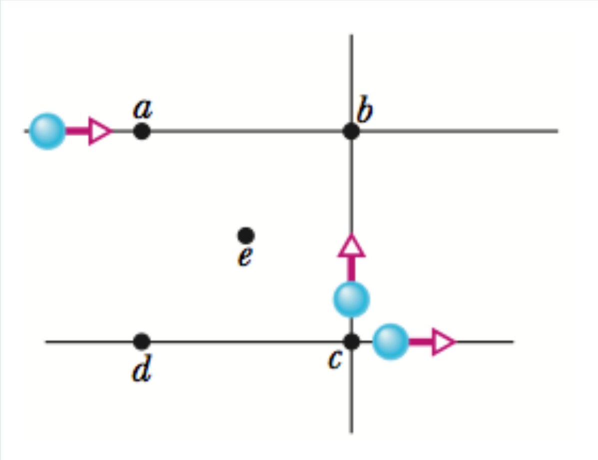
Three particles of the same mass m and the same constant speed v moving as indicated by the velocity vectors. Points a, b, c, and d form a square of size h with point e at the center. What is $\vec{L}$ with respect to point d?

$A)\ \vec{L} = (mvh)\,\vec{k}$

$B)\ \vec{L} = (mvh/2)\,\vec{k}$

$C)\ \vec{L} = 0$

$D)\ \vec{L} = -(mvh/2)\,\vec{k}$



Three particles of the same mass m and the same constant speed v moving as indicated by the velocity vectors. Points a, b, c, and d form a square of size h with point e at the center. What is $\vec{L}$ with respect to point d?

$\vec{k}$ points out of the page

A) $\vec{L} = (mvh)\,\vec{k}$

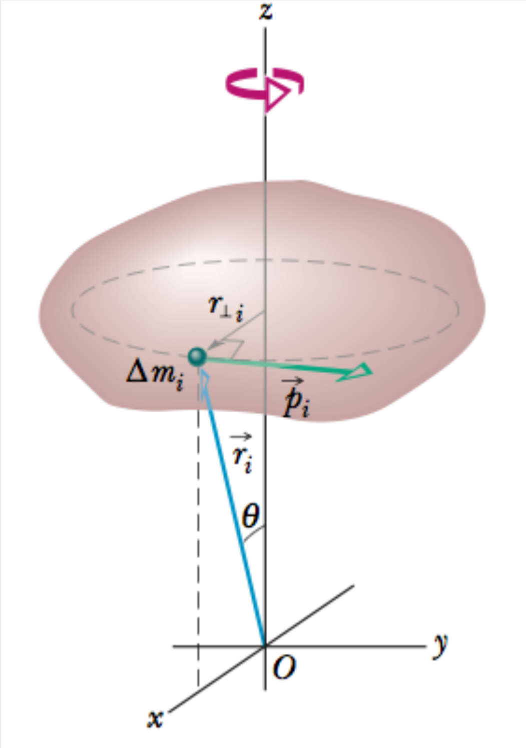
B) $\vec{L} = (mvh/2)\,\vec{k}$

C) $\vec{L} = 0$

D) $\vec{L} = -(mvh/2)\,\vec{k}$

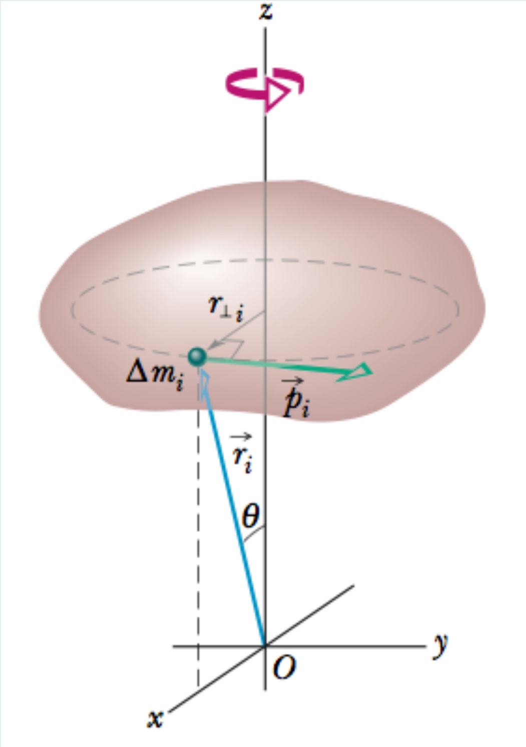
$$\vec{L} = -\,mvh\,\vec{k} + mvh\,\vec{k} + 0$$

• Angular momentum of a rigid body rotating about a fixed axis



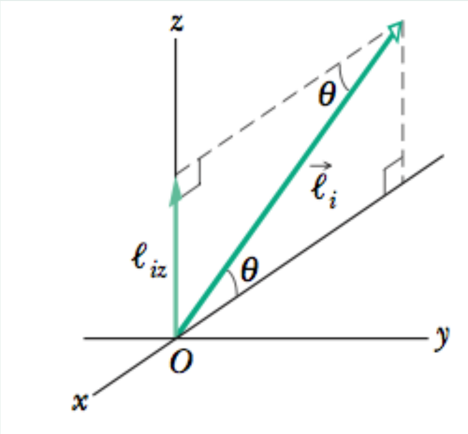
- A rigid body rotates about the z-axis with angular speed ω.
- A mass mass element Δm_i within the body moves about the z-axis on a circle with radius $r_{\perp i}$.
- The mass element has linear momentum $\vec{p_i}$ and it is located relative to the origin O by the position vector $\vec{r_i}$.

