1 Physics 613: Problem Set 2 (*due Monday Feb 19*)

1.1 Spin and the Dirac Equation

1. Verify that $[L_i, P_j] = i\epsilon_{ijk}P_k$ where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum operator.

2. Verify that $[L_z, H_{Dirac}] = i(\alpha_x P_y - \alpha_y P_x)$. What are $[L_x, H_{Dirac}]$ and $[L_y, H_{Dirac}]$?

3. Verify that with $\mathbf{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \equiv \frac{1}{2}\boldsymbol{\Sigma}$, the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ commutes with H_{Dirac}.

1.2 Solutions to the Dirac Equation

In class we introduced the solutions to the Dirac equation $\psi(x) = u_s(k)e^{-ikx}$ and $\psi(x) = v_s(k)e^{ikx}$.

1. Verify by plugging into the Dirac equation that u_s and v_s satisfy

$$(\not{k} - m)u_s(k) = 0 \tag{1}$$

and

$$(\not{k} + m)v_s(k) = 0 \tag{2}$$

2. By considering the eigenvalues of $\not{k} - m$ and $\not{k} + m$, prove that there are exactly two independent u_s and two independent v_s solutions for every k (so $s = 1, 2$).

3. Show that the *helicity operator* $h = \frac{\mathbf{P}\cdot\boldsymbol{\Sigma}}{|\mathbf{P}|}$ commutes with the Dirac Hamiltonian $H_{Dirac} = \boldsymbol{\alpha} \cdot \mathbf{P} + \beta m$ and find the eigenvalues of h.

4. For momentum in the z direction (i.e. $\mathbf{k} = (0, 0, k)$), find explicitly the solutions u_s and v_s classified by eigenvalues of the helicity operator.

1.3 Dirac Hamiltonian and charge

1. Substitute the mode expansion

$$\Psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} \sum_s \left(b_s(\mathbf{k})u_s(\mathbf{k})e^{-ikx} + d_s^\dagger(\mathbf{k})v_s(\mathbf{k})e^{ikx}\right) \tag{3}$$

into the Dirac Hamiltonian

$$H = \int d^3x \left(-i\bar{\Psi}\gamma^i\partial_i\Psi(x) + m\bar{\Psi}\Psi(x)\right) \tag{4}$$

and, using the canonical commutation relations for b_s and d_s, derive

$$H = \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} (b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) + d_s^\dagger(\mathbf{k}) d_s(\mathbf{k})) \tag{5}$$

2. Do the same for the charge operator

$$Q = \int d^3x \Psi^\dagger \Psi \tag{6}$$

and derive

$$Q = \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} (b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) - d_s^\dagger(\mathbf{k}) d_s(\mathbf{k})) \tag{7}$$