1 Physics 613: Problem Set 1 (*due Wednesday, Feb 25*)

1.1 Covariant Formulation of E&M

Maxwell's equations are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} - \dot{\mathbf{E}} &= 0 \\ \nabla \times \mathbf{E} + \dot{\mathbf{B}} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{1}$$

with $\mathbf{E}$ and $\mathbf{B}$ given in terms of the scalar and vector potential as:

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \dot{\mathbf{A}} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{2}$$

We can combine Φ and $\mathbf{A}$ into the Lorentz-covariant gauge potential $A_\mu = (\Phi, \mathbf{A})$ and define the gauge invariant field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Show how $F_{\mu\nu}$ contains the $\mathbf{E}$ and $\mathbf{B}$ fields.
2. Show how the covariant form of Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$, together with the relations between the gauge potential and the $\mathbf{E}$ and $\mathbf{B}$ fields (2), imply the usual component form of Maxwell's equations (1).
3. In class we argued that $F_{\mu\nu}F^{\mu\nu}$ was the unique object which satisfies the following requirements: Lorentz invariant, gauge invariant, quadratic in A_μ and second order in derivatives. There is one other possible object that satisfies these requirements: $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric 4-index tensor. Show that this term (called the θ-term) is in fact a total derivative.

1.2 Canonical Quantization of a Scalar Field

A much simpler field theory than Maxwell theory is that of a real scalar field $\phi(x)$, with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{3}$$

1. Derive the equation of motion for ϕ:

$$\Box\phi = m^2\phi \tag{4}$$

   This is called the Klein-Gordon equation.

2. Derive the conjugate momentum $\Pi = \dot{\phi}$ and show that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{5}$$

3. The mode expansion for ϕ is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \left(a_{\mathbf{k}} e^{ikx} + a^\dagger_{\mathbf{k}} e^{-ikx} \right) \tag{6}$$

   Remember that $kx = k^\mu x_\mu = E_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x}$. How are E_k and $\mathbf{k}$ related?

4. Show that if $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ satisfy the commutation relations of an infinite set of simple harmonic oscillators

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0 \\ [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] &= (2\pi)^3 (2E_{\mathbf{k}}) \delta^3(\mathbf{k}-\mathbf{k}') \end{aligned} \tag{7}$$

   then one obtains the canonical commutation relations for ϕ and Π:

$$\begin{aligned} [\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] &= [\Pi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = 0 \\ [\phi(\mathbf{x},t), \Pi(\mathbf{x}',t)] &= i\delta^3(\mathbf{x}-\mathbf{x}') \end{aligned} \tag{8}$$

5. Plug the mode expansion (6) into the Hamiltonian density (5) and show that the Hamiltonian reduces to that of an infinite set of decoupled simple harmonic oscillators (with energy $E_{\mathbf{k}}$):

$$H = \int d^3x\, \mathcal{H} = \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \tag{9}$$

   up to an overall (infinite) zero point energy.

1.3 Dirac Matrices

In class we derived the Dirac equation by starting from the requirement that there exist $N \times N$ Hermitian matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β satisfying

$$(\mathbf{p}^2 + m^2)\mathbb{1}_N = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)^2 \tag{10}$$

Here we will verify a number of the steps that we skipped over.

1. Verify that (10) implies $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$ and $\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0$ (for $i \neq j$).

2. Use part 1 to prove that the $\boldsymbol{\alpha}$ and β matrices must be traceless and have eigenvalues ± 1. (Together these imply that N must be even.)

3. Prove that there is no solution to (10) with $N = 2$.

4. For $N = 4$ verify that $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$ satisfy (10).

5. Verify that $\gamma^0 = \beta$ and $\gamma^i = \beta \alpha^i$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

1.4 LEP and the Higgs

In the first lecture we described the LEP e^+e^- circular collider experiment at CERN which reached a maximum center of mass energy of 209 GeV. This allowed it to reach a sensitivity in the Higgs mass of around 114 GeV through the process $e^+e^- \to ZH$. The main thing limiting LEP's energy was energy loss due to synchotron radiation – every revolution of the beam around the LEP ring would lose energy

$$\Delta E \propto \frac{E^4}{m_e^4 R} \tag{11}$$

where E is the COM energy and $R = 4.2$ km is the radius of the LEP ring.

1. For a fixed power budget, approximately how much bigger would the LEP ring have to have been to reach a COM energy that could have been sensitive to the actual Higgs mass (125 GeV)?

2. Again for a fixed amount of power, and re-using the LEP tunnel, what is the maximum energy one could reach by colliding protons instead of electrons, if synchotron radiation was the only limiting factor?

3. What if we collided muons instead?