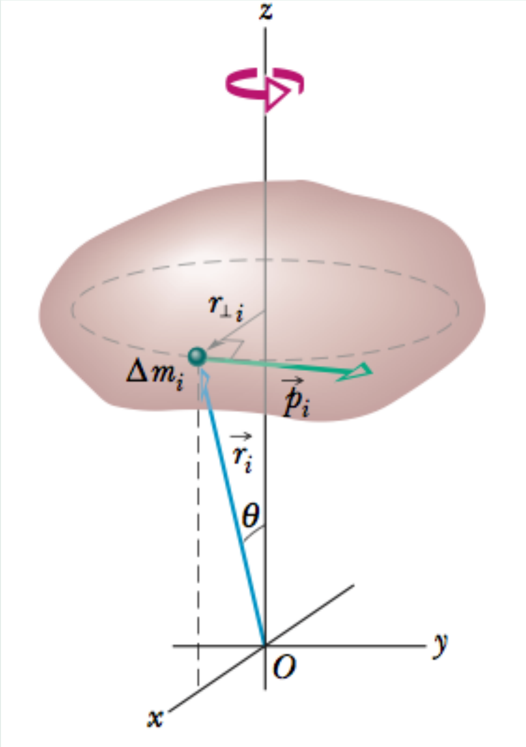
- The angular momentum $\vec{\ell}_i$ of the mass element with respect to O



$$\vec{\ell}_i = \vec{r}_i \times \vec{p}_i = \vec{r}_i \times (\Delta m_i)\vec{v}_i$$

$$\ell_{i,z} = \ell_i \sin\theta = r_{\perp i}(\Delta m_i)v_i$$



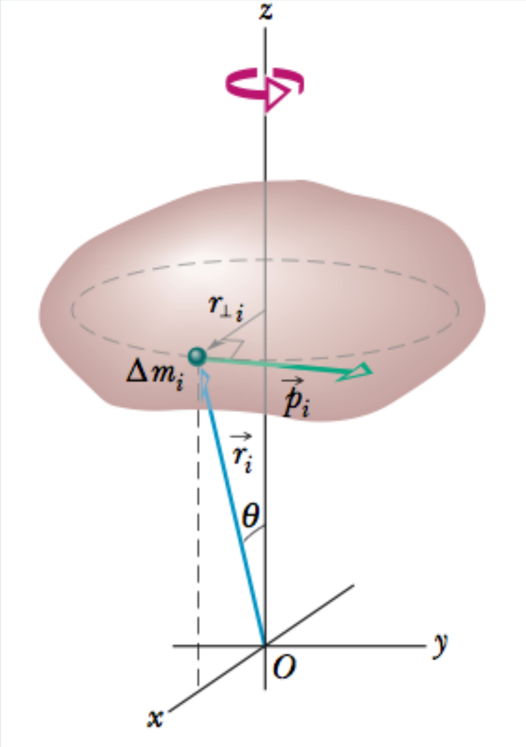


- The z-component of the angular momentum of a rigid body:

$$\begin{aligned} L_z &= \sum_i \Delta m_i v_i r_{\perp i} \\ &= \sum_i \Delta m_i r_{\perp i}^2 \omega \\ &= I\omega \end{aligned}$$

Rigid body, fixed axis

$$L_z = I\omega$$



Note: If mass distribution is **symmetric** about the axis of rotation:

$$\vec{L}_O = L_z \vec{k}$$

Conservation of Angular Momentum

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}$$

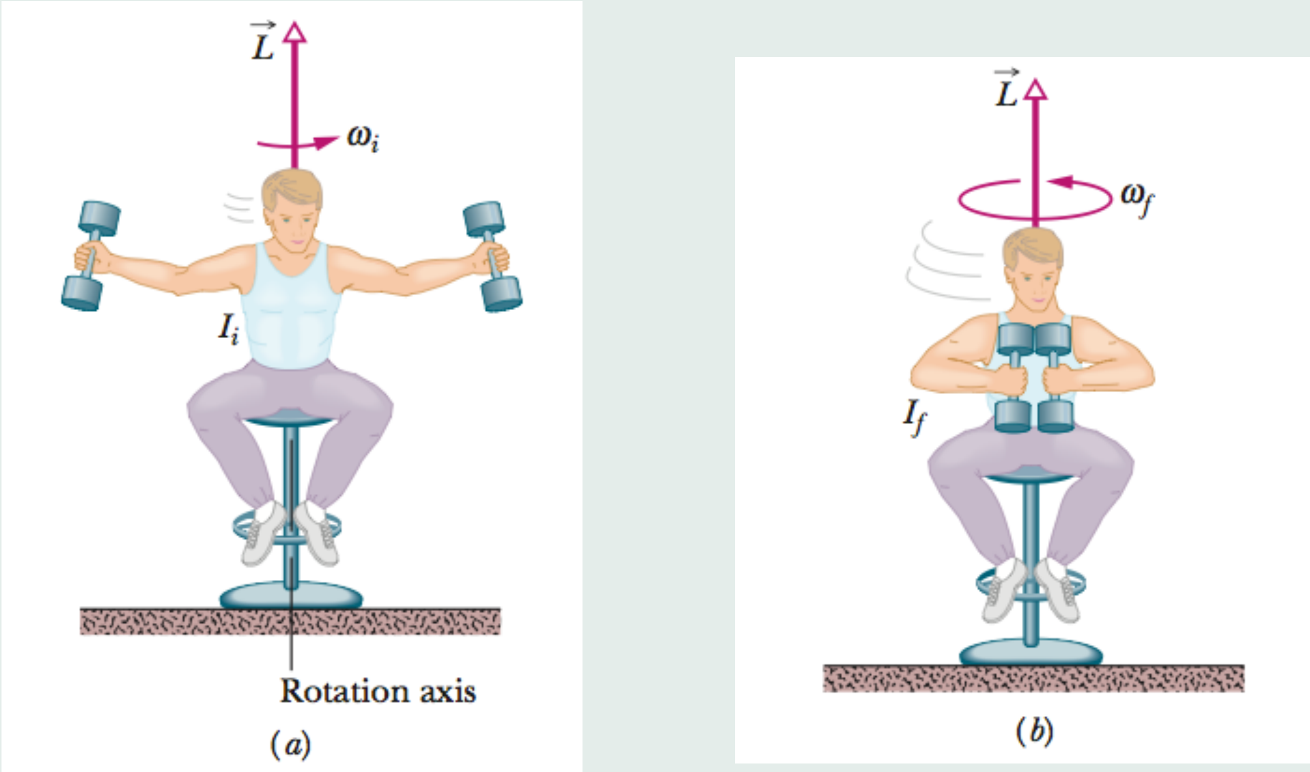
$$\vec{\tau}_{\text{net}} = 0 \;\Rightarrow\; \frac{d\vec{L}}{dt} = 0$$

If the net external torque acting on a system is zero, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

Table 11-1

More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constan



$$L = I_i \omega_i = I_f \omega_f$$

$$I_i > I_f \;\; \Rightarrow \;\; \omega_i < \omega_f$$

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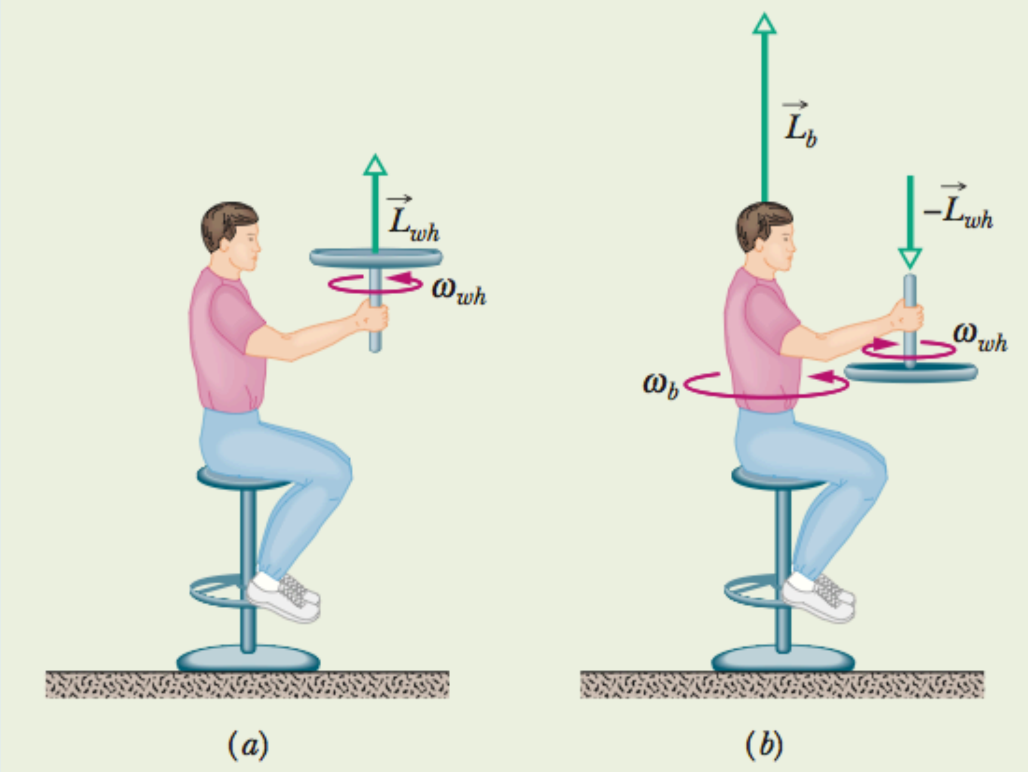


Angular momentum conservation:

$$L_{b,z} = 2L_{\text{wh}}$$

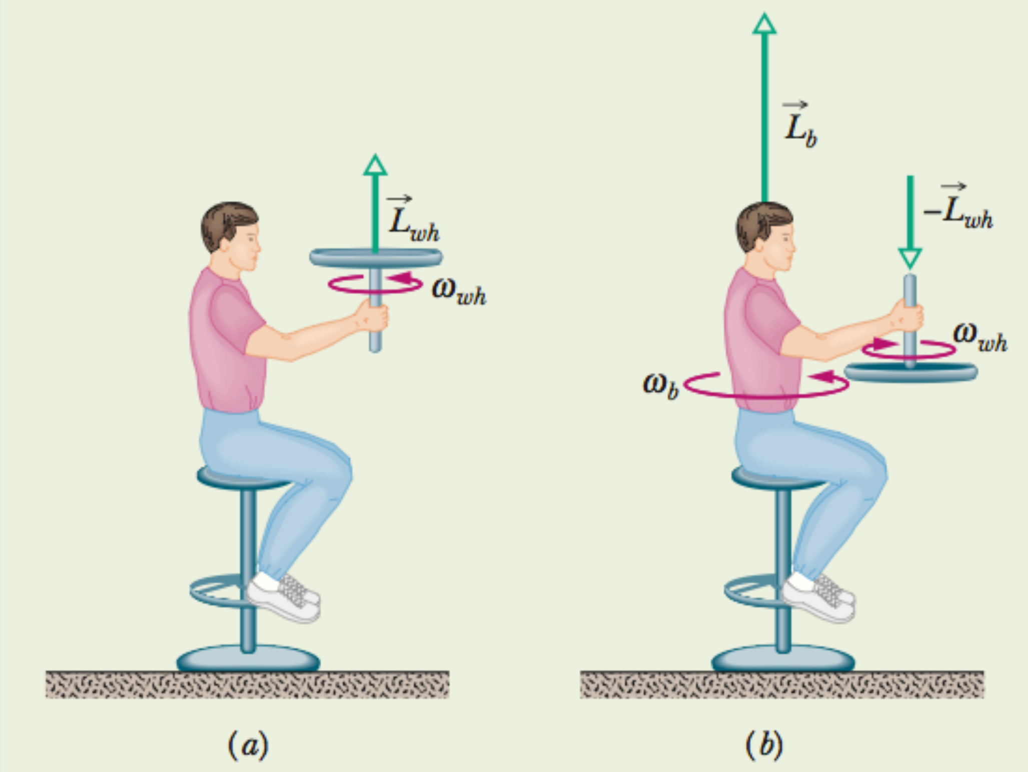
- Initially: student stationary, wheel rotating:

$$L_{a,z} = 0$$

$$L_{\text{total},z} = L_{\text{wh}}$$

- Finally: both student and wheel rotating:

$$L_{\text{total},z} = L_{b,z} - L_{\text{wh}}$$

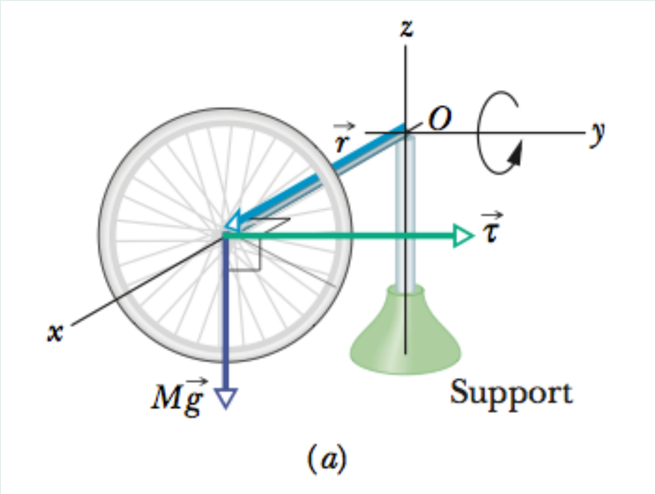


$$L_{b,z} = 2\vec{L}_{\text{wh}}$$

$$I_b\omega_b = 2I_{\text{wh}}\omega_{\text{wh}}$$

Note: I_b is the rotational inertia of the **combined system** about the vertical axis (not only the student)

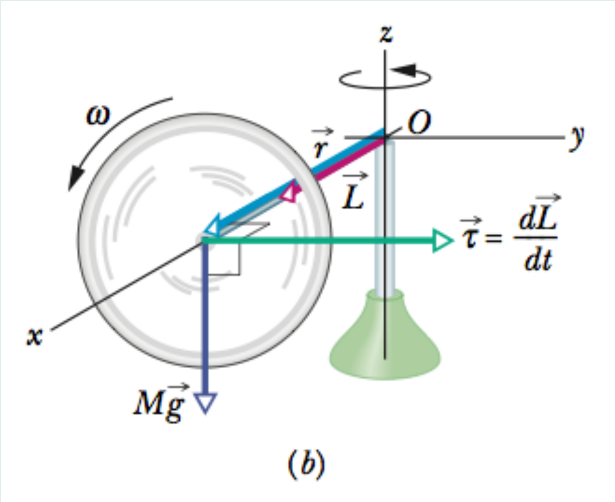
- **Precession of a Gyroscope**



- **Gyroscope:** a wheel fixed to a shaft free to spin about the axis of the shaft
- One end of the shaft is placed on a support.
- If **non-spinning**:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{F_g}$$

$$\tau_{F_g} = Mrg$$



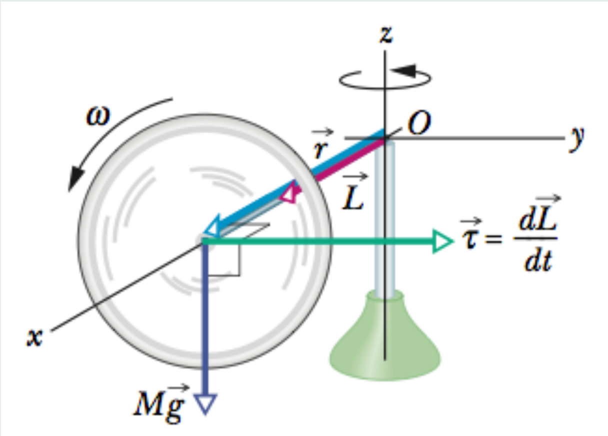
- **Fast spinning:**

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{F_g}$$

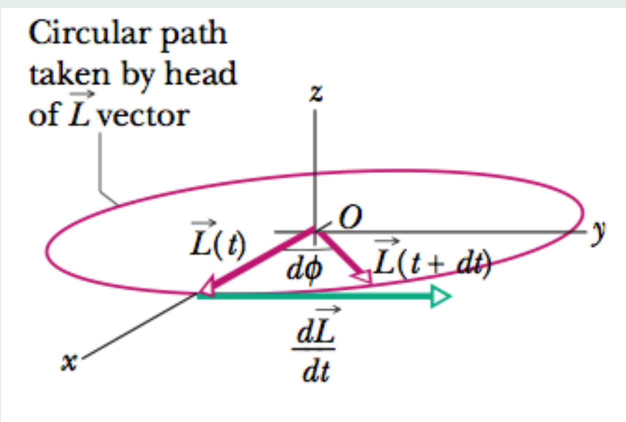
- The **magnitude** of $\vec{L}$ is **constant** (good approximation.)

$$\vec{\tau} \neq 0 \;\; \Rightarrow d\vec{L}/dt \neq 0$$

What is going on ?



(b)



(c)

- $d\vec{L}/dt \neq 0$ and **constant magnitude**

$$\Downarrow$$

The **direction** of $\vec{L}$ must vary.

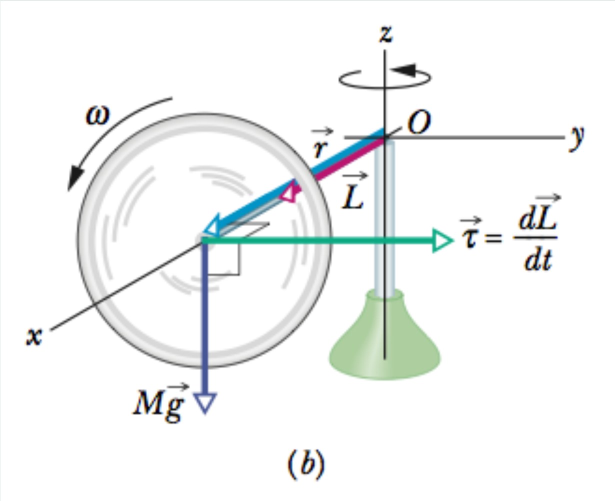
$$\Downarrow$$

Precession

$$|d\vec{L}| = \tau dt = Mgrdt$$

$$|d\vec{L}| = Ld\phi$$

$$\frac{d\phi}{dt} = \frac{Mgr}{L} = \frac{Mgr}{I\omega}$$



Precession angular velocity

$$\Omega = \frac{Mgr}{I\omega}$$

12. Equilibrium and Elasticity

- How do objects behave under applied external forces? Under what conditions can they remain static or stationary?

- Under what conditions do objects deform and what are the effects of their deformations?

- **Equilibrium**

An object is in **equilibrium** if:

- The linear momentum $\vec{P}$ of its center of mass is constant.

- Its angular momentum about its center of mass, or about any other point, is also constant.

$$\vec{P}, \quad \vec{L} \quad \textbf{constant}$$

An object is in

static equilibrium if

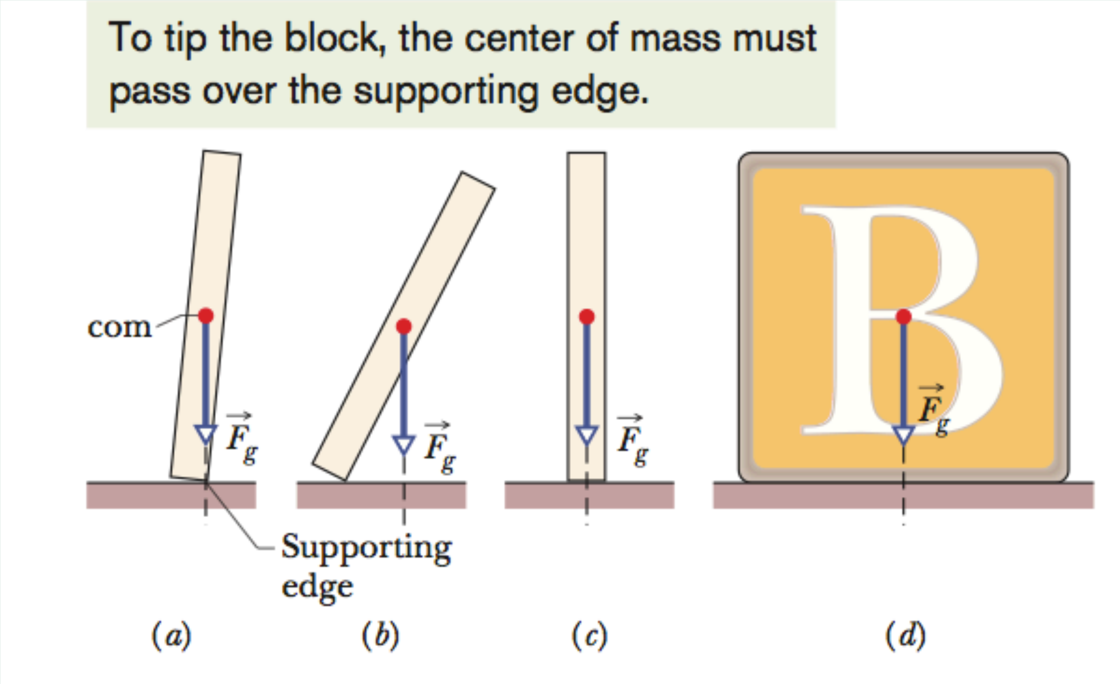
$$\vec{P} = 0, \quad \vec{L} = 0$$

No translation, no rotation.

Static equilibrium is:

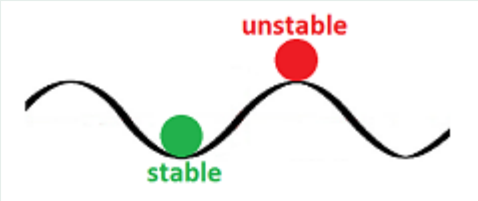
- **Stable** if the body returns to the state of static equilibrium after having been displaced from that state by a **small** force.

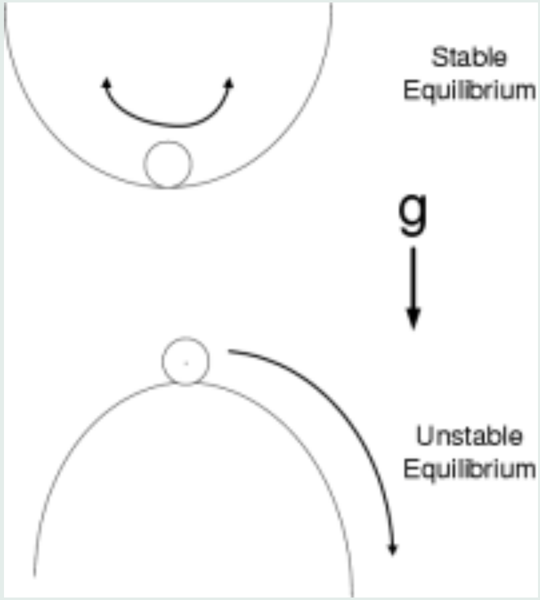
- **Unstable** if any small force can displace the body and end the equilibrium.



(a) **unstable** static equilibrium

$(c), (d)$ **stable** static equilibrium


unstable
stable


Stable
Equilibrium
g
Unstable
Equilibrium

- **Conditions for equilibrium**

$$\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}} \qquad \vec{P}\ \textbf{constant}\ \Rightarrow \vec{F}_{\text{net}} = 0$$

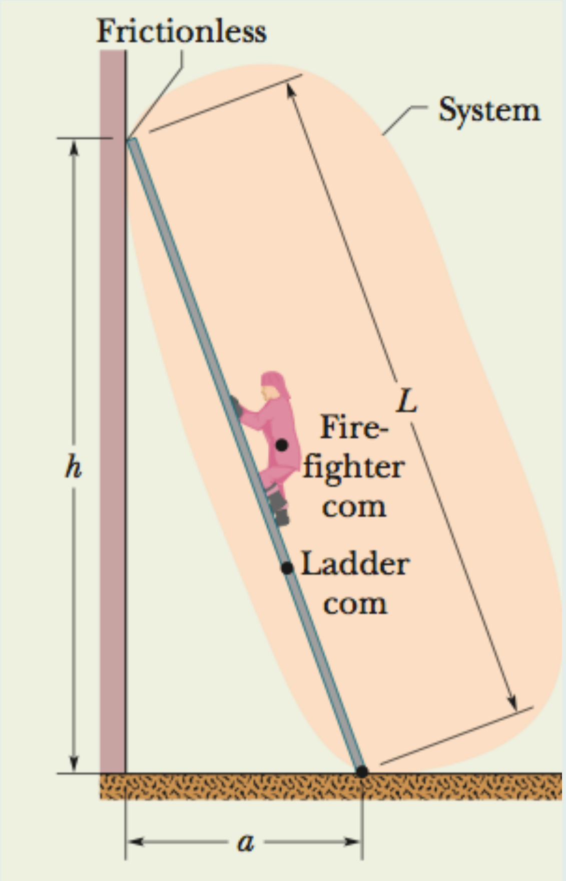
$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} \qquad \vec{L}\ \textbf{constant}\ \Rightarrow \vec{\tau}_{\text{net}} = 0$$

1. The vector sum of all the external forces that act on the body must be zero.

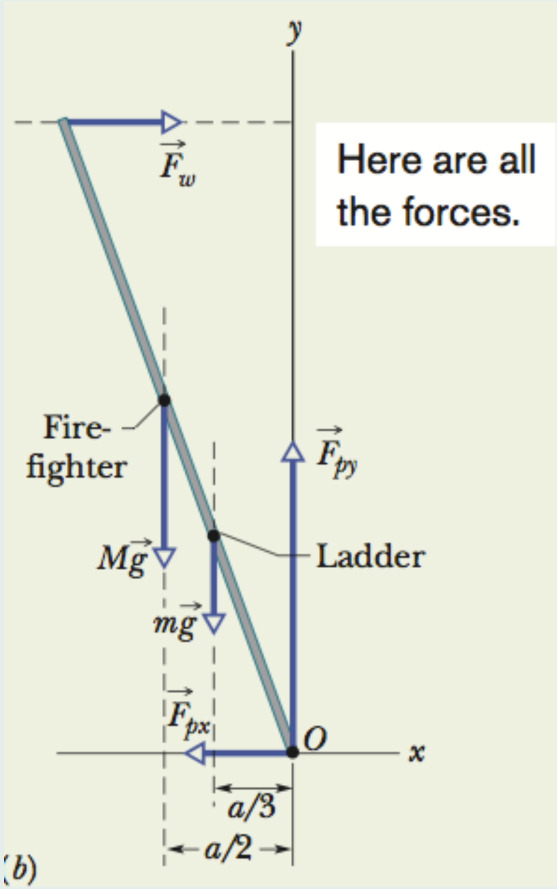
2. The vector sum of all external torques that act on the body, measured about any possible point, must also be zero.

- **Conditions for equilibrium**

$$F_{\text{net},x} = 0 \qquad \tau_{\text{net},x} = 0$$
$$F_{\text{net},y} = 0 \qquad \tau_{\text{net},y} = 0$$
$$F_{\text{net},z} = 0 \qquad \tau_{\text{net},z} = 0$$



- A ladder of length $L = 12\text{m}$ and mass $m = 45\text{kg}$ leans against a frictionless wall. Its upper end is at height $h = 9.3\text{m}$ above the pavement.
- The ladder's center of mass is $L/3$ from the lower end.
- A firefighter of mass $M = 72\text{kg}$ climbs the ladder until her center of mass is $L/2$ from the lower end.
- What then are the magnitudes of the forces on the ladder from the wall and the pavement?



Forces on the ladder:

- Ladder's weight:

$$m\vec{g} = -mg\hat{j}$$

- Firefighter's weight:

$$M\vec{g} = -Mg\hat{j}$$

- Normal to wall:

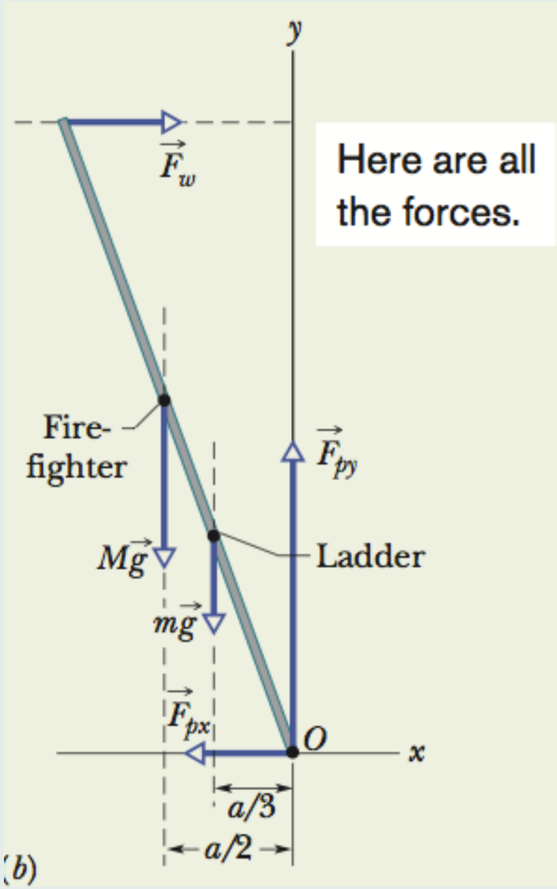
$$\vec{N}_w = N_w\hat{i}$$

- Normal to pavement:

$$\vec{N}_p = N_p\hat{j}$$

- Static friction:

$$\vec{f}_s = -f_s\hat{i}$$



Force balance:

$$x - \text{axis}: \quad N_w - f_s = 0$$

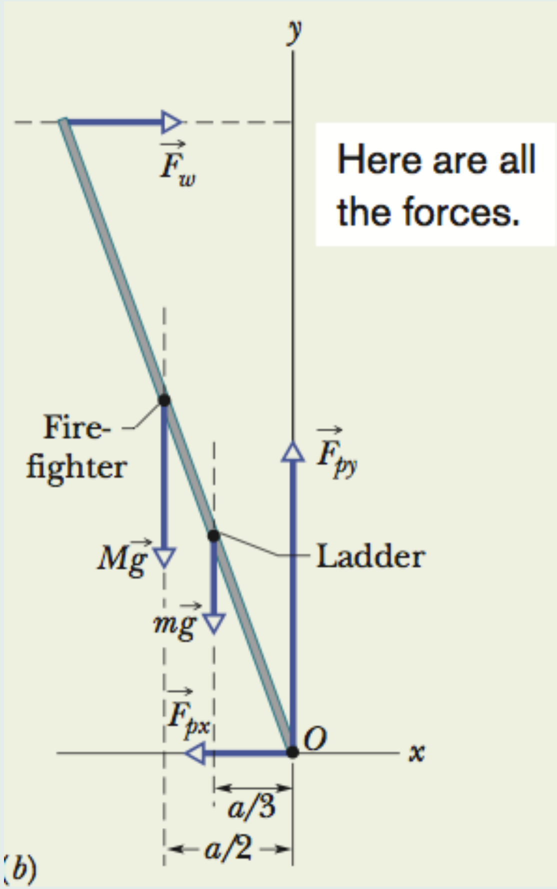
$$y - \text{axis}: \quad N_p - Mg - mg = 0$$

Torque balance about O:

$$Mg(a/2) + mg(a/3) - N_w h = 0$$

a = length of projection of ladder onto pavement

$$a = \sqrt{L^2 - h^2}$$

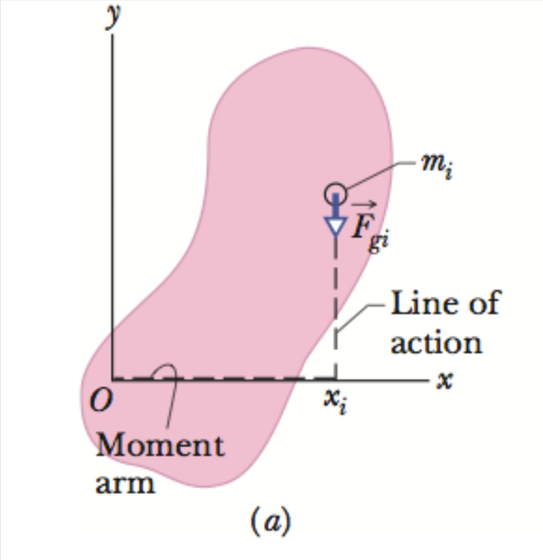


$$N_w = \frac{ag}{h}\left(\frac{M}{2} + \frac{m}{3}\right)$$

$$f_s = N_w$$

$$N_p = (M + m)g$$

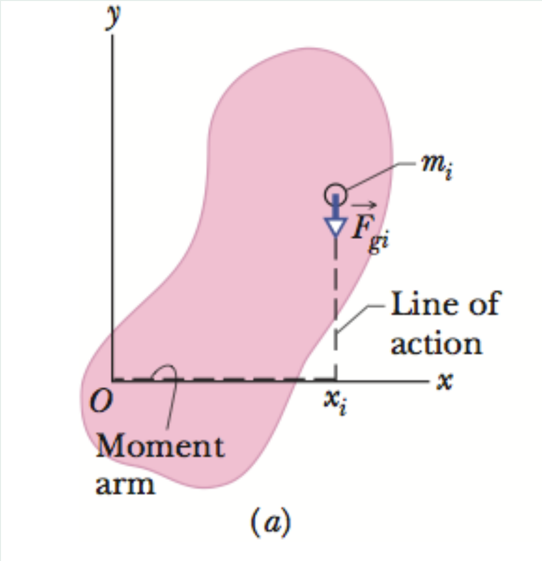
- **Center of gravity**



- Gravitational force acting on a rigid body:

$$\vec{F}_g = \sum_i (\Delta m_i)\vec{g} = M\vec{g}$$

provided that the gravitational field is **uniform** i.e. $\vec{g}$ is the same for **all** mass elements Δm_i

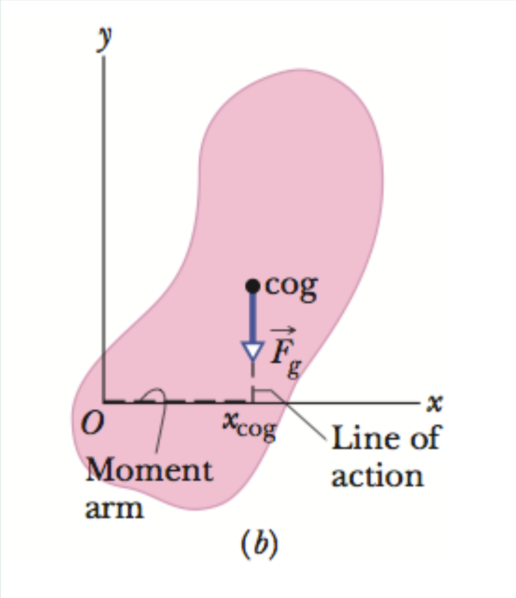


- Torque of gravitational force acting on a rigid body:

$$\vec{\tau}_{F_g} = \sum_i (\Delta m_i)\vec{r}_i \times \vec{g}$$
$$= \vec{r}_{\text{com}} \times (M\vec{g})$$

provided that the gravitational field is **uniform** i.e. $\vec{g}$ is the same for **all** mass elements Δm_i

Conclusions:



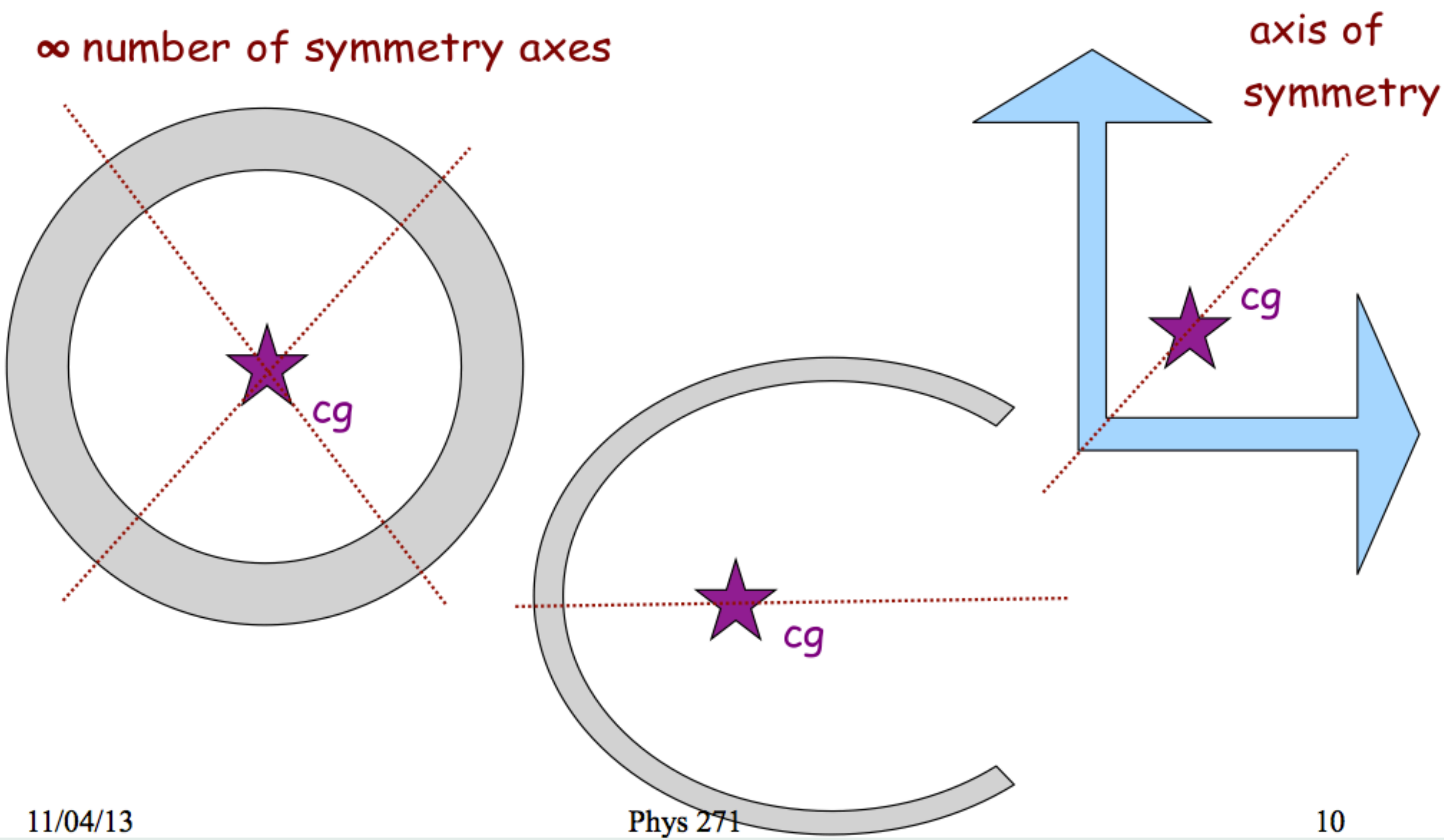
1. The gravitational force $\vec{F}_g$ on a body effectively acts at a single point, called the **center of gravity (cog)** of the body.

2. If $\vec{g}$ is the same for all elements of a body, then the body's **center of gravity (cog)** is coincident with the body's **center of mass (com)**.

3. If $\vec{g}$ is **not** the same for all mass elements

$$\text{COG} \neq \text{COM}$$

∞ number of symmetry axes



- Hang it, twice, from different points (if 2 dimensional object)
  - C of G must be under each pivot point, for it to be in static equilibrium - other $F_{gravity}$ and $F_{support}$ do not line up, and there is a torque (more on this in a few minutes)