1 Physics 613: Problem Set 1 (*due Monday, Feb 5*)

1.1 Covariant Formulation of E&M

Maxwell's equations are

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{B} - \dot{\mathbf{E}} &= 0 \\
\nabla \times \mathbf{E} + \dot{\mathbf{B}} &= 0 \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned} \tag{1}$$

with $\mathbf{E}$ and $\mathbf{B}$ given in terms of the scalar and vector potential as:

$$\begin{aligned}
\mathbf{E} &= -\nabla \Phi - \dot{\mathbf{A}} \\
\mathbf{B} &= \nabla \times \mathbf{A}
\end{aligned} \tag{2}$$

We can combine Φ and $\mathbf{A}$ into the Lorentz-covariant gauge potential $A_\mu = (\Phi, \mathbf{A})$ and define the gauge invariant field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Show how $F_{\mu\nu}$ contains the $\mathbf{E}$ and $\mathbf{B}$ fields.

2. Show how the covariant form of Maxwell's equations $\partial_\mu F^{\mu\nu} = 0$, together with the relations between the gauge potential and the $\mathbf{E}$ and $\mathbf{B}$ fields (2), imply the usual component form of Maxwell's equations (1).

3. In class we argued that $F_{\mu\nu}F^{\mu\nu}$ was the unique object which satisfies the following requirements: Lorentz invariant, gauge invariant, quadratic in A_μ and second order in derivatives. There is one other possible object that satisfies these requirements: $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric 4-index tensor. Show that this term (called the θ-term) is in fact a total derivative.

1.2 Canonical Quantization of a Scalar Field

A much simpler field theory than Maxwell theory is that of a real scalar field $\phi(x)$, with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \tag{3}$$

1. Derive the equation of motion for ϕ:

$$\Box \phi = m^2 \phi \tag{4}$$

This is called the Klein-Gordon equation.

2. Derive the conjugate momentum $\Pi = \dot{\phi}$ and show that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{5}$$

3. The mode expansion for ϕ is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \left(a_{\mathbf{k}} e^{ikx} + a_{\mathbf{k}}^\dagger e^{-ikx} \right) \tag{6}$$

   Remember that $kx = k^\mu x_\mu = E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x}$. How are E_k and $\mathbf{k}$ related?

4. Show that if $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ satisfy the commutation relations of an infinite set of simple harmonic oscillators

$$\begin{aligned} &[a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger] = 0 \\ &[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 (2E_{\mathbf{k}}) \delta^3(\mathbf{k} - \mathbf{k}') \end{aligned} \tag{7}$$

   then one obtains the canonical commutation relations for ϕ and Π:

$$\begin{aligned} &[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\Pi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = 0 \\ &[\phi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = i\delta^3(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{8}$$

5. Plug the mode expansion (6) into the Hamiltonian density (5) and show that the Hamiltonian reduces to that of an infinite set of decoupled simple harmonic oscillators (with energy $E_{\mathbf{k}}$):

$$H = \int d^3x \, \mathcal{H} = \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{9}$$

   up to an overall (infinite) zero point energy.

1.3 Dirac Matrices

In class we derived the Dirac equation by starting from the requirement that there exist $N \times N$ Hermitian matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β satisfying

$$(\mathbf{p}^2 + m^2)\mathbb{1}_N = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)^2 \tag{10}$$

Here we will verify a number of the steps that we skipped over.

1. Verify that (10) implies $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$ and $\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0$ (for $i \neq j$).

2. Use part 1 to prove that the $\boldsymbol{\alpha}$ and β matrices must be traceless and have eigenvalues ± 1. (Together these imply that N must be even.)

3. Prove that there is no solution to (10) with $N = 2$.

4. For $N = 4$ verify that $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$ satisfy (10).

5. Verify that $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.