1 Physics 613: Problem Set 5 (*due Thursday April 11*)

1.1 Spin operator revisited

In our derivation of the electron magnetic moment from QED, we had the formula

$$\langle e|H_{eff}|e\rangle = -\frac{eB}{m}\langle e|\bar{\Psi}S^{12}\Psi|e\rangle \tag{1}$$

where S^{12} came from $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. I said in class that S^{12} was the generator of spin in the z direction, i.e. $S^{12} = S_z$. Using the explicit form of the γ matrices, verify more generally that $S^{ij} = \epsilon^{ijk}S_k$.

1.2 Lie algebra facts

1. Prove that $[T^a, T^b]$ is traceless and anti-Hermitian.
2. Prove that the structure constants f^{abc} are totally antisymmetric.
3. Prove the Jacobi identity $[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0$
4. Prove that the adjoint commutation relations $[T^a_{adj}, T^b_{adj}] = if^{abc}T^c_{adj}$ are equivalent to the Jacobi identity.
5. Prove that $[T^aT^a, T^b] = 0$ (therefore T^aT^a must be proportional to the identity in every representation; the proportionality constant is called the Casimir invariant).

1.3 $SU(3)$ generators

Look up the Gell-mann basis of the $SU(3)$ generators and explicitly find all the nonzero structure constants for $SU(3)$ in this basis (please use a computer, e.g. Mathematica, for this, don't do it by hand as it would be way too tedious!)

1.4 $SU(3)$ representations

$SU(3)$ irreducible representations (irreps) are labeled by two integers (n, m) and can be thought of as all multiple-index tensors of the form $A^{i_1 \ldots i_n}_{j_1,\ldots j_m}$, with indices running from $1, 2, 3$, which are totally *symmetric* in all upper and all lower indices and are *traceless* under contraction of any upper with any lower index. For example, the fundamental rep corresponds to $(1, 0)$ and the anti-fundamental to $(0, 1)$. The adjoint corresponds to $(1, 1)$.

1. Show that $\dim(n, 0) = \dim(0, n) = \frac{1}{2}(n + 2)(n + 1)$.

Irreps can be multiplied (tensor product) and decomposed (direct sum) into smaller irreps by symmetrizing and tracing over indices.

For example we can multiply a fundamental and antifundamental $A^a B_b = C^a_b + \frac{1}{3}\delta^a_b A^c B_c$ where C is the traceless part of AB. In this way we obtain the sum of the adjoint (octet) and the trivial (singlet) representation. We can express this more mathematically as $(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0)$, or in terms of the dimensions, $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

One can derive (it takes a bit of thought) the following multiplication rule $(n, 0) \otimes (m, 0) = (n+m, 0) \oplus (n+m-2, 1) \oplus (n+m-4, 2) \oplus \ldots$ and similarly for $(0, n) \otimes (0, m)$.

2. The (u, d, s) quarks transform in the fundamental of $SU(3)$ and their antiquarks transform in the anti-fundamental of $SU(3)$. Using the facts above, show that the mesons (which are quark anti-quark bound states) must transform in either the adjoint (octet) or the singlet representation.

3. Using the facts above, explain how the baryon octet and decuplet arise from multiplication of $SU(3)$ representations.