4 Physics 613: Problem Set 4 (*due Thursday March 21*)

4.1 Feynman Parameters

Starting from the Feynman parameter trick we introduced in class:

$$\frac{1}{AB} = \int_0^1 dxdy\, \delta(x+y-1) \frac{1}{(xA+yB)^2} \tag{1}$$

Derive the following additional identities

1. $\frac{1}{AB^n} = \int dxdy\, \delta(x+y-1) \frac{ny^{n-1}}{(xA+yB)^{n+1}}$
2. $\frac{1}{ABC} = 2\int dxdydz\, \delta(x+y+z-1) \frac{1}{(xA+yB+zC)^3}$

4.2 QED with multiple fermions

Consider QED with N_f fermions Ψ_i with charge Q_i and mass m_i, $i = 1, \dots N_f$. (The basic case we have been studying in class is $N_f = 1$ with $Q_1 = 1$ and $m_1 = m$.)

1. What is the vacuum polarization function $\Pi(p^2)$ in this theory? You don't need to derive this from scratch or evaluate any integrals; rather you should start from the result we derived in class for the basic $N_f = 1$ case, which is

$$\Pi(p^2) = -\frac{e^2}{2\pi^2} \int_0^1 dx\, x(1-x) \log\left(\frac{m^2 - p^2 x(1-x)}{m^2}\right) \tag{2}$$

2. In the Standard Model, there are 3 leptons with charge -1 (electron, muon, tau); and 9 quarks with charge 2/3 (up,charm,top with 3 colors each); and 9 quarks with charge $-1/3$ (down, strange, bottom with 3 colors each). Assuming these all have a common mass (let's say 1 GeV for simplicity), use your answer in part 1 to figure out where is the Landau pole of QED for the Standard Model.

4.3 $Z_1 = Z_2$

In this problem we will prove that $Z_1 = Z_2$ at one-loop including the finite parts, starting from the expressions for the electron self energy and vertex function that we derived in class (with some typos and conventions fixed – in this problem we are in the mostly plus signature):

$$\Sigma(\not{p}) = -\frac{\alpha}{2\pi}\int_0^1 dx((4-\epsilon)m + (2-\epsilon)x\not{p})\left(\frac{1}{\epsilon}+\frac{1}{2}\log\frac{\tilde{\mu}^2}{x(1-x)p^2+(1-x)m^2+xm_\gamma^2}\right) \\ -(Z_2-1)\not{p}-(Z_m-1)m \tag{3}$$

and

$$\begin{aligned} V^\mu &= \frac{e^3}{8\pi^2}\left(\left(\frac{1}{\epsilon}-1-\frac{1}{2}\int dF_3\log\frac{D}{\tilde{\mu}^2}\right)\gamma^\mu + \frac{1}{4}\int dF_3\frac{N^\mu}{D}\right)+eZ_1\gamma^\mu \\ D &= x_1(1-x_1)p^2+x_2(1-x_2)p'^2-2x_1x_2p\cdot p'+(x_1+x_2)m^2+x_3m_\gamma^2 \\ N^\mu &= \gamma_\nu(x_1\not{p}-(1-x_2)\not{p}'+m)\gamma^\mu(-(1-x_1)\not{p}+x_2\not{p}'+m)\gamma^\nu \end{aligned} \tag{4}$$

where $\int dF_3 = 2\int dx_1dx_2dx_3\delta(x_1+x_2+x_3-1)$. Impose the renormalization conditions

$$\Sigma'(\not{p})\Big|_{\not{p}=-m} = 0 \tag{5}$$

and

$$\bar{u}(p')V^\mu(p,p')u(p)\Big|_{p=p',p^2=-m^2} = e\bar{u}(p)\gamma^\mu u(p) \tag{6}$$

to determine Z_1 and Z_2, and show that $Z_1 = Z_2$. (You only need to show this for the singular and finite terms in the $\epsilon \to 0$ and $m_\gamma \to 0$ limits. The calculation is quite messy, you are encouraged to use Mathematica to evaluate the necessary integrals and expansions.)

4.4 Gordon Identity

In this problem we will prove the Gordon identity:

$$2m\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu u(p) \tag{7}$$

where $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu]$. This identity plays a key part in the derivation of the anomalous magnetic moment of the electron.

1. Prove the identities $\gamma^\mu\not{p} = -p^\mu - 2iS^{\mu\nu}p_\nu$ and $\not{p}'\gamma^\mu = -p'^\mu + 2iS^{\mu\nu}p'_\nu$.

2. Use part 1 to prove the Gordon identity